Chapter 5

Linear Models and Matrix Algebra (Continued)

In Chap. 4, it was shown that a linear-equation system, however large, may be written in a compact matrix notation. Furthermore, such an equation system can be solved by finding the inverse of the coefficient matrix, provided the inverse exists. Now we must address ourselves to the questions of how to test for the existence of the inverse and how to find that inverse. Only after we have answered these questions will it be possible to apply matrix algebra meaningfully to economic models.

5.1 Conditions for Nonsingularity of a Matrix

A given coefficient matrix A can have an inverse (i.e., can be "nonsingular") only if it is square. As was pointed out earlier, however, the squareness condition is necessary but not sufficient for the existence of the inverse A^{-1}. A matrix can be square, but singular (without an inverse) nonetheless.

Necessary versus Sufficient Conditions

The concepts of "necessary condition" and "sufficient condition" are used frequently in economics. It is important that we understand their precise meanings before proceeding further.

A necessary condition is in the nature of a prerequisite: Suppose that a statement p is true *only if* another statement q is true; then q constitutes a necessary condition of p. Symbolically, we express this as follows:

$$p \Rightarrow q \tag{5.1}$$

which is read as "p *only if* q," or alternatively, "if p, then q." It is also logically correct to interpret (5.1) to mean "p *implies* q." It may happen, of course, that we also have $p \Rightarrow w$ at the same time. Then both q and w are necessary conditions for p.

Example 1 If we let p be the statement "a person is a father" and q be the statement "a person is male," then the logical statement $p \Rightarrow q$ applies. A person is a father *only if* he is male, and to be male is a necessary condition for fatherhood. Note, however, that the converse is not true: fatherhood is not a necessary condition for maleness.

A different type of situation is one in which a statement p is true if q is true, but p can also be true when q is not true. In this case, q is said to be a sufficient condition for p. The truth of q suffices to establish the truth of p, but it is not a necessary condition for p. This case is expressed symbolically by

$$p \Leftarrow q \tag{5.2}$$

which is read: "*p if q*" (without the word *only*)—or alternatively, "if q, then p," as if reading (5.2) backward. It can also be interpreted to mean "q implies p."

Example 2

If we let p be the statement "one can get to Europe" and q be the statement "one takes a plane to Europe," then $p \Leftarrow q$. Flying can serve to get one to Europe, but since ocean transportation is also feasible, flying is not a prerequisite. We can write $p \Leftarrow q$, but not $p \Rightarrow q$.

In a third possible situation, q is *both* necessary and sufficient for p. In such an event, we write

$$p \Leftrightarrow q \tag{5.3}$$

which is read: "*p if and only if q*" (also written as "p iff q"). The double-headed arrow is really a combination of the two types of arrow in (5.1) and (5.2), hence the joint use of the two terms "if" and "only if." Note that (5.3) states not only that p implies q but also that q implies p.

Example 3

If we let p be the statement "there are less than 30 days in the month" and q be the statement "it is the month of February," then $p \Leftrightarrow q$. To have less than 30 days in the month, it is necessary that it be February. Conversely, the specification of February is sufficient to establish that there are less than 30 days in the month. Thus q is a necessary-and-sufficient condition for p.

In order to prove $p \Rightarrow q$, it needs to be shown that q follows logically from p. Similarly, to prove $p \Leftarrow q$ requires a demonstration that p follows logically from q. But to prove $p \Leftrightarrow q$ necessitates a demonstration that p and q follow from each other.

Necessary conditions and sufficient conditions are important as screening devices. Consider a pool of applicants being considered for scholarship awards, or for job positions. Since necessary conditions are in the nature of prerequisites, they serve to separate the candidates into two groups: Those who fail to meet the necessary conditions are automatically disqualified; those who satisfy the necessary conditions remain as admissible candidates. To remain as an admissible candidate, however, carries no guarantee that the candidate will eventually be successful. Thus, necessary conditions are more conclusive in screening out the unsuccessful candidates than in identifying the successful ones. In general, we should bear in mind that necessary conditions are *not* in themselves *sufficient*.

In contrast to necessary conditions, sufficient conditions serve directly to identify successful candidates. A candidate that satisfies a sufficient condition is automatically a successful one. Just as necessary conditions are not in themselves sufficient, sufficient conditions are not in themselves necessary. This is because, along with any given sufficient condition, there may exist other, less stringent, sufficient conditions, and the candidate who fails to satisfy the given sufficient condition may yet qualify under an easier sufficient condition. For example, a grade of A is sufficient for passing a course, but it is not a necessary condition since a grade of B is also sufficient.

The most effective screening device is found in the necessary-and-sufficient conditions. Failure to satisfy such a condition means the candidate is definitely out, and satisfaction of such a condition means the candidate is definitely in. We can find an immediate application of this in our present discussion of nonsingularity of a matrix.

Conditions for Nonsingularity

After the squareness condition (a necessary condition) is already met, a sufficient condition for the nonsingularity of a matrix is that its rows be linearly independent (or, what amounts to the same thing, that its *columns* be linearly independent). When the dual conditions of squareness and linear independence are taken together, they constitute the necessary-and-sufficient condition for nonsingularity (nonsingularity $\Leftrightarrow$ squareness *and* linear independence).

An $n \times n$ coefficient matrix A can be considered as an ordered set of row vectors, i.e., as a column vector whose elements are themselves row vectors:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} v_1' \\ v_2' \\ \vdots \\ v_n' \end{bmatrix}$$

where $v_i' = [a_{i1} \quad a_{i2} \quad \cdots \quad a_{in}]$, $i = 1, 2, \ldots, n$. For the rows (row vectors) to be linearly independent, none must be a linear combination of the rest. More formally, as was mentioned in Sec. 4.3, linear row independence requires that the only set of scalars k_i which can satisfy the vector equation

$$\sum_{i=1}^{n} k_i v_i' = \underset{(1\times n)}{0} \tag{5.4}$$

be $k_i = 0$ for all i.

Example 4

If the coefficient matrix is

$$A = \begin{bmatrix} 3 & 4 & 5 \\ 0 & 1 & 2 \\ 6 & 8 & 10 \end{bmatrix} = \begin{bmatrix} v_1' \\ v_2' \\ v_3' \end{bmatrix}$$

then, since $[6 \quad 8 \quad 10] = 2[3 \quad 4 \quad 5]$, we have $v_3' = 2v_1' = 2v_1' + 0v_2'$. Thus the third row is expressible as a linear combination of the first two, and the rows are *not* linearly independent. Alternatively, we may write the previous equation as

$$2v_1' + 0v_2' - v_3' = [6 \quad 8 \quad 10] + [0 \quad 0 \quad 0] - [6 \quad 8 \quad 10] = [0 \quad 0 \quad 0]$$

Inasmuch as the set of scalars that led to the zero vector of (5.4) is not $k_i = 0$ for all i, it follows that the rows are linearly dependent.

Unlike the squareness condition, the linear-independence condition cannot normally be ascertained at a glance. Thus a method of testing linear independence among rows (or columns) needs to be developed. Before we concern ourselves with that task, however, it would strengthen our motivation first to have an intuitive understanding of why the linear-independence condition is heaped together with the squareness condition at all. From the discussion of counting equations and unknowns in Sec. 3.4, we recall the general conclusion that, for a system of equations to possess a unique solution, it is not sufficient to have the same number of equations as unknowns. In addition, the *equations* must be consistent with and functionally independent (meaning, in the present context of linear systems, *linearly* independent) of one another. There is a fairly obvious tie-in between the "same number of equations as unknowns" criterion and the *squareness* (same number of rows and columns) of the coefficient matrix. What the "linear independence among the rows" requirement does is to preclude the inconsistency and the linear dependence *among the equations* as well. Taken together, therefore, the dual requirement of squareness and row independence in the coefficient matrix is tantamount to the conditions for the existence of a unique solution enunciated in Sec. 3.4.

Let us illustrate how the linear dependence *among the rows* of the coefficient matrix can cause inconsistency or linear dependence *among the equations* themselves. Let the equation system $Ax = d$ take the form

$$\begin{bmatrix} 10 & 4 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

where the coefficient matrix A contains linearly dependent rows: $v_1' = 2v_2'$. (Note that its columns are also dependent, the first being $\frac{5}{2}$ of the second.) We have not specified the values of the constant terms d_1 and d_2, but there are only *two* distinct possibilities regarding their relative values: (1) $d_1 = 2d_2$ and (2) $d_1 \neq 2d_2$. Under the first—with, say, $d_1 = 12$ and $d_2 = 6$—the two equations are consistent but *linearly dependent* (just as the two rows of matrix A are), for the first equation is merely the second equation times 2. One equation is then redundant, and the system reduces in effect to a single equation, $5x_1 + 2x_2 = 6$, with an infinite number of solutions. For the second possibility—with, say, $d_1 = 12$ but $d_2 = 0$—the two equations are *inconsistent,* because if the first equation $(10x_1 + 4x_2 = 12)$ is true, then, by halving each term, we can deduce that $5x_1 + 2x_2 = 6$; consequently the second equation $(5x_1 + 2x_2 = 0)$ cannot possibly be true also. Thus no solution exists.

The upshot is that no unique solution will be available (under either possibility) so long as the rows in the coefficient matrix A are linearly dependent. In fact, the only way to have a unique solution is to have linearly independent rows (or columns) in the coefficient matrix. In that case, matrix A will be nonsingular, which means that the inverse A^{-1} does exist, and that a unique solution $x^* = A^{-1}d$ can be found.

Rank of a Matrix

Even though the concept of row independence has been discussed only with regard to square matrices, it is equally applicable to any $m \times n$ rectangular matrix. If the maximum number of linearly independent rows that can be found in such a matrix is r, the matrix is said to be of *rank r.* (The rank also tells us the maximum number of linearly independent *columns* in the said matrix.) The rank of an $m \times n$ matrix can be at most m or n, whichever is smaller.

Given a matrix with only two rows (or two columns), row independence (or column independence) is easily verified by visual inspection—one only has to check whether one row (column) is the exact multiple of the other. But for a matrix of larger dimension, visual inspection may not be feasible, and a more formal method is needed. One method for finding the rank of a matrix A (not necessarily square), i.e., for determining the number of independent rows in A, involves transforming A into a so-called *echelon matrix* by using certain "elementary row operations." A particular structural feature of the echelon matrix will then tell us the rank of matrix A.

There are only three types of *elementary row operations* on a matrix:†

1. Interchange of any two rows in the matrix.
2. Multiplication (or division) of a row by any scalar $k \neq 0$.
3. Addition of "k times any row" to another row.

While each of these operations converts a given matrix A into a different form, none of them alters the rank. It is this characteristic of elementary row operations that enables us to read the rank of A from its echelon matrix. The easiest way to explain the method of echelon matrix is by a specific example.

Example 5

Find the rank of the matrix

$$A = \begin{bmatrix} 0 & -11 & -4 \\ 2 & 6 & 2 \\ 4 & 1 & 0 \end{bmatrix}$$

from its echelon form. First, we check the first column of A for the presence of zero elements. If there are zero elements in column 1, we move those zero elements to the bottom of the matrix. In the case of A, we want to move the 0 (first element of column 1) to the bottom of that column, which can be accomplished by interchanging row 1 and row 3 (using the first elementary row operation). The result is

$$A_1 = \begin{bmatrix} 4 & 1 & 0 \\ 2 & 6 & 2 \\ 0 & -11 & -4 \end{bmatrix}$$

Our next objective is to reshape the first column of A_1 into a unit vector e_1 as defined in (4.7). To transform the element 4 into unity, we divide row 1 of A_1 by the scalar 4 (applying the second elementary row operation), which yields

$$A_2 = \begin{bmatrix} 1 & \frac{1}{4} & 0 \\ 2 & 6 & 2 \\ 0 & -11 & -4 \end{bmatrix}$$

Then, to transform the element 2 in column 1 of A_2 into 0, we multiply row 1 of A_2 by −2, and then add the result to row 2 of A_2 (applying the third elementary row operation). The resulting matrix,

$$A_3 = \begin{bmatrix} 1 & \frac{1}{4} & 0 \\ 0 & 5\frac{1}{2} & 2 \\ 0 & -11 & -4 \end{bmatrix}$$

† Similarly to elementary row operations, there can be defined elementary column operations. For our purposes, row operations are sufficient.

now has the desired unit vector e_1 as its first column. Having achieved this, we now exclude the first row of A_3 from further consideration, and continue to work only on the remaining two rows, where we want to create a two-element unit vector in the second column—by transforming the element $5\frac{1}{2}$ into 1, and the element -11 into 0. To this end, we need to divide row 2 of A_3 by $5\frac{1}{2}$, thereby changing the row into the vector $[0 \quad 1 \quad \frac{4}{11}]$, and then add 11 times this vector to row 3 of A_3. The end result, in the form of

$$A_4 = \begin{bmatrix} 1 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{4}{11} \\ 0 & 0 & 0 \end{bmatrix}$$

exemplifies the echelon matrix, which, by definition, possesses three structural features. First, nonzero rows (rows with at least one nonzero element) appear above the zero rows (rows that contain only 0s). Second, in every nonzero row, the first nonzero element is unity. Third, the unit element (the first nonzero element) in any row must appear to the left of the counterpart unit element of the immediately following row. It should be clear by now that all the elementary row operations we have undertaken are designed to produce these features in A_4.

Now, we can simply read the rank of A from the number of nonzero rows present in the echelon matrix A_4. Since A_4 contains two nonzero rows, we can conclude that $r(A) = 2$. This is, of course, also the rank of matrices A_1 through A_4, because elementary row operations do not alter the rank of a matrix.

The method of echelon matrix transformation applies to nonsquare as well as square matrices. We have chosen a square matrix for Example 5 because our immediate objective is to address the question of nonsingularity, which pertains only to square matrices. By definition, for an $n \times n$ matrix A to be nonsingular, it must have n linearly independent rows (or columns); consequently, it must be of rank n, and its echelon matrix must contain exactly n nonzero rows, with no zero rows at all. Conversely, an $n \times n$ matrix having rank n must be nonsingular. Thus an $n \times n$ echelon matrix with no zero rows must be nonsingular, as is the matrix from which the echelon matrix is derived via elementary row operations. In Example 5, the matrix A is 3×3, but $r(A) = 2$; hence, A is not nonsingular.

EXERCISE 5.1

1. In the following paired statements, let p be the first statement and q the second. Indicate for each case whether (5.1), (5.2), or (5.3) applies.
   (*a*) It is a holiday; it is Thanksgiving Day.
   (*b*) A geometric figure has four sides; it is a rectangle.
   (*c*) Two ordered pairs (a, b) and (b, a) are equal; a is equal to b.
   (*d*) A number is rational; it can be expressed as a ratio of two integers.
   (*e*) A 4×4 matrix is nonsingular; the rank of the 4×4 matrix is 4.
   (*f*) The gasoline tank in my car is empty; I cannot start my car.
   (*g*) The letter is returned to the sender with the marking "addressee unknown"; the sender wrote the wrong address on the envelope.

2. Let p be the statement "a geometric figure is a square," and let q be as follows:
   (*a*) It has four sides.
   (*b*) It has four equal sides.
   (*c*) It has four equal sides each perpendicular to the adjacent one.
   Which is true for each case: $p \Rightarrow q$, $p \Leftarrow q$, or $p \Leftrightarrow q$?
3. Are the rows linearly independent in each of the following?
   (*a*) $\begin{bmatrix} 24 & 8 \\ 9 & -3 \end{bmatrix}$ (*b*) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ (*c*) $\begin{bmatrix} 0 & 4 \\ 3 & 2 \end{bmatrix}$ (*d*) $\begin{bmatrix} -1 & 5 \\ 2 & -10 \end{bmatrix}$
4. Check whether the columns of each matrix in Prob. 3 are also linearly independent. Do you get the same answer as for row independence?
5. Find the rank of each of the following matrices from its echelon matrix, and comment on the question of nonsingularity.
   (*a*) $A = \begin{bmatrix} 1 & 5 & 1 \\ 0 & 3 & 9 \\ -1 & 0 & 0 \end{bmatrix}$ (*c*) $C = \begin{bmatrix} 7 & 6 & 3 & 3 \\ 0 & 1 & 2 & 1 \\ 8 & 0 & 0 & 8 \end{bmatrix}$
   (*b*) $B = \begin{bmatrix} 0 & -1 & -4 \\ 3 & 1 & 2 \\ 6 & 1 & 0 \end{bmatrix}$ (*d*) $D = \begin{bmatrix} 2 & 7 & 9 & -1 \\ 1 & 1 & 0 & 1 \\ 0 & 5 & 9 & -3 \end{bmatrix}$
6. By definition of linear dependence among rows of a matrix, one or more rows can be expressed as a linear combination of some other rows. In the echelon matrix, linear dependence is signified by the presence of one or more zero rows. What provides the link between the presence of a linear combination of rows in a given matrix and the presence of zero rows in the echelon matrix?

5.2 Test of Nonsingularity by Use of Determinant

To ascertain whether a square matrix is nonsingular, we can also make use of the concept of determinant.

Determinants and Nonsingularity

The determinant of a square matrix A, denoted by $|A|$, is a uniquely defined scalar (number) associated with that matrix. Determinants are defined only for *square* matrices. The smallest possible matrix is, of course, the 1×1 matrix $A = [a_{11}]$. By definition, its determinant is equal to the single element a_{11} itself: $|A| = |a_{11}| = a_{11}$. The symbol $|a_{11}|$ here must not be confused with the look-alike symbol for the absolute value of a number. In the absolute-value context, we have, for instance, not only $|5| = 5$, but also $|-5| = 5$, because the absolute value of a number is its numerical value without regard to the algebraic sign. In contrast, the determinant symbol preserves the sign of the element, so while $|8| = 8$ (a positive number), we have $|-8| = -8$ (a negative number). This distinction proves to be crucial in the later discussion when we apply determinantal tests whose results depend critically on the signs of determinants of various dimensions, including 1×1 ones, such as $|a_{11}| = a_{11}$.

For a 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, its determinant is defined to be the sum of two terms as follows:

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12} \qquad [= \text{a scalar}] \qquad \textbf{(5.5)}$$

which is obtained by multiplying the two elements in the principal diagonal of A and then subtracting the product of the two remaining elements. In view of the dimension of matrix A, the determinant $|A|$ given in (5.5) is called a *second-order determinant*.

Example 1 Given $A = \begin{bmatrix} 10 & 4 \\ 8 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 0 & -1 \end{bmatrix}$, their determinants are

$$|A| = \begin{vmatrix} 10 & 4 \\ 8 & 5 \end{vmatrix} = 10(5) - 8(4) = 18$$

and

$$|B| = \begin{vmatrix} 3 & 5 \\ 0 & -1 \end{vmatrix} = 3(-1) - 0(5) = -3$$

While a determinant (enclosed by two vertical bars rather than brackets) is by definition a scalar, a matrix as such does not have a numerical value. In other words, a determinant is reducible to a number, but a matrix is, in contrast, a whole block of numbers. It should also be emphasized that a determinant is defined only for a square matrix, whereas a matrix as such does not have to be square.

Even at this early stage of discussion, it is possible to have an inkling of the relationship between the linear dependence of the rows in a matrix A, on the one hand, and its determinant $|A|$, on the other. The two matrices

$$C = \begin{bmatrix} c'_1 \\ c'_2 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 3 & 8 \end{bmatrix} \qquad \text{and} \qquad D = \begin{bmatrix} d'_1 \\ d'_2 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 8 & 24 \end{bmatrix}$$

both have linearly dependent rows, because $c'_1 = c'_2$ and $d'_2 = 4d'_1$. Both of their determinants also turn out to be equal to zero:

$$|C| = \begin{vmatrix} 3 & 8 \\ 3 & 8 \end{vmatrix} = 3(8) - 3(8) = 0$$

$$|D| = \begin{vmatrix} 2 & 6 \\ 8 & 24 \end{vmatrix} = 2(24) - 8(6) = 0$$

This result strongly suggests that a "vanishing" determinant (a zero-value determinant) may have something to do with linear dependence. We shall see that this is indeed the case. Furthermore, the value of a determinant $|A|$ can serve not only as a criterion for testing the linear independence of the rows (hence the nonsingularity) of matrix A, but also as an input in the calculation of the inverse A^{-1}, if it exists.

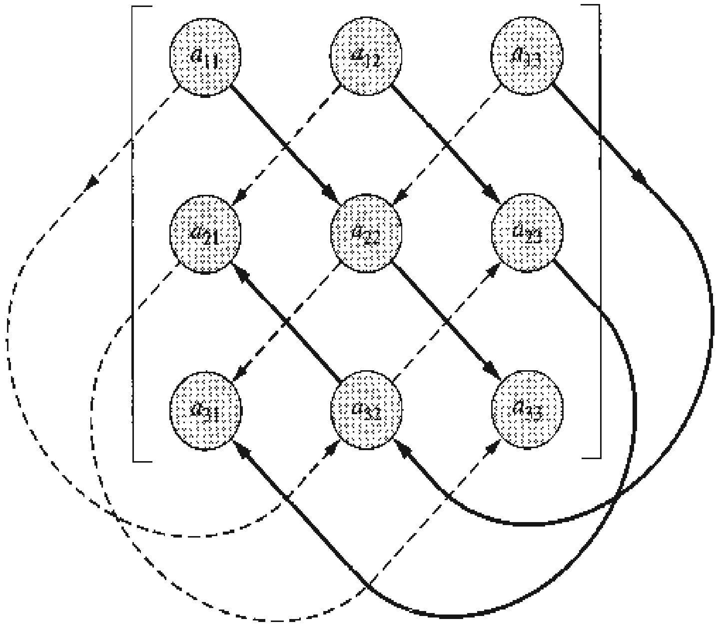
First, however, we must widen our vista by a discussion of higher-order determinants.

Evaluating a Third-Order Determinant

A determinant of order 3 is associated with a 3×3 matrix. Given

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

FIGURE 5.1



its determinant has the value

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \qquad [= \text{a scalar}] \tag{5.6}$$

Looking first at the lower line of (5.6), we see the value of $|A|$ expressed as a sum of six product terms, three of which are prefixed by minus signs and three by plus signs. Complicated as this sum may appear, there is nonetheless a very easy way of "catching" all these six terms from a given third-order determinant. This is best explained diagrammatically (Fig. 5.1). In the determinant shown in Fig. 5.1, each element in the top row has been linked with two other elements via two *solid* arrows as follows: $a_{11} \to a_{22} \to a_{33}$, $a_{12} \to a_{23} \to a_{31}$, and $a_{13} \to a_{32} \to a_{21}$. Each triplet of elements so linked can be multiplied out, and their product taken as one of the six product terms in (5.6). The solid-arrow product terms are to be prefixed with plus signs.

On the other hand, each top-row element has also been connected with two other elements via two *broken* arrows as follows: $a_{11} \to a_{32} \to a_{23}$, $a_{12} \to a_{21} \to a_{33}$, and $a_{13} \to a_{22} \to a_{31}$. Each triplet of elements so connected can also be multiplied out, and their product taken as one of the six terms in (5.6). Such products are prefixed by minus signs. The sum of all the six products will then be the value of the determinant.

Example 2

$$\begin{vmatrix} 2 & 1 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = (2)(5)(9) + (1)(6)(7) + (3)(8)(4) - (2)(8)(6) - (1)(4)(9) - (3)(5)(7) = -9$$

Example 3

$$\begin{vmatrix} -7 & 0 & 3 \\ 9 & 1 & 4 \\ 0 & 6 & 5 \end{vmatrix} = (-7)(1)(5) + (0)(4)(0) + (3)(6)(9) - (-7)(6)(4) - (0)(9)(5) - (3)(1)(0)$$

$$= 295$$

This method of cross-diagonal multiplication provides a handy way of evaluating a third-order determinant, but unfortunately it is *not* applicable to determinants of orders higher than 3. For the latter, we must resort to the so-called Laplace expansion of the determinant.

Evaluating an *n*th-Order Determinant by Laplace Expansion

Let us first explain the *Laplace-expansion* process for a third-order determinant. Returning to the first line of (5.6), we see that the value of $|A|$ can also be regarded as a sum of *three* terms, each of which is a product of a first-row element and a particular *second*-order determinant. This latter process of evaluating $|A|$—by means of certain lower-order determinants—illustrates the Laplace expansion of the determinant.

The three second-order determinants in (5.6) are not arbitrarily determined, but are specified by means of a definite rule. The first one, $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$, is a *sub*determinant of $|A|$ obtained by deleting the *first* row and *first* column of $|A|$. This is called the *minor* of the element a_{11} (the element at the intersection of the deleted row and column) and is denoted by $|M_{11}|$. In general, the symbol $|M_{ij}|$ can be used to represent the minor obtained by deleting the *i*th row and *j*th column of a given determinant. Since a minor is itself a determinant, it has a value. As the reader can verify, the other two second-order determinants in (5.6) are, respectively, the minors $|M_{12}|$ and $|M_{13}|$; that is,

$$|M_{11}| \equiv \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \qquad |M_{12}| \equiv \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \qquad |M_{13}| \equiv \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

A concept closely related to the minor is that of the *cofactor*. A cofactor, denoted by $|C_{ij}|$, is a minor with a prescribed algebraic sign attached to it.† The rule of sign is as follows. If the sum of the two subscripts i and j in the minor $|M_{ij}|$ is even, then the cofactor takes the same sign as the minor; that is, $|C_{ij}| \equiv |M_{ij}|$. If it is odd, then the cofactor takes the opposite sign to the minor; that is, $|C_{ij}| \equiv -|M_{ij}|$. In short, we have

$$|C_{ij}| \equiv (-1)^{i+j}|M_{ij}|$$

where it is obvious that the expression $(-1)^{i+j}$ can be positive if and only if $(i+j)$ is even. The fact that a cofactor has a specific sign is of extreme importance and should always be borne in mind.

Example 4

In the determinant $\begin{vmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{vmatrix}$, the minor of the element 8 is

$$|M_{12}| = \begin{vmatrix} 6 & 4 \\ 3 & 1 \end{vmatrix} = -6$$

but the cofactor of the same element is

$$|C_{12}| = -|M_{12}| = 6$$

because $i + j = 1 + 2 = 3$ is odd. Similarly, the cofactor of the element 4 is

$$|C_{23}| = -|M_{23}| = -\begin{vmatrix} 9 & 8 \\ 3 & 2 \end{vmatrix} = 6$$

† Many writers use the symbols M_{ij} and C_{ij} (without the vertical bars) for minors and cofactors. We add the vertical bars to give visual emphasis to the fact that minors and cofactors are in the nature of determinants and, as such, have scalar values.

Using these new concepts, we can express a third-order determinant as

$$|A| = a_{11}|M_{11}| - a_{12}|M_{12}| + a_{13}|M_{13}|$$
$$= a_{11}|C_{11}| + a_{12}|C_{12}| + a_{13}|C_{13}| = \sum_{j=1}^{3} a_{1j}|C_{1j}| \qquad \textbf{(5.7)}$$

i.e., as a sum of three terms, each of which is the product of a first-row element and its corresponding cofactor. Note the difference in the signs of the $a_{12}|M_{12}|$ and $a_{12}|C_{12}|$ terms in (5.7). This is because $1 + 2$ gives an odd number.

The Laplace expansion of a *third*-order determinant serves to reduce the evaluation problem to one of evaluating only certain *second*-order determinants. A similar reduction is achieved in the Laplace expansion of higher-order determinants. In a fourth-order determinant $|B|$, for instance, the top row will contain four elements, $b_{11} \ldots b_{14}$; thus, in the spirit of (5.7), we may write

$$|B| = \sum_{j=1}^{4} b_{1j}|C_{1j}|$$

where the cofactors $|C_{1j}|$ are of order 3. Each third-order cofactor can then be evaluated as in (5.6). In general, the Laplace expansion of an nth-order determinant will reduce the problem to one of evaluating n cofactors, each of which is of the $(n-1)$st order, and the repeated application of the process will methodically lead to lower and lower orders of determinants, eventually culminating in the basic second-order determinants as defined in (5.5). Then the value of the original determinant can be easily calculated.

Although the process of Laplace expansion has been couched in terms of the cofactors of the first-row elements, it is also feasible to expand a determinant by the cofactor of any row or, for that matter, of any column. For instance, if the first column of a third-order determinant $|A|$ consists of the elements a_{11}, a_{21}, and a_{31}, expansion by the cofactors of these elements will also yield the value of $|A|$:

$$|A| = a_{11}|C_{11}| + a_{21}|C_{21}| + a_{31}|C_{31}| = \sum_{i=1}^{3} a_{i1}|C_{i1}|$$

Example 5

Given $|A| = \begin{vmatrix} 5 & 6 & 1 \\ 2 & 3 & 0 \\ 7 & -3 & 0 \end{vmatrix}$, expansion by the first *row* produces the result

$$|A| = 5\begin{vmatrix} 3 & 0 \\ -3 & 0 \end{vmatrix} - 6\begin{vmatrix} 2 & 0 \\ 7 & 0 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 7 & -3 \end{vmatrix} = 0 + 0 - 27 = -27$$

But expansion by the first *column* yields the identical answer:

$$|A| = 5\begin{vmatrix} 3 & 0 \\ -3 & 0 \end{vmatrix} - 2\begin{vmatrix} 6 & 1 \\ -3 & 0 \end{vmatrix} + 7\begin{vmatrix} 6 & 1 \\ 3 & 0 \end{vmatrix} = 0 - 6 - 21 = -27$$

Insofar as numerical calculation is concerned, this fact affords us an opportunity to choose some "easy" row or column for expansion. A row or column with the largest number of 0s or 1s is always preferable for this purpose, because a 0 times its cofactor is simply 0, so that the term will drop out, and a 1 times its cofactor is simply the cofactor itself, so that at least one multiplication step can be saved. In Example 5, the easiest way to expand the determinant is by the third column, which consists of the elements 1, 0, and 0. We could therefore have evaluated it thus:

$$|A| = 1\begin{vmatrix} 2 & 3 \\ 7 & -3 \end{vmatrix} = -6 - 21 = -27$$

To sum up, the value of a determinant $|A|$ of order n can be found by the Laplace expansion of *any row* or *any column* as follows:

$$|A| = \sum_{j=1}^{n} a_{ij}|C_{ij}| \qquad \text{[expansion by the } i\text{th row]}$$

$$= \sum_{i=1}^{n} a_{ij}|C_{ij}| \qquad \text{[expansion by the } j\text{th column]} \qquad \textbf{(5.8)}$$

EXERCISE 5.2

1. Evaluate the following determinants:

(a) $\begin{vmatrix} 8 & 1 & 3 \\ 4 & 0 & 1 \\ 6 & 0 & 3 \end{vmatrix}$ (c) $\begin{vmatrix} 4 & 0 & 2 \\ 6 & 0 & 3 \\ 8 & 2 & 3 \end{vmatrix}$ (e) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

(b) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 7 & 5 \\ 3 & 6 & 9 \end{vmatrix}$ (d) $\begin{vmatrix} 1 & 1 & 4 \\ 8 & 11 & -2 \\ 0 & 4 & 7 \end{vmatrix}$ (f) $\begin{vmatrix} x & 5 & 0 \\ 3 & y & 2 \\ 9 & -1 & 8 \end{vmatrix}$

2. Determine the signs to be attached to the relevant minors in order to get the following cofactors of a determinant: $|C_{13}|$, $|C_{23}|$, $|C_{33}|$, $|C_{41}|$, and $|C_{34}|$.

3. Given $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$, find the minors and cofactors of the elements a, b, and f.

4. Evaluate the following determinants:

(a) $\begin{vmatrix} 1 & 2 & 0 & 9 \\ 2 & 3 & 4 & 6 \\ 1 & 6 & 0 & -1 \\ 0 & -5 & 0 & 8 \end{vmatrix}$ (b) $\begin{vmatrix} 2 & 7 & 0 & 1 \\ 5 & 6 & 4 & 8 \\ 0 & 0 & 9 & 0 \\ 1 & -3 & 1 & 4 \end{vmatrix}$

5. In the first determinant of Prob. 4, find the value of the cofactor of the element 9.

6. Find the minors and cofactors of the third row, given

$$A = \begin{bmatrix} 9 & 11 & 4 \\ 3 & 2 & 7 \\ 6 & 10 & 4 \end{bmatrix}$$

7. Use Laplace expansion to find the determinant of

$$A = \begin{bmatrix} 15 & 7 & 9 \\ 2 & 5 & 6 \\ 9 & 0 & 12 \end{bmatrix}$$

5.3 Basic Properties of Determinants

We can now discuss some properties of determinants which will enable us to "discover" the connection between linear dependence among the rows of a square matrix and the vanishing of the determinant of that matrix.

Five basic properties will be discussed here. These are properties common to determinants of all orders, although we shall illustrate mostly with second-order determinants:

Property I The interchange of rows and columns does not affect the value of a determinant. In other words, the determinant of a matrix A has the same value as that of its transpose A', that is, $|A| = |A'|$.

Example 1

$$\begin{vmatrix} 4 & 3 \\ 5 & 6 \end{vmatrix} = \begin{vmatrix} 4 & 5 \\ 3 & 6 \end{vmatrix} = 9$$

Example 2

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc$$

Property II The interchange of any *two* rows (or any *two* columns) will alter the sign, but not the numerical value, of the determinant. (This property is obviously related to the first elementary row operation on a matrix.)

Example 3 $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$, but the interchange of the two rows yields

$$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad = -(ad - bc)$$

Example 4 $\begin{vmatrix} 0 & 1 & 3 \\ 2 & 5 & 7 \\ 3 & 0 & 1 \end{vmatrix} = -26$, but the interchange of the first and third columns yields

$$\begin{vmatrix} 3 & 1 & 0 \\ 7 & 5 & 2 \\ 1 & 0 & 3 \end{vmatrix} = 26.$$

Property III The multiplication of any *one* row (or *one* column) by a scalar k will change the value of the determinant k-fold. (This property is related to the second elementary row operation on a matrix.)

Example 5 By multiplying the top row of the determinant in Example 3 by k, we get

$$\begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = kad - kbc = k(ad - bc) = k\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

It is important to distinguish between the two expressions kA and $k|A|$. In multiplying a *matrix* A by a scalar k, all the elements in A are to be multiplied by k. But, if we read the equation in the present example from right to left, it should be clear that, in multiplying a *determinant* $|A|$ by k, only a single row (or column) should be multiplied by k. This equation, therefore, in effect gives us a rule for factoring a determinant: whenever any single row or column contains a common divisor, it may be factored out of the determinant.

Example 6 Factoring the first column and the second row in turn, we have

$$\begin{vmatrix} 15a & 7b \\ 12c & 2d \end{vmatrix} = 3\begin{vmatrix} 5a & 7b \\ 4c & 2d \end{vmatrix} = 3(2)\begin{vmatrix} 5a & 7b \\ 2c & d \end{vmatrix} = 6(5ad - 14bc)$$

The direct evaluation of the original determinant will, of course, produce the same answer.

In contrast, the factoring of a *matrix* requires the presence of a common divisor for *all* its elements, as in

$$\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix} = k\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Property IV The addition (subtraction) of a multiple of any row to (from) another row will leave the value of the determinant unaltered. The same holds true if we replace the word *row* by *column* in the previous statement. (This property is related to the third elementary row operation on a matrix.)

Example 7 Adding k times the top row of the determinant in Example 3 to its second row, we end up with the original determinant:

$$\begin{vmatrix} a & b \\ c+ka & d+kb \end{vmatrix} = a(d+kb) - b(c+ka) = ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Property V If one row (or column) is a multiple of another row (or column), the value of the determinant will be zero. As a special case of this, when two rows (or two columns) are *identical*, the determinant will vanish.

Example 8

$$\begin{vmatrix} 2a & 2b \\ a & b \end{vmatrix} = 2ab - 2ab = 0 \qquad \begin{vmatrix} c & c \\ d & d \end{vmatrix} = cd - cd = 0$$

Additional examples of this type of "vanishing" determinant can be found in Exercise 5.2-1.

This important property is, in fact, a logical consequence of Property IV. To understand this, let us apply Property IV to the two determinants in Example 8 and watch the outcome. For the first one, try to subtract twice the second row from the top row; for the second determinant, subtract the second column from the first column. Since these operations do not alter the values of the determinants, we can write

$$\begin{vmatrix} 2a & 2b \\ a & b \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ a & b \end{vmatrix} \qquad \begin{vmatrix} c & c \\ d & d \end{vmatrix} = \begin{vmatrix} 0 & c \\ 0 & d \end{vmatrix}$$

The new (reduced) determinants now contain, respectively, a row and a column of zeros; thus their Laplace expansion must yield a value of zero in both cases. In general, when one row (column) is a multiple of another row (column), the application of Property IV can always reduce all elements of that row (column) to zero, and Property V therefore follows.

The basic properties just discussed are useful in several ways. For one thing, they can be of great help in simplifying the task of evaluating determinants. By subtracting multiples of one row (or column) from another, for instance, the elements of the determinant may be reduced to much smaller and simpler numbers. Factoring, if feasible, can also accomplish the same. If we can indeed apply these properties to transform some row or column into a form containing mostly 0s or 1s, Laplace expansion of the determinant will become a much more manageable task.

Determinantal Criterion for Nonsingularity

Our present concern, however, is primarily to link the linear dependence of rows with the vanishing of a determinant. For this purpose, Property V can be invoked. Consider an equation system $Ax = d$:

$$\begin{bmatrix} 3 & 4 & 2 \\ 15 & 20 & 10 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

This system can have a unique solution if and only if the rows in the coefficient matrix A are linearly independent, so that A is nonsingular. But the second row is five times the first; the rows are indeed *dependent,* and hence no unique solution exists. The detection of this row dependence was by visual inspection, but by virtue of Property V we could also have discovered it through the fact that $|A| = 0$.

The row dependence in a matrix may, of course, assume a more intricate and secretive pattern. For instance, in the matrix

$$B = \begin{bmatrix} 4 & 1 & 2 \\ 5 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} v_1' \\ v_2' \\ v_3' \end{bmatrix}$$

there exists row dependence because $2v_1' - v_2' - 3v_3' = 0$; yet this fact defies visual detection. Even in this case, however, Property V will give us a vanishing determinant, $|B| = 0$, since by adding three times v_3' to v_2' and subtracting twice v_1' from it, the second row can be reduced to a zero vector. In general, *any* pattern of linear dependence among rows will be reflected in a vanishing determinant—and herein lies the beauty of Property V! Conversely, if the rows are linearly independent, the determinant must have a nonzero value.

We have, in the previous two paragraphs, tied the nonsingularity of a matrix principally to the linear independence among *rows.* But, on occasion, we have made the claim that, for a *square* matrix A, row independence $\Leftrightarrow$ column independence. We are now equipped to prove that claim:

> According to Property I, we know that $|A| = |A'|$. Since row independence in $A \Leftrightarrow |A| \neq 0$, we may also state that row independence in $A \Leftrightarrow |A'| \neq 0$. But $|A'| \neq 0 \Leftrightarrow$ row independence in the transpose $A' \Leftrightarrow$ column independence in A (rows of A' are by definition the columns of A). Therefore, *row* independence in $A \Leftrightarrow$ *column* independence in A.

Our discussion of the test of nonsingularity can now be summarized. Given a linear-equation system $Ax = d$, where A is an $n \times n$ coefficient matrix,

$$\begin{aligned} |A| \neq 0 &\Leftrightarrow \text{there is row (column) independence in matrix } A \\ &\Leftrightarrow A \text{ is nonsingular} \\ &\Leftrightarrow A^{-1} \text{ exists} \\ &\Leftrightarrow \text{a unique solution } x^* = A^{-1}d \text{ exists} \end{aligned}$$

Thus the value of the determinant of the coefficient matrix, $|A|$, provides a convenient criterion for testing the nonsingularity of matrix A and the existence of a unique solution to the equation system $Ax = d$. Note, however, that the determinantal criterion says nothing about the algebraic signs of the solution values; i.e., even though we are assured of a unique solution when $|A| \neq 0$, we may sometimes get negative solution values that are economically inadmissible.

Example 9

Does the equation system

$$\begin{aligned} 7x_1 - 3x_2 - 3x_3 &= 7 \\ 2x_1 + 4x_2 + \ \ x_3 &= 0 \\ -2x_2 - \ \ x_3 &= 2 \end{aligned}$$

possess a unique solution? The determinant $|A|$ is

$$\begin{vmatrix} 7 & -3 & -3 \\ 2 & 4 & 1 \\ 0 & -2 & -1 \end{vmatrix} = -8 \neq 0$$

Therefore a unique solution does exist.

Rank of a Matrix Redefined

The rank of a matrix A was earlier defined to be the maximum number of linearly independent rows in A. In view of the link between row independence and the nonvanishing of the determinant, we can redefine the rank of an $m \times n$ matrix as the maximum order of a nonvanishing determinant that can be constructed from the rows and columns of that matrix. The rank of any matrix is a unique number.

Obviously, the rank can at most be m or n, whichever is smaller, because a determinant is defined only for a square matrix, and from a matrix of dimension, say, 3×5, the largest possible determinants (vanishing or not) will be of order 3. Symbolically, this fact may be expressed as follows:

$$r(A) \leq \min \{m, n\}$$

which is read: "The rank of A is less than or equal to the minimum of the set of two numbers m and n." The rank of an $n \times n$ nonsingular matrix A must be n; in that case, we may write $r(A) = n$.

Sometimes, one may be interested in the rank of the product of two matrices. In that case, the following rule can be of use:

$$r(AB) \leq \min \{r(A), r(B)\} \tag{5.9}$$

While this rule does not yield a unique value of $r(AB)$, the application of the rule can nevertheless lead to unique results. In particular, we can use (5.9) to show that if a matrix A, with $r(A) = j$, is multiplied by any (conformable) nonsingular matrix B, the rank of the product matrix AB (or BA, as the case may be), must be j. We shall prove this for the product AB (the case of BA is analogous). First, looking at the right-hand side of (5.9), we see only three possible cases: (i) $r(A) < r(B)$, (ii) $r(A) = r(B)$, and (iii) $r(A) > r(B)$.

For cases (i) and (ii), (5.9) reduces directly to $r(AB) \le r(A) = j$. For case (iii), we find that $r(AB) \le r(B) < r(A) = j$. Thus, either way, we get

$$r(AB) \le r(A) = j \tag{5.10}$$

Now consider the identity $(AB)B^{-1} = A$. By (5.9), we can write

$$r[(AB)B^{-1}] \le \min\{r(AB), r(B^{-1})\}$$

Applying the same reasoning that led us to (5.10), we can conclude from this that

$$r[(AB)B^{-1}] \le r(AB)$$

Since the left-side expression of this inequality is equal to $r(A) = j$, we may write

$$j \le r(AB) \tag{5.11}$$

But (5.10) and (5.11) cannot be satisfied simultaneously unless $r(AB) = j$. Thus the rank of the product matrix AB must be j, as asserted.

EXERCISE 5.3

1. Use the determinant $\begin{vmatrix} 4 & 0 & -1 \\ 2 & 1 & -7 \\ 3 & 3 & 9 \end{vmatrix}$ to verify the first four properties of determinants.
2. Show that, when all the elements of an nth-order determinant $|A|$ are multiplied by a number k, the result will be $k^n|A|$.
3. Which properties of determinants enable us to write the following?

   (*a*) $\begin{vmatrix} 9 & 18 \\ 27 & 56 \end{vmatrix} = \begin{vmatrix} 9 & 18 \\ 0 & 2 \end{vmatrix}$ (*b*) $\begin{vmatrix} 9 & 27 \\ 4 & 2 \end{vmatrix} = 18\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}$
4. Test whether the following matrices are nonsingular:

   (*a*) $\begin{bmatrix} 4 & 0 & 1 \\ 19 & 1 & -3 \\ 7 & 1 & 0 \end{bmatrix}$ (*c*) $\begin{bmatrix} 7 & -1 & 0 \\ 1 & 1 & 4 \\ 13 & -3 & -4 \end{bmatrix}$

   (*b*) $\begin{bmatrix} 4 & -2 & 1 \\ -5 & 6 & 0 \\ 7 & 0 & 3 \end{bmatrix}$ (*d*) $\begin{bmatrix} -4 & 9 & 5 \\ 3 & 0 & 1 \\ 10 & 8 & 6 \end{bmatrix}$
5. What can you conclude about the rank of each matrix in Prob. 4?
6. Can any of the given sets of 3-vectors below span the 3-space? Why or why not?

   (*a*) $[1 \;\; 2 \;\; 1]$ $[2 \;\; 3 \;\; 1]$ $[3 \;\; 4 \;\; 2]$

   (*b*) $[8 \;\; 1 \;\; 3]$ $[1 \;\; 2 \;\; 8]$ $[-7 \;\; 1 \;\; 5]$
7. Rewrite the simple national-income model (3.23) in the $Ax = d$ format (with Y as the first variable in the vector x), and then test whether the coefficient matrix A is nonsingular.
8. Comment on the validity of the following statements:

   (*a*) "Given any matrix A, we can always derive from it a transpose, and a determinant."

   (*b*) "Multiplying each element of an $n \times n$ determinant by 2 will double the value of that determinant."

   (*c*) "If a square matrix A vanishes, then we can be sure that the equation system $Ax = d$ is nonsingular."

5.4 Finding the Inverse Matrix

If the matrix A in the linear-equation system $Ax = d$ is nonsingular, then A^{-1} exists, and the solution of the system will be $x^* = A^{-1}d$. We have learned to test the nonsingularity of A by the criterion $|A| \neq 0$. The next question is, How can we find the inverse A^{-1} if A does pass that test?

Expansion of a Determinant by Alien Cofactors

Before answering this query, let us discuss another important property of determinants.

Property VI The expansion of a determinant by *alien cofactors* (the cofactors of a "wrong" row or column) always yields a value of zero.

Example 1

If we expand the determinant $\begin{vmatrix} 4 & 1 & 2 \\ 5 & 2 & 1 \\ 1 & 0 & 3 \end{vmatrix}$ by using its *first*-row elements but the cofactors of the *second*-row elements

$$|C_{21}| = -\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = -3 \qquad |C_{22}| = \begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix} = 10 \qquad |C_{23}| = -\begin{vmatrix} 4 & 1 \\ 1 & 0 \end{vmatrix} = 1$$

we get $a_{11}|C_{21}| + a_{12}|C_{22}| + a_{13}|C_{23}| = 4(-3) + 1(10) + 2(1) = 0$.

More generally, applying the same type of expansion by alien cofactors as described in Example 1 to the determinant $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ will yield a zero sum of products as follows:

$$\begin{aligned} \sum_{i=1}^{3} a_{1j}|C_{2j}| &= a_{11}|C_{21}| + a_{12}|C_{22}| + a_{13}|C_{23}| \\ &= -a_{11}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{13}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \qquad (5.12) \\ &= -a_{11}a_{12}a_{33} + a_{11}a_{13}a_{32} + a_{11}a_{12}a_{33} - a_{12}a_{13}a_{31} \\ &\quad - a_{11}a_{13}a_{32} + a_{12}a_{13}a_{31} = 0 \end{aligned}$$

The reason for this outcome lies in the fact that the sum of products in (5.12) can be considered as the result of the *regular* expansion by the second row of another determinant $|A^*| \equiv \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, which differs from $|A|$ only in its second row and whose first two rows are identical. As an exercise, write out the cofactors of the second rows of $|A^*|$ and verify that these are precisely the cofactors which appeared in (5.12) and with the correct signs. Since $|A^*| = 0$, because of its two identical rows, the expansion by alien cofactors shown in (5.12) will of necessity yield a value of zero also.

Property VI is valid for determinants of all orders and applies when a determinant is expanded by the alien cofactors of any row or any column. Thus we may state, in general, that for a determinant of order n the following holds:

$$\left.\begin{aligned} &\sum_{j=1}^{n} a_{ij}|C_{i'j}| = 0 \qquad (i \neq i') \qquad \text{[expansion by } i\text{th row and cofactors of } i'\text{th row]} \\ &\sum_{i=1}^{n} a_{ij}|C_{ij'}| = 0 \qquad (j \neq j') \qquad \text{[expansion by } j\text{th column and cofactors of } j'\text{th column]} \end{aligned}\right\} \tag{5.13}$$

Carefully compare (5.13) with (5.8). In the latter (regular Laplace expansion), the subscripts of a_{ij} and of $|C_{ij}|$ must be identical in each product term in the sum. In the expansion by alien cofactors, such as in (5.13), on the other hand, one of the two subscripts (a chosen value of i' or j') is inevitably "out of place."

Matrix Inversion

Property VI, as summarized in (5.13), is of direct help in developing a method of matrix inversion, i.e., of finding the inverse of a matrix.

Assume that an $n \times n$ nonsingular matrix A is given:

$$\underset{(n\times n)}{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad (|A| \neq 0) \tag{5.14}$$

Since each element of A has a cofactor $|C_{ij}|$, it is possible to form a matrix of cofactors by replacing each element a_{ij} in (5.14) with its cofactor $|C_{ij}|$. Such a cofactor matrix, denoted by $C = [|C_{ij}|]$, must also be $n \times n$. For our present purposes, however, the transpose of C is of more interest. This transpose C' is commonly referred to as the *adjoint* of A and is symbolized by adj A. Written out, the adjoint takes the form

$$\underset{(n\times n)}{C'} \equiv \text{adj } A \equiv \begin{bmatrix} |C_{11}| & |C_{21}| & \cdots & |C_{n1}| \\ |C_{12}| & |C_{22}| & \cdots & |C_{n2}| \\ \cdots & \cdots & \cdots & \cdots \\ |C_{1n}| & |C_{2n}| & \cdots & |C_{nn}| \end{bmatrix} \tag{5.15}$$

The matrices A and C' are conformable for multiplication, and their product AC' is another $n \times n$ matrix in which each element is a sum of products. By utilizing the formula for Laplace expansion as well as Property VI of determinants, the product AC' may be expressed as follows:

$$\underset{(n\times n)}{AC'} = \begin{bmatrix} \sum_{j=1}^{n} a_{1j}|C_{1j}| & \sum_{j=1}^{n} a_{1j}|C_{2j}| & \cdots & \sum_{j=1}^{n} a_{1j}|C_{nj}| \\ \sum_{j=1}^{n} a_{2j}|C_{1j}| & \sum_{j=1}^{n} a_{2j}|C_{2j}| & \cdots & \sum_{j=1}^{n} a_{2j}|C_{nj}| \\ \vdots & \vdots & & \vdots \\ \sum_{j=1}^{n} a_{nj}|C_{1j}| & \sum_{j=1}^{n} a_{nj}|C_{2j}| & \cdots & \sum_{j=1}^{n} a_{nj}|C_{nj}| \end{bmatrix}$$

$$= \begin{bmatrix} |A| & 0 & \cdots & 0 \\ 0 & |A| & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & |A| \end{bmatrix} \qquad \text{[by (5.8) and (5.13)]}$$

$$= |A| \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = |A| I_n \qquad \text{[factoring]}$$

As the determinant $|A|$ is a nonzero scalar, it is permissible to divide both sides of the equation $AC' = |A|I$ by $|A|$. The result is

$$\frac{AC'}{|A|} = I \qquad \text{or} \qquad A\frac{C'}{|A|} = I$$

Premultiplying both sides of the last equation by A^{-1}, and using the result that $A^{-1}A = I$, we can get $\frac{C'}{|A|} = A^{-1}$, or

$$A^{-1} = \frac{1}{|A|} \text{adj } A \qquad \text{[by (5.15)]} \tag{5.16}$$

Now, we have found a way to invert the matrix A!

The general procedure for finding the inverse of a square matrix A thus involves the following steps: (1) find $|A|$ [we need to proceed with the subsequent steps if and only if $|A| \neq 0$, for if $|A| = 0$, the inverse in (5.16) will be undefined]; (2) find the cofactors of all the elements of A, and arrange them as a matrix $C = [|C_{ij}|]$; (3) take the transpose of C to get adj A; and (4) divide adj A by the determinant $|A|$. The result will be the desired inverse A^{-1}.

Example 2

Find the inverse of $A = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$. Since $|A| = -2 \neq 0$, the inverse A^{-1} exists. The cofactor of each element is in this case a 1×1 determinant, which is simply defined as the scalar element of that determinant itself (that is, $|a_{ij}| \equiv a_{ij}$). Thus, we have

$$C = \begin{bmatrix} |C_{11}| & |C_{12}| \\ |C_{21}| & |C_{22}| \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -2 & 3 \end{bmatrix}$$

Observe the minus signs attached to 1 and 2, as required for cofactors. Transposing the cofactor matrix yields

$$\text{adj } A = \begin{bmatrix} 0 & -2 \\ -1 & 3 \end{bmatrix}$$

so the inverse A^{-1} can be written as

$$A^{-1} = \frac{1}{|A|} \text{adj } A = -\frac{1}{2} \begin{bmatrix} 0 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix}$$

Example 3

Find the inverse of $B = \begin{bmatrix} 4 & 1 & -1 \\ 0 & 3 & 2 \\ 3 & 0 & 7 \end{bmatrix}$. Since $|B| = 99 \neq 0$, the inverse B^{-1} also exists. The cofactor matrix is

$$\begin{bmatrix} \begin{vmatrix} 3 & 2 \\ 0 & 7 \end{vmatrix} & -\begin{vmatrix} 0 & 2 \\ 3 & 7 \end{vmatrix} & \begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} \\ -\begin{vmatrix} 1 & -1 \\ 0 & 7 \end{vmatrix} & \begin{vmatrix} 4 & -1 \\ 3 & 7 \end{vmatrix} & -\begin{vmatrix} 4 & 1 \\ 3 & 0 \end{vmatrix} \\ \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 4 & -1 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 4 & 1 \\ 0 & 3 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 21 & 6 & -9 \\ -7 & 31 & 3 \\ 5 & -8 & 12 \end{bmatrix}$$

Therefore,

$$\text{adj } B = \begin{bmatrix} 21 & -7 & 5 \\ 6 & 31 & -8 \\ -9 & 3 & 12 \end{bmatrix}$$

and the desired inverse matrix is

$$B^{-1} = \frac{1}{|B|} \text{adj } B = \frac{1}{99} \begin{bmatrix} 21 & -7 & 5 \\ 6 & 31 & -8 \\ -9 & 3 & 12 \end{bmatrix}$$

You can check that the results in Examples 2 and 3 do satisfy $AA^{-1} = A^{-1}A = I$ and $BB^{-1} = B^{-1}B = I$, respectively.

EXERCISE 5.4

1. Suppose that we expand a fourth-order determinant by its *third column* and the cofactors of the *second-column* elements. How would you write the resulting sum of products in $\sum$ notation? What will be the sum of products in $\sum$ notation if we expand it by the *second row* and the cofactors of the *fourth-row* elements?
2. Find the inverse of each of the following matrices:

   (*a*) $A = \begin{bmatrix} 5 & 2 \\ 0 & 1 \end{bmatrix}$ (*b*) $B = \begin{bmatrix} -1 & 0 \\ 9 & 2 \end{bmatrix}$ (*c*) $C = \begin{bmatrix} 3 & 7 \\ 3 & -1 \end{bmatrix}$ (*d*) $D = \begin{bmatrix} 7 & 6 \\ 0 & 3 \end{bmatrix}$
3. (*a*) Drawing on your answers to Prob. 2, formulate a two-step rule for finding the adjoint of a given 2×2 matrix A: In the first step, indicate what should be done to the two diagonal elements of A in order to get the diagonal elements of adj A; in the second step, indicate what should be done to the two off-diagonal elements of A. (*Warning:* This rule applies only to 2×2 matrices.)

   (*b*) Add a third step which, in conjunction with the previous two steps, yields the 2×2 inverse matrix A^{-1}.
4. Find the inverse of each of the following matrices:

   (*a*) $E = \begin{bmatrix} 4 & -2 & 1 \\ 7 & 3 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ (*c*) $G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

   (*b*) $F = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 3 \\ 4 & 0 & 2 \end{bmatrix}$ (*d*) $H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

5. Find the inverse of

$$A = \begin{bmatrix} 4 & 1 & -5 \\ -2 & 3 & 1 \\ 3 & -1 & 4 \end{bmatrix}$$

6. Solve the system $Ax = d$ by matrix inversion, where

(*a*) $4x + 3y = 28$
$2x + 5y = 42$

(*b*) $4x_1 + x_2 - 5x_3 = 8$
$-2x_1 + 3x_2 + x_3 = 12$
$3x_1 - x_2 + 4x_3 = 5$

7. Is it possible for a matrix to be its own inverse?

5.5 Cramer's Rule

The method of matrix inversion discussed in Sec. 5.4 enables us to derive a practical, if not always efficient, way of solving a linear-equation system, known as *Cramer's rule.*

Derivation of the Rule

Given an equation system $Ax = d$, where A is $n \times n$, the solution can be written as

$$x^* = A^{-1}d = \frac{1}{|A|}(\text{adj } A)d \qquad \text{[by (5.16)]}$$

provided A is nonsingular. According to (5.15), this means that

$$\begin{bmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_n^* \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} |C_{11}| & |C_{21}| & \cdots & |C_{n1}| \\ |C_{12}| & |C_{22}| & \cdots & |C_{n2}| \\ \cdots & \cdots & \cdots & \cdots \\ |C_{1n}| & |C_{2n}| & \cdots & |C_{nn}| \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$

$$= \frac{1}{|A|}\begin{bmatrix} d_1|C_{11}| + d_2|C_{21}| + \cdots + d_n|C_{n1}| \\ d_1|C_{12}| + d_2|C_{22}| + \cdots + d_n|C_{n2}| \\ \cdots\cdots\cdots\cdots \\ d_1|C_{1n}| + d_2|C_{2n}| + \cdots + d_n|C_{nn}| \end{bmatrix}$$

$$= \frac{1}{|A|}\begin{bmatrix} \sum_{i=1}^{n} d_i|C_{i1}| \\ \sum_{i=1}^{n} d_i|C_{i2}| \\ \vdots \\ \sum_{i=1}^{n} d_i|C_{in}| \end{bmatrix}$$

Equating the corresponding elements on the two sides of the equation, we obtain the solution values

$$x_1^* = \frac{1}{|A|}\sum_{i=1}^{n} d_i|C_{i1}| \qquad x_2^* = \frac{1}{|A|}\sum_{i=1}^{n} d_i|C_{i2}| \qquad \text{(etc.)} \tag{5.17}$$

The $\sum$ terms in (5.17) look unfamiliar. What do they mean? From (5.8), we see that the Laplace expansion of a determinant $|A|$ by its first column can be expressed in the form $\sum_{i=1}^{n} a_{i1}|C_{i1}|$. If we replace the first column of $|A|$ by the column vector d but keep all the other columns intact, then a new determinant will result, which we can call $|A_1|$ —the subscript 1 indicating that the first column has been replaced by d. The expansion of $|A_1|$ by its first column (the d column) will yield the expression $\sum_{i=1}^{n} d_i|C_{i1}|$, because the elements d_i now take the place of the elements a_{i1}. Returning to (5.17), we see therefore that

$$x_1^* = \frac{1}{|A|}|A_1|$$

Similarly, if we replace the second column of $|A|$ by the column vector d, while retaining all the other columns, the expansion of the new determinant $|A_2|$ by its second column (the d column) will result in the expression $\sum_{i=1}^{n} d_i|C_{i2}|$. When divided by $|A|$, this latter sum will give us the solution value x_2^*, and so on.

This procedure can now be generalized. To find the solution value of the jth variable x_j^*, we can merely replace the jth column of the determinant $|A|$ by the constant terms $d_1 \cdots d_n$ to get a new determinant $|A_j|$ and then divide $|A_j|$ by the original determinant $|A|$. Thus, the solution of the system $Ax = d$ can be expressed as

$$x_j^* = \frac{|A_j|}{|A|} = \frac{1}{|A|}\begin{vmatrix} a_{11} & a_{12} & \cdots & d_1 & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & d_2 & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & d_n & \cdots & a_{nn} \end{vmatrix} \tag{5.18}$$

(jth column replaced by d)

The result in (5.18) is the statement of Cramer's rule. Note that, whereas the matrix inversion method yields the solution values of *all* the endogenous variables at once (x^* is a vector), Cramer's rule can give us the solution value of only a single endogenous variable at a time (x_j^* is a scalar); this is why it may not be efficient.

Example 1

Find the solution of the equation system

$$5x_1 + 3x_2 = 30$$
$$6x_1 - 2x_2 = 8$$

The coefficients and the constant terms give the following determinants:

$$|A| = \begin{vmatrix} 5 & 3 \\ 6 & -2 \end{vmatrix} = -28 \qquad |A_1| = \begin{vmatrix} 30 & 3 \\ 8 & -2 \end{vmatrix} = -84$$

$$|A_2| = \begin{vmatrix} 5 & 30 \\ 6 & 8 \end{vmatrix} = -140$$

Therefore, by virtue of (5.18), we can immediately write

$$x_1^* = \frac{|A_1|}{|A|} = \frac{-84}{-28} = 3 \qquad \text{and} \qquad x_2^* = \frac{|A_2|}{|A|} = \frac{-140}{-28} = 5$$

Example 2 Find the solution of the equation system

$$\begin{aligned} 7x_1 - \ x_2 - \ x_3 &= 0 \\ 10x_1 - 2x_2 + \ x_3 &= 8 \\ 6x_1 + 3x_2 - 2x_3 &= 7 \end{aligned}$$

The relevant determinants $|A|$ and $|A_j|$ are found to be

$$|A| = \begin{vmatrix} 7 & -1 & -1 \\ 10 & -2 & 1 \\ 6 & 3 & -2 \end{vmatrix} = -61 \qquad |A_1| = \begin{vmatrix} 0 & -1 & -1 \\ 8 & -2 & 1 \\ 7 & 3 & -2 \end{vmatrix} = -61$$

$$|A_2| = \begin{vmatrix} 7 & 0 & -1 \\ 10 & 8 & 1 \\ 6 & 7 & -2 \end{vmatrix} = -183 \qquad |A_3| = \begin{vmatrix} 7 & -1 & 0 \\ 10 & -2 & 8 \\ 6 & 3 & 7 \end{vmatrix} = -244$$

thus the solution values of the variables are

$$x_1^* = \frac{|A_1|}{|A|} = \frac{-61}{-61} = 1 \qquad x_2^* = \frac{|A_2|}{|A|} = \frac{-183}{-61} = 3 \qquad x_3^* = \frac{|A_3|}{|A|} = \frac{-244}{-61} = 4$$

Notice that in each of these examples we find $|A| \neq 0$. This is a necessary condition for the application of Cramer's rule, as it is for the existence of the inverse A^{-1}. Cramer's rule is, after all, based upon the concept of the inverse matrix, even though in practice it bypasses the process of matrix inversion.

Note on Homogeneous-Equation Systems

The equation systems $Ax = d$ considered before can have any constants in the vector d. If $d = 0$, that is, if $d_1 = d_2 = \cdots = d_n = 0$, however, the equation system will become

$$Ax = 0$$

where 0 is a zero vector. This special case is referred to as a *homogeneous-equation system*. The word *homogeneous* here relates to the property that when all the variables, $x_1, \ldots, x_n$ are multiplied by the same number, the equation system will remain valid. This is possible only if the constant terms of the system—those unattached to any x_i are all zero.

If the matrix A is nonsingular, a homogeneous-equation system can yield only a "trivial solution," namely, $x_1^* = x_2^* = \cdots = x_n^* = 0$. This follows from the fact that the solution $x^* = A^{-1}d$ will in this case become

$$\underset{(n\times 1)}{x^*} = \underset{(n\times n)}{A^{-1}} \underset{(n\times 1)}{0} = \underset{(n\times 1)}{0}$$

Alternatively, this outcome can be derived from Cramer's rule. The fact that $d = 0$ implies that $|A_j|$, for all j, must contain a whole column of zeros, and thus the solution will turn out to be

$$x_j^* = \frac{|A_j|}{|A|} = \frac{0}{|A|} = 0 \qquad (j = 1, 2, \ldots, n)$$

Curiously enough, the *only* way to get a *non*trivial solution from a homogeneous-equation system is to have $|A| = 0$, that is, to have a *singular* coefficient matrix A! In that event, we have

$$x_j^* = \frac{|A_j|}{|A|} = \frac{0}{0}$$

where the 0/0 expression is not equal to zero but is, rather, something undefined. Consequently, Cramer's rule is not applicable. This does not mean that we cannot obtain solutions; it means only that we cannot get a unique solution.

Consider the homogeneous-equation system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= 0 \\ a_{21}x_1 + a_{22}x_2 &= 0 \end{aligned} \tag{5.19}$$

It is self-evident that $x_1^* = x_2^* = 0$ is a solution, but that solution is trivial. Now, assume that the coefficient matrix A is singular, so that $|A| = 0$. This implies that the row vector $[a_{11} \quad a_{12}]$ is a multiple of the row vector $[a_{21} \quad a_{22}]$; consequently, one of the two equations is redundant. By deleting, say, the second equation from (5.19), we end up with one (the first) equation in two variables, the solution of which is $x_1^* = (-a_{12}/a_{11})x_2^*$. This solution is nontrivial and well defined if $a_{11} \neq 0$, but it really represents an infinite number of solutions because, for every possible value of x_2^*, there is a corresponding value x_1^* such that the pair constitutes a solution. Thus no unique nontrivial solution exists for this homogeneous-equation system. This last statement is also generally valid for the n-variable case.

Solution Outcomes for a Linear-Equation System

Our discussion of the several variants of the linear-equation system $Ax = d$ reveals that as many as four different types of solution outcome are possible. For a better overall view of these variants, we list them in tabular form in Table 5.1.

As a first possibility, the system may yield a unique, nontrivial solution. This type of outcome can arise only when we have a nonhomogeneous system with a nonsingular coefficient matrix A. The second possible outcome is a unique, trivial solution, and this is

TABLE 5.1 **Solution Outcomes for a Linear-Equation System $Ax = d$**

	Vector d	
Determinant $\|A\|$	**$d \neq 0$ (nonhomogeneous system)**	**$d = 0$ (homogeneous system)**
$\|A\| \neq 0$ (matrix A nonsingular)	There exists a unique, nontrivial solution $x^* \neq 0$.	There exists a unique, trivial solution $x^* = 0$.
$\|A\| = 0$ (matrix A singular)		
Equations dependent	There exist an infinite number of solutions (not including the trivial one).	There exist an infinite number of solutions (including the trivial one).
Equations inconsistent	No solution exists.	[Not possible.]

associated with a homogeneous system with a nonsingular matrix A. As a third possibility, we may have an infinite number of solutions. This eventuality is linked exclusively to a system in which the equations are dependent (i.e., in which there are redundant equations). Depending on whether the system is homogeneous, the trivial solution may or may not be included in the set of infinite number of solutions. Finally, in the case of an inconsistent equation system, there exists no solution at all. From the point of view of a model builder, the most useful and desirable outcome is, of course, that of a unique, nontrivial solution $x^* \neq 0$.

EXERCISE 5.5

1. Use Cramer's rule to solve the following equation systems:

   (*a*) $3x_1 - 2x_2 = 6$, $2x_1 + x_2 = 11$

   (*b*) $-x_1 + 3x_2 = -3$, $4x_1 - x_2 = 12$

   (*c*) $8x_1 - 7x_2 = 9$, $x_1 + x_2 = 3$

   (*d*) $5x_1 + 9x_2 = 14$, $7x_1 - 3x_2 = 4$

2. For each of the equation systems in Prob. 1, find the inverse of the coefficient matrix, and get the solution by the formula $x^* = A^{-1}d$.
3. Use Cramer's rule to solve the following equation systems:

   (*a*) $8x_1 - x_2 = 16$, $2x_2 + 5x_3 = 5$, $2x_1 + 3x_3 = 7$

   (*b*) $-x_1 + 3x_2 + 2x_3 = 24$, $x_1 + x_3 = 6$, $5x_2 - x_3 = 8$

   (*c*) $4x + 3y - 2z = 1$, $x + 2y = 6$, $3x + z = 4$

   (*d*) $-x + y + z = a$, $x - y + z = b$, $x + y - z = c$

4. Show that Cramer's rule can be derived alternatively by the following procedure. Multiply both sides of the first equation in the system $Ax = d$ by the cofactor $|C_{1j}|$, and then multiply both sides of the second equation by the cofactor $|C_{2j}|$, etc. Add all the newly obtained equations. Then assign the values $1, 2, \ldots, n$ to the index j, successively, to get the solution values $x_1^*, x_2^*, \ldots, x_n^*$ as shown in (5.17).

5.6 Application to Market and National-Income Models

Simple equilibrium models such as those discussed in Chap. 3 can be solved with ease by Cramer's rule or by matrix inversion.

Market Model

The two-commodity model described in (3.12) can be written (after eliminating the quantity variables) as a system of two linear equations, as in (3.13′):

$$c_1 P_1 + c_2 P_2 = -c_0$$
$$\gamma_1 P_1 + \gamma_2 P_2 = -\gamma_0$$

The three determinants needed—$|A|$, $|A_1|$, and $|A_2|$—have the following values:

$$|A| = \begin{vmatrix} c_1 & c_2 \\ \gamma_1 & \gamma_2 \end{vmatrix} = c_1\gamma_2 - c_2\gamma_1$$

$$|A_1| = \begin{vmatrix} -c_0 & c_2 \\ -\gamma_0 & \gamma_2 \end{vmatrix} = -c_0\gamma_2 + c_2\gamma_0$$

$$|A_2| = \begin{vmatrix} c_1 & -c_0 \\ \gamma_1 & -\gamma_0 \end{vmatrix} = -c_1\gamma_0 + c_0\gamma_1$$

Therefore the equilibrium prices must be

$$P_1^* = \frac{|A_1|}{|A|} = \frac{c_2\gamma_0 - c_0\gamma_2}{c_1\gamma_2 - c_2\gamma_1} \qquad P_2^* = \frac{|A_2|}{|A|} = \frac{c_0\gamma_1 - c_1\gamma_0}{c_1\gamma_2 - c_2\gamma_1}$$

which are precisely those obtained in (3.14) and (3.15). The equilibrium quantities can be found, as before, by setting $P_1 = P_1^*$ and $P_2 = P_2^*$ in the demand or supply functions.

National-Income Model

The simple national-income model cited in (3.23) can also be solved by the use of Cramer's rule. As written in (3.23), the model consists of the following two simultaneous equations:

$$Y = C + I_0 + G_0$$
$$C = a + bY \qquad (a > 0, \quad 0 < b < 1)$$

These can be rearranged into the form

$$Y - C = I_0 + G_0$$
$$-bY + C = a$$

so that the endogenous variables Y and C appear only on the left of the equals signs, whereas the exogenous variables and the unattached parameter appear only on the right. The coefficient matrix now takes the form $\begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix}$, and the column vector of constants (data), $\begin{bmatrix} I_0 + G_0 \\ a \end{bmatrix}$. Note that the sum $I_0 + G_0$ is considered as a single entity, i.e., a single element in the constant vector.

Cramer's rule now leads immediately to the following solution:

$$Y^* = \frac{\begin{vmatrix} (I_0 + G_0) & -1 \\ a & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ -b & 1 \end{vmatrix}} = \frac{I_0 + G_0 + a}{1 - b}$$

$$C^* = \frac{\begin{vmatrix} 1 & (I_0 + G_0) \\ -b & a \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ -b & 1 \end{vmatrix}} = \frac{a + b(I_0 + G_0)}{1 - b}$$

You should check that the solution values just obtained are identical with those shown in (3.24) and (3.25).

Let us now try to solve this model by inverting the coefficient matrix. Since the coefficient matrix is $A = \begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix}$, its cofactor matrix is $\begin{bmatrix} 1 & b \\ 1 & 1 \end{bmatrix}$, and we therefore have $\text{adj } A = \begin{bmatrix} 1 & 1 \\ b & 1 \end{bmatrix}$. It follows that the inverse matrix is

$$A^{-1} = \frac{1}{|A|}\text{adj}A = \frac{1}{1-b}\begin{bmatrix} 1 & 1 \\ b & 1 \end{bmatrix}$$

We know that, for the equation system $Ax = d$, the solution is expressible as $x^* = A^{-1}d$. Applied to the present model, this means that

$$\begin{bmatrix} Y^* \\ C^* \end{bmatrix} = \frac{1}{1-b}\begin{bmatrix} 1 & 1 \\ b & 1 \end{bmatrix}\begin{bmatrix} I_0 + G_0 \\ a \end{bmatrix} = \frac{1}{1-b}\begin{bmatrix} I_0 + G_0 + a \\ b(I_0 + G_0) + a \end{bmatrix}$$

It is easy to see that this is again the same solution as obtained before.

IS-LM Model: Closed Economy

As another linear model of the economy, we can think of the economy as being made up of two sectors: the real goods sector and the monetary sector.

The goods market involves the following equations:

$$Y = C + I + G$$
$$C = a + b(1-t)Y$$
$$I = d - ei$$
$$G = G_0$$

The endogenous variables are Y, C, I, and i (where i is the rate of interest). The exogenous variable is G_0, while a, d, e, b, and t are structural parameters.

In the newly introduced money market, we have:

Equilibrium condition: $M_d = M_s$

Money demand: $M_d = kY - li$

Money supply: $M_s = M_0$

where M_0 is the exogenous stock of money and k and l are parameters. These three equations can be condensed into:

$$M_0 = kY - li$$

Together, the two sectors give us the following system of equations:

$$Y - C - I = G_0$$
$$b(1-t)Y - C = -a$$
$$I + ei = d$$
$$kY - li = M_0$$

Note that by further substitution the system could be further reduced to a 2×2 system of equations. For now, we will leave it as a 4×4 system. In matrix form, we have

$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ b(1-t) & -1 & 0 & 0 \\ 0 & 0 & 1 & e \\ k & 0 & 0 & -l \end{bmatrix}\begin{bmatrix} Y \\ C \\ I \\ i \end{bmatrix} = \begin{bmatrix} G_0 \\ -a \\ d \\ M_0 \end{bmatrix}$$

To find the determinant of the coefficient matrix, we can use Laplace expansion on one of the columns (preferably one with the most zeros). Expanding the fourth column, we find

$$|A| = (-e)\begin{vmatrix} 1 & -1 & -1 \\ b(1-t) & -1 & 0 \\ k & 0 & 0 \end{vmatrix} - l\begin{vmatrix} 1 & -1 & -1 \\ b(1-t) & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (-e)(k)\begin{vmatrix} -1 & -1 \\ -1 & 0 \end{vmatrix} - l\begin{vmatrix} 1 & -1 \\ b(1-t) & -1 \end{vmatrix}$$

$$= ek - l[(-1) - (-1)b(1-t)]$$

$$= ek + l[1 - b(1-t)]$$

We can use Cramer's rule to find the equilibrium income Y^*. This is done by replacing the first column of the coefficient matrix A with the vector of exogenous variables and taking the ratio of the determinant of the new matrix to the original determinant, or

$$Y^* = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} G_0 & -1 & -1 & 0 \\ -a & -1 & 0 & 0 \\ d & 0 & 1 & e \\ M_0 & 0 & 0 & -l \end{vmatrix}}{ek + l[1 - b(1-t)]}$$

Using Laplace expansion on the second column of the numerator produces

$$Y^* = \frac{(-1)(-1)^3\begin{vmatrix} -a & 0 & 0 \\ d & 1 & e \\ M_0 & 0 & -l \end{vmatrix}}{ek + l[1 - b(1-t)]} + \frac{(-1)(-1)^4\begin{vmatrix} G_0 & -1 & 0 \\ d & 1 & e \\ M_0 & 0 & -l \end{vmatrix}}{ek + l[1 - b(1-t)]}$$

$$= \frac{\begin{vmatrix} -a & 0 & 0 \\ d & 1 & e \\ M_0 & 0 & -l \end{vmatrix} - \begin{vmatrix} G_0 & -1 & 0 \\ d & 1 & e \\ M_0 & 0 & -l \end{vmatrix}}{ek + l[1 - b(1-t)]}$$

By further expansion, we obtain

$$Y^* = \frac{(1)\begin{vmatrix} -a & 0 \\ M_0 & -l \end{vmatrix} - \left\{(-1)(-1)^3\begin{vmatrix} d & e \\ M_0 & -l \end{vmatrix} + (-1)^4\begin{vmatrix} G_0 & 0 \\ M_0 & -l \end{vmatrix}\right\}}{ek + l[1 - b(1-t)]}$$

$$= \frac{al - [d(-l) - eM_0] - G_0(-l)}{ek + l[1 - b(1-t)]}$$

$$= \frac{l(a + d + G_0) + eM_0}{ek + l[1 - b(1-t)]}$$

Since the solution to Y^* is linear with respect to the exogenous variables, we can rewrite Y^* as

$$Y^* = \left(\frac{e}{ek + l[1 - b(1-t)]}\right) M_0 + \left(\frac{l}{ek + l[1 - b(1-t)]}\right)(a + d + G_0)$$

In this form, we can see that the Keynesian policy multipliers with respect to the money supply and government expenditure are the coefficients of M_0 and G_0, that is,

Money-supply multiplier: $$\frac{e}{ek + l[1 - b(1 - t)]}$$

and

Government-expenditure multiplier: $$\frac{l}{ek + l[l - b(1 - t)]}$$

Matrix Algebra versus Elimination of Variables

The economic models used for illustration above involve two or four equations only, and thus only fourth or lower-order determinants need to be evaluated. For large equation systems, higher-order determinants will appear, and their evaluation will be more complicated. And so will be the inversion of large matrices. From the computational point of view, in fact, matrix inversion and Cramer's rule are not necessarily more efficient than the method of successive eliminations of variables.

However, matrix methods have other merits. As we have seen from the preceding pages, matrix algebra gives us a compact notation for any linear-equation system, and also furnishes a determinantal criterion for testing the existence of a unique solution. These are advantages not otherwise available. In addition, it should be noted that, unlike the elimination-of-variable method, which affords no means of analytically expressing the solution, the matrix-inversion method and Cramer's rule do provide the handy solution expressions $x^* = A^{-1}d$ and $x_j^* = |A_j|/|A|$. Such analytical expressions of the solution are useful not only because they are in themselves a summary statement of the actual solution procedure, but also because they make possible the performance of further mathematical operations on the solution as written, if called for.

Under certain circumstances, matrix methods can even claim a computational advantage, such as when the task is to solve at the same time several equation systems having an identical coefficient matrix A but different constant-term vectors. In such cases, the elimination-of-variable method would require that the computational procedure be repeated each time a new equation system is considered. With the matrix-inversion method, however, we are required to find the common inverse matrix A^{-1} *only once;* then the same inverse can be used to premultiply all the constant-term vectors pertaining to the various equation systems involved, in order to obtain their respective solutions. This particular computational advantage will take on great practical significance when we consider the solution of the Leontief input-output models in Sec. 5.7.

EXERCISE 5.6

1. Solve the national-income model in Exercise 3.5-1:
   (*a*) By matrix inversion (*b*) By Cramer's rule
   (List the variables in the order *Y*, *C*, *T*.)
2. Solve the national-income model in Exercise 3.5-2:
   (*a*) By matrix inversion (*b*) By Cramer's rule
   (List the variables in the order *Y*, *C*, *G*.)