Chapter 6

# Comparative Statics and the Concept of Derivative

This chapter and Chaps. 7 and 8 will be devoted to the methods of comparative-static analysis.

## 6.1 The Nature of Comparative Statics

Comparative statics, as the name suggests, is concerned with the comparison of different equilibrium states that are associated with different sets of values of parameters and exogenous variables. For purposes of such a comparison, we always start by assuming a given initial equilibrium state. In the isolated-market model, for example, such an initial equilibrium will be represented by a determinate price $P^*$ and a corresponding quantity $Q^*$. Similarly, in the simple national-income model of (3.23), the initial equilibrium will be specified by a determinate $Y^*$ and a corresponding $C^*$. Now if we let a disequilibrating change occur in the model—in the form of a change in the value of some parameter or exogenous variable—the initial equilibrium will, of course, be upset. As a result, the various endogenous variables must undergo certain adjustments. If it is assumed that a new equilibrium state relevant to the new values of the data can be defined and attained, the question posed in the comparative-static analysis is: How would the new equilibrium compare with the old?

It should be noted that in comparative statics we still disregard the process of adjustment of the variables; we merely compare the initial (*pre*change) equilibrium state with the final (*post*change) equilibrium state. Also, we still preclude the possibility of instability of equilibrium, *for we assume the new equilibrium to be attainable, just as we do for the old.*

A comparative-static analysis can be either qualitative or quantitative in nature. If we are interested only in the question of, say, whether an increase in investment $I_0$ will increase or decrease the equilibrium income $Y^*$, the analysis will be qualitative because the *direction* of change is the only matter considered. But if we are concerned with the *magnitude* of the change in $Y^*$ resulting from a given change in $I_0$ (that is, the size of the investment multiplier), the analysis will obviously be quantitative. By obtaining a quantitative answer, however, we can automatically tell the direction of change from its algebraic sign. Hence the quantitative analysis always embraces the qualitative.

It should be clear that the problem under consideration is essentially one of finding a *rate of change:* the rate of change of the equilibrium value of an endogenous variable with respect to the change in a particular parameter or exogenous variable. For this reason, the mathematical concept of *derivative* takes on preponderant significance in comparative statics, because that concept—the most fundamental one in the branch of mathematics known as *differential calculus*—is directly concerned with the notion of rate of change! Later on, moreover, we shall find the concept of derivative to be of extreme importance for optimization problems as well.

## 6.2 Rate of Change and the Derivative

Even though our present context is concerned only with the rates of change of the equilibrium values of the variables in a model, we may carry on the discussion in a more general manner by considering the rate of change of any variable $y$ in response to a change in another variable $x$, where the two variables are related to each other by the function

$$y = f(x)$$

Applied to the comparative-static context, the variable $y$ will represent the equilibrium value of an endogenous variable, and $x$ will be some parameter. Note that, for a start, we are restricting ourselves to the simple case where there is only a single parameter or exogenous variable in the model. Once we have mastered this simplified case, however, the extension to the case of more parameters will prove relatively easy.

### The Difference Quotient

Since the notion of "change" figures prominently in the present context, a special symbol is needed to represent it. When the variable $x$ changes from the value $x_0$ to a new value $x_1$, the change is measured by the difference $x_1 - x_0$. Hence, using the symbol $\Delta$ (the Greek capital delta, for "difference") to denote the change, we write $\Delta x = x_1 - x_0$. Also needed is a way of denoting the value of the function $f(x)$ at various values of $x$. The standard practice is to use the notation $f(x_i)$ to represent the value of $f(x)$ when $x = x_i$. Thus, for the function $f(x) = 5 + x^2$, we have $f(0) = 5 + 0^2 = 5$; and similarly, $f(2) = 5 + 2^2 = 9$, etc.

When $x$ changes from an initial value $x_0$ to a new value $(x_0 + \Delta x)$, the value of the function $y = f(x)$ changes from $f(x_0)$ to $f(x_0 + \Delta x)$. The change in $y$ per unit of change in $x$ can be represented by the *difference quotient.*

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \tag{6.1}$$

This quotient, which measures the average rate of change of $y$, can be calculated if we know the initial value of $x$, or $x_0$, and the magnitude of change in $x$, or $\Delta x$. That is, $\Delta y/\Delta x$ is a function of $x_0$ and $\Delta x$.

**Example 1** Given $y = f(x) = 3x^2 - 4$, we can write

$$f(x_0) = 3(x_0)^2 - 4 \qquad f(x_0 + \Delta x) = 3(x_0 + \Delta x)^2 - 4$$

Therefore, the difference quotient is

$$\frac{\Delta y}{\Delta x} = \frac{3(x_0 + \Delta x)^2 - 4 - \left(3x_0^2 - 4\right)}{\Delta x} = \frac{6x_0\,\Delta x + 3(\Delta x)^2}{\Delta x}$$
$$= 6x_0 + 3\,\Delta x \tag{6.2}$$

which can be evaluated if we are given $x_0$ and $\Delta x$. Let $x_0 = 3$ and $\Delta x = 4$; then the average rate of change of $y$ is $6(3) + 3(4) = 30$. This means that, on the average, as $x$ changes from 3 to 7, the change in $y$ is 30 units per unit change in $x$.

## The Derivative

Frequently, we are interested in the rate of change of $y$ when $\Delta x$ is very small. In such a case, it is possible to obtain an approximation of $\Delta y/\Delta x$ by dropping all the terms in the difference quotient involving the expression $\Delta x$. In (6.2), for instance, if $\Delta x$ is very small, we may simply take the term $6x_0$ on the right as an approximation of $\Delta y/\Delta x$. The smaller the value of $\Delta x$, of course, the closer is the approximation to the true value of $\Delta y/\Delta x$.

As $\Delta x$ approaches zero (meaning that it gets closer and closer to, but never actually reaches, zero), $(6x_0 + 3\,\Delta x)$ will approach the value $6x_0$, and by the same token, $\Delta y/\Delta x$ will approach $6x_0$ also. Symbolically, this fact is expressed either by the statement $\Delta y/\Delta x \to 6x_0$ as $\Delta x \to 0$, or by the equation

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (6x_0 + 3\Delta x) = 6x_0 \tag{6.3}$$

where the symbol $\lim\limits_{\Delta x \to 0}$ is read as "The limit of . . . as $\Delta x$ approaches 0." If, as $\Delta x \to 0$, the limit of the difference quotient $\Delta y/\Delta x$ indeed exists, that limit is called the derivative of the function $y = f(x)$.

Several points should be noted about the derivative if it exists. First, a derivative is a *function:* in fact, in this usage the word *derivative* really means a derived function. The original function $y = f(x)$ is a *primitive function,* and the derivative is another function derived from it. Whereas the difference quotient is a function of $x_0$ and $\Delta x$, you should observe—from (6.3), for instance- that the derivative is a function of $x_0$ only. This is because $\Delta x$ is already compelled to approach zero, and therefore it should not be regarded as another variable in the function. Let us also add that so far we have used the subscripted symbol $x_0$ only in order to stress the fact that a change in $x$ must start from some specific value of $x$. Now that this is understood, we may delete the subscript and simply state that the derivative, like the primitive function, is itself a function of the independent variable $x$. That is, for each value of $x$, there is a unique corresponding value for the derivative function.

Second, since the derivative is merely a limit of the difference quotient, which measures a rate of change of $y$, the derivative must of necessity also be a measure of some rate of change. In view of the fact that the change in $x$ envisaged in the derivative concept is infinitesimal (that is, $\Delta x \to 0$), the rate measured by the derivative is in the nature of an *instantaneous* rate of change.

Third, there is the matter of notation. Derivative functions are commonly denoted in two ways. Given a primitive function $y = f(x)$, one way of denoting its derivative (if it exists) is to use the symbol $f'(x)$, or simply $f'$; this notation is attributed to the mathematician

Lagrange. The other common notation is $dy/dx$, devised by the mathematician Leibniz. [Actually there is a third notation, $Dy$, or $Df(x)$, but we shall not use it in the following discussion.] The notation $f'(x)$, which resembles the notation for the primitive function $f(x)$, has the advantage of conveying the idea that the derivative is itself a function of $x$. The reason for expressing it as $f'(x)$—rather than, say, $\phi(x)$—is to emphasize that the function $f'$ is derived from the primitive function $f$. The alternative notation, $dy/dx$, serves instead to emphasize that the value of a derivative measures a rate of change. The letter $d$ is the counterpart of the Greek $\Delta$, and $dy/dx$ differs from $\Delta y/\Delta x$ chiefly in that the former is the limit of the latter as $\Delta x$ approaches zero. In the subsequent discussion, we shall use both of these notations, depending on which seems the more convenient in a particular context.

Using these two notations, we may define the derivative of a given function $y = f(x)$ as follows:

$$\frac{dy}{dx} \equiv f'(x) \equiv \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

**Example 2**

Referring to the function $y = 3x^2 - 4$ again, we have shown its difference quotient to be (6.2), and the limit of that quotient to be (6.3). On the basis of the latter, we may now write (replacing $x_0$ with $x$):

$$\frac{dy}{dx} = 6x \qquad \text{or} \qquad f'(x) = 6x$$

Note that different values of $x$ will give the derivative correspondingly different values. For instance, when $x = 3$, we find, by substituting $x = 3$ in the $f'(x)$ expression, that $f'(3) = 6(3) = 18$; similarly, when $x = 4$, we have $f'(4) = 6(4) = 24$. Thus, whereas $f'(x)$ denotes a *derivative function,* the expressions $f'(3)$ and $f'(4)$ each represents a specific *derivative value.*

## EXERCISE 6.2

1. Given the function $y = 4x^2 + 9$:
   (*a*) Find the difference quotient as a function of $x$ and $\Delta x$. (Use $x$ in lieu of $x_0$.)
   (*b*) Find the derivative $dy/dx$.
   (*c*) Find $f'(3)$ and $f'(4)$.
2. Given the function $y = 5x^2 - 4x$:
   (*a*) Find the difference quotient as a function of $x$ and $\Delta x$.
   (*b*) Find the derivative $dy/dx$.
   (*c*) Find $f'(2)$ and $f'(3)$.
3. Given the function $y = 5x - 2$:
   (*a*) Find the difference quotient $\Delta y/\Delta x$. What type of function is it?
   (*b*) Since the expression $\Delta x$ does not appear in the function $\Delta y/\Delta x$ in part (*a*), does it make any difference to the value of $\Delta y/\Delta x$ whether $\Delta x$ is large or small? Consequently, what is the limit of the difference quotient as $\Delta x$ approaches zero?

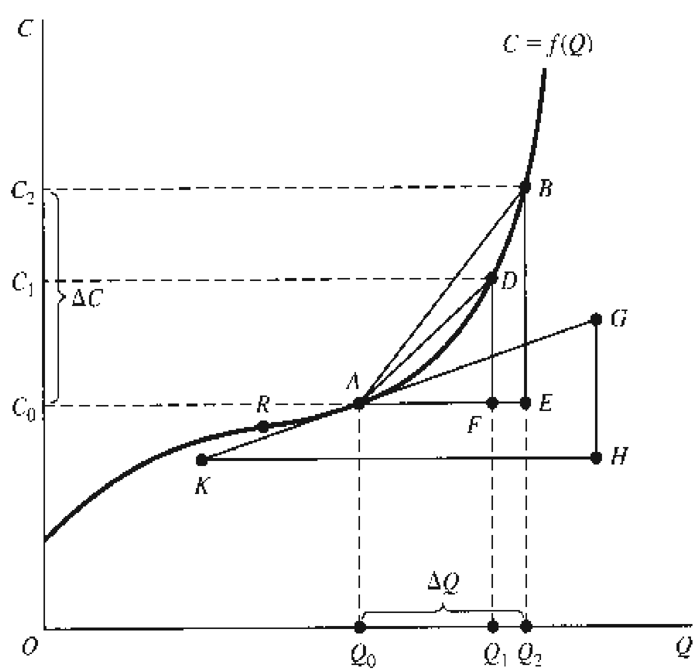
## 6.3 The Derivative and the Slope of a Curve

Elementary economics tells us that, given a total-cost function $C = f(Q)$, where $C$ denotes total cost and $Q$ the output, the marginal cost (MC) is defined as the change in total cost resulting from a unit increase in output; that is, $\text{MC} = \Delta C/\Delta Q$. It is understood that $\Delta Q$ is an extremely small change. For the case of a product that has discrete units (integers only), a change of one unit is the smallest change possible; but for the case of a product whose quantity is a continuous variable, $\Delta Q$ can refer to an infinitesimal change. In this latter case, it is well known that the marginal cost can be measured by the slope of the total-cost curve. But the slope of the total-cost curve is nothing but the limit of the ratio $\Delta C/\Delta Q$, when $\Delta Q$ approaches zero. Thus the concept of the slope of a curve is merely the geometric counterpart of the concept of the derivative. Both have to do with the "marginal" notion so extensively used in economics.

In Fig. 6.1, we have drawn a total-cost curve $C$, which is the graph of the (primitive) function $C = f(Q)$. Suppose that we consider $Q_0$ as the initial output level from which an increase in output is measured; then the relevant point on the cost curve is the point $A$. If output is to be raised to $Q_0 + \Delta Q = Q_2$, the total cost will be increased from $C_0$ to $C_0 + \Delta C = C_2$; thus $\Delta C/\Delta Q = (C_2 - C_0)/(Q_2 - Q_0)$. Geometrically, this is the ratio of two line segments, $EB/AE$, or the *slope* of the line $AB$. This particular ratio measures an average rate of change—the *average* marginal cost for the particular $\Delta Q$ pictured—and represents a difference quotient. As such, it is a function of the initial value $Q_0$ and the amount of change $\Delta Q$.

What happens when we vary the magnitude of $\Delta Q$? If a smaller output increment is contemplated (say, from $Q_0$ to $Q_1$ only), then the average marginal cost will be measured by the slope of the line $AD$ instead. Moreover, as we reduce the output increment further and further, flatter and flatter lines will result until, in the limit (as $\Delta Q \to 0$), we obtain the line $KG$ (which is the *tangent line* to the cost curve at point $A$) as the relevant line. The slope

**FIGURE 6.1**



of $KG$ ($= HG/KH$) measures the slope of the total-cost curve at point $A$ and represents the limit of $\Delta C/\Delta Q$, as $\Delta Q \to 0$, when initial output is at $Q = Q_0$. Therefore, in terms of the derivative, the slope of the $C = f(Q)$ curve at point $A$ corresponds to the particular derivative value $f'(Q_0)$.

What if the initial output level is changed from $Q_0$ to, say, $Q_2$? In that case, point $B$ on the curve will replace point $A$ as the relevant point, and the slope of the curve at the new point $B$ will give us the derivative value $f'(Q_2)$. Analogous results are obtainable for alternative initial output levels. In general, the derivative $f'(Q)$—a function of $Q$—will vary as $Q$ changes.

# 6.4 The Concept of Limit

The derivative $dy/dx$ has been defined as the limit of the difference quotient $\Delta y/\Delta x$ as $\Delta x \to 0$. If we adopt the shorthand symbols $q \equiv \Delta y/\Delta x$ ($q$ for quotient) and $v \equiv \Delta x$ ($v$ for variation in the value of $x$), we have

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{v \to 0} q$$

In view of the fact that the derivative concept relies heavily on the notion of limit, it is imperative that we get a clear idea about that notion.

## Left-Side Limit and Right-Side Limit

The concept of limit is concerned with the question: "What value does one variable (say, $q$) approach as another variable (say, $v$) approaches a specific value (say, zero)?" In order for this question to make sense, $q$ must, of course, be a function of $v$; say, $q = g(v)$. Our immediate interest is in finding the limit of $q$ as $v \to 0$, but we may just as easily explore the more general case of $v \to N$, where $N$ is any finite real number. Then, $\lim_{v \to 0} q$ will be merely a special case of $\lim_{v \to N} q$ where $N = 0$. In the course of the discussion, we shall actually also consider the limit of $q$ as $v \to +\infty$ (plus infinity) or as $v \to -\infty$ (minus infinity).

When we say $v \to N$, the variable $v$ can approach the number $N$ either from values greater than $N$, or from values less than $N$. If, as $v \to N$ from the left side (from values less than $N$), $q$ approaches a finite number $L$, we call $L$ the *left-side limit* of $q$. On the other hand, if $L$ is the number that $q$ tends to as $v \to N$ from the right side (from values greater than $N$), we call $L$ the *right-side limit* of $q$. The left- and right-side limits may or may not be equal.

The left-side limit of $q$ is symbolized by $\lim_{v \to N^-} q$ (the minus sign signifies from values less than $N$), and the right-side limit is written as $\lim_{v \to N^+} q$. When—and only when—the two limits have a common finite value (say, $L$), we consider the limit of $q$ to exist and write it as $\lim_{v \to N} q = L$. Note that $L$ must be a *finite* number. If we have the situation of $\lim_{v \to N} q = \infty$ (or $-\infty$), we shall consider $q$ to possess *no* limit, because $\lim_{v \to N} q = \infty$ means that $q \to \infty$ as $v \to N$, and if $q$ will assume *ever-increasing* values as $v$ tends to $N$, it would be contradictory to say that $q$ has a limit. As a convenient way of expressing the fact that $q \to \infty$ as $v \to N$, however, some people do indeed write $\lim_{v \to N} q = \infty$ and speak of $q$ as having an "infinite limit."

In certain cases, only the limit of one side needs to be considered. In taking the limit of $q$ as $v \to +\infty$, for instance, only the left-side limit of $q$ is relevant, because $v$ can approach $+\infty$ only from the left. Similarly, for the case of $v \to -\infty$, only the right-side limit is relevant. Whether the limit of $q$ exists in these cases will depend only on whether $q$ approaches a finite value as $v \to +\infty$, or as $v \to -\infty$.

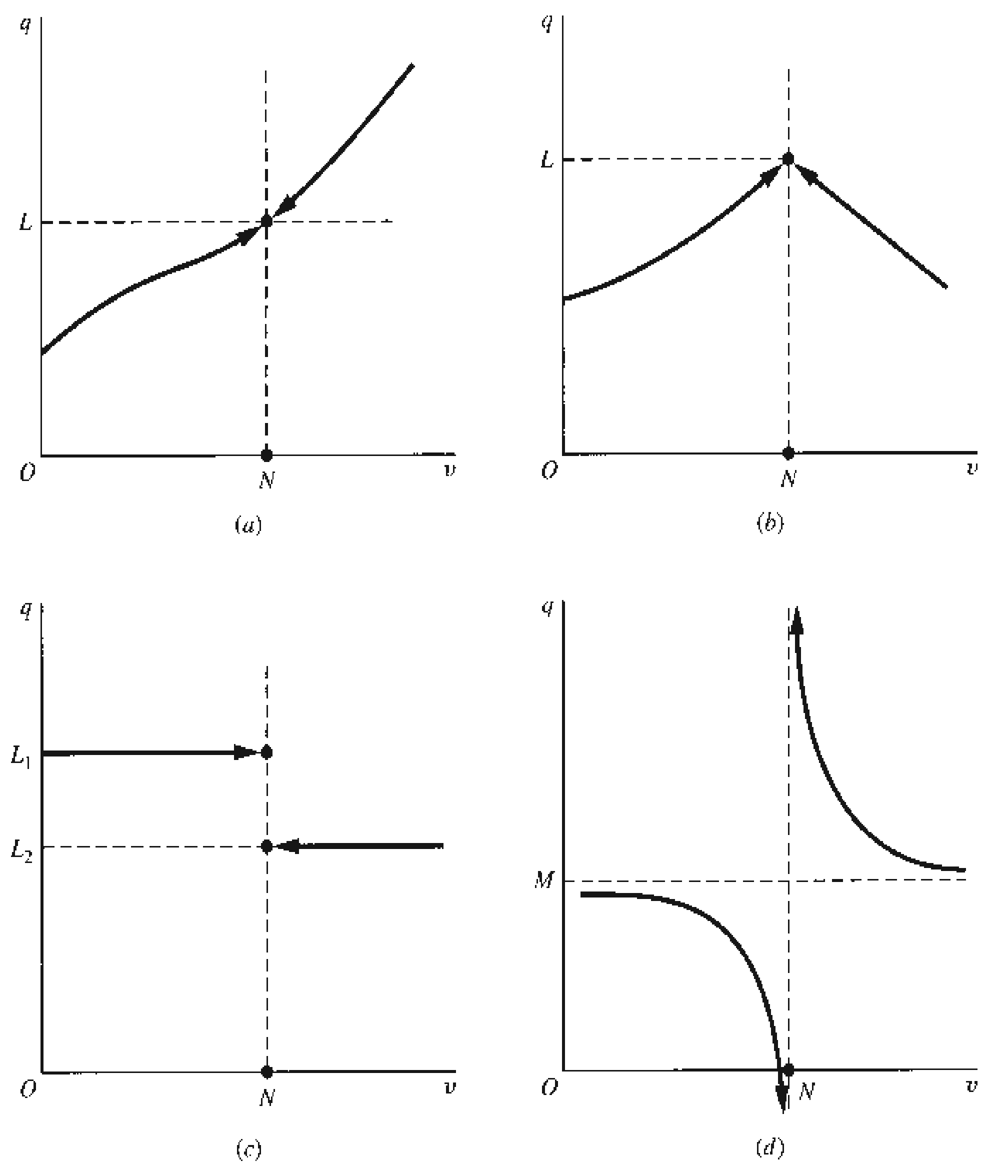
It is important to realize that the symbol $\infty$ (infinity) is not a number, and therefore it cannot be subjected to the usual algebraic operations. We cannot have $3 + \infty$ or $1/\infty$; nor can we write $q = \infty$, which is not the same as $q \to \infty$. However, it is acceptable to express the *limit* of $q$ as "=" (as against $\to$) $\infty$, for this merely indicates that $q \to \infty$.

## Graphical Illustrations

Let us illustrate, in Fig. 6.2, several possible situations regarding the limit of a function $q = g(v)$.

Figure 6.2*a* shows a smooth curve. As the variable $v$ tends to the value $N$ from *either* side on the horizontal axis, the variable $q$ tends to the value $L$. In this case, the left-side limit is identical with the right-side limit; therefore we can write $\lim\limits_{v \to N} q = L$.

**FIGURE 6.2**



The curve drawn in Fig. 6.2*b* is not smooth; it has a sharp turning point directly above the point $N$. Nevertheless, as $v$ tends to $N$ from either side, $q$ again tends to an identical value $L$. The limit of $q$ again exists and is equal to $L$.

Figure 6.2*c* shows what is known as a *step function*.† In this case, as $v$ tends to $N$, the left-side limit of $q$ is $L_1$, but the right-side limit is $L_2$, a different number. Hence, $q$ does not have a limit as $v \to N$.

Lastly, in Fig. 6.2*d*, as $v$ tends to $N$, the left-side limit of $q$ is $-\infty$, whereas the right-side limit is $+\infty$, because the two parts of the (hyperbolic) curve will fall and rise indefinitely while approaching the broken vertical line as an asymptote. Again, $\lim_{v \to N} q$ does not exist. On the other hand, if we are considering a different sort of limit in diagram *d*, namely, $\lim_{v \to +\infty} q$, then only the left-side limit has relevance, and we do find that limit to exist: $\lim_{v \to +\infty} q = M$. Analogously, you can verify that $\lim_{v \to -\infty} q = M$ as well.

It is also possible to apply the concepts of left-side and right-side limits to the discussion of the marginal cost in Fig. 6.1. In that context, the variables $q$ and $v$ will refer, respectively, to the quotient $\Delta C/\Delta Q$ and to the magnitude of $\Delta Q$, with all changes being measured from point $A$ on the curve. In other words, $q$ will refer to the slope of such lines as $AB$, $AD$, and $KG$, whereas $v$ will refer to the length of such lines as $Q_0Q_2$ (= line $AE$) and $Q_0Q_1$ (= line $AF$). We have already seen that, as $v$ approaches zero from a positive value, $q$ will approach a value equal to the slope of line $KG$. Similarly, we can establish that, if $\Delta Q$ approaches zero from a negative value (i.e., as the *decrease* in output becomes less and less), the quotient $\Delta C/\Delta Q$, as measured by the slope of such lines as $RA$ (not drawn), will also approach a value equal to the slope of line $KG$. Indeed, the situation here is very much akin to that illustrated in Fig. 6.2*a*. Thus the slope of $KG$ in Fig. 6.1 (the counterpart of $L$ in Fig. 6.2) is indeed the limit of the quotient $q$ as $v$ tends to zero, and as such it gives us the marginal cost at the output level $Q = Q_0$.

## Evaluation of a Limit

Let us now illustrate the algebraic evaluation of a limit of a given function $q = g(v)$.

**Example 1**

Given $q = 2 + v^2$, find $\lim_{v \to 0} q$. To take the left-side limit, we substitute the series of negative values $-1, -\frac{1}{10}, -\frac{1}{100}, \ldots$ (in that order) for $v$ and find that $(2 + v^2)$ will decrease steadily and approach 2 (because $v^2$ will gradually approach 0). Next, for the right-side limit, we substitute the series of positive values $1, \frac{1}{10}, \frac{1}{100}, \ldots$ (in that order) for $v$ and find the same limit as before. Inasmuch as the two limits are identical, we consider the limit of $q$ to exist and write $\lim_{v \to 0} q = 2$.

† This name is easily explained by the shape of the curve. But step functions can be expressed algebraically, too. The one illustrated in Fig. 6.2c can be expressed by the equation

$$q = \begin{cases} L_1 & \text{(for } 0 \le v < N) \\ L_2 & \text{(for } N \le v) \end{cases}$$

Note that, in each subset of its domain as described, the function appears as a distinct constant function, which constitutes a "step" in the graph.

In economics, step functions can be used, for instance, to show the various prices charged for different quantities purchased (the curve shown in Fig. 6.2c pictures *quantity discount*) or the various tax rates applicable to different income brackets.

It is tempting to regard the answer obtained in Example 1 as the outcome of setting $v = 0$ in the equation $q = 2 + v^2$, but this temptation should in general be resisted. In evaluating $\lim_{v \to N} q$, we only let $v$ *tend to* $N$, but, as a rule, do not let $v = N$. Indeed, we can quite legitimately speak of the limit of $q$ as $v \to N$, even if $N$ is *not* in the domain of the function $q = g(v)$. In this latter case, if we try to set $v = N$, $q$ will clearly be undefined.

**Example 2**

Given $q = (1 - v^2)/(1 - v)$, find $\lim_{v \to 1} q$. Here, $N = 1$ is not in the domain of the function, and we cannot set $v = 1$ because that would involve division by zero. Moreover, even the limit-evaluation procedure of letting $v \to 1$, as used in Example 1, will cause difficulty, for the denominator $(1 - v)$ will approach zero when $v \to 1$, and we will still have no way of performing the division in the limit.

One way out of this difficulty is to try to transform the given ratio to a form in which $v$ will not appear in the denominator. Since $v \to 1$ implies that $v \neq 1$, so that $(1 - v)$ is nonzero, it is legitimate to divide the expression $(1 - v^2)$ by $(1 - v)$, and write†

$$q = \frac{1 - v^2}{1 - v} = 1 + v \qquad (v \neq 1)$$

In this new expression for $q$, there is no longer a denominator with $v$ in it. Since $(1 + v) \to 2$ as $v \to 1$ from *either* side, we may then conclude that $\lim_{v \to 1} q = 2$.

**Example 3**

Given $q = (2v + 5)/(v + 1)$, find $\lim_{v \to +\infty} q$. The variable $v$ again appears in *both* the numerator and the denominator. If we let $v \to +\infty$ in both, the result will be a ratio between two infinitely large numbers, which does not have a clear meaning. To get out of the difficulty, we try this time to transform the given ratio to a form in which the variable $v$ will not appear in the numerator.‡ This, again, can be accomplished by dividing out the given ratio. Since $(2v + 5)$ is not evenly divisible by $(v + 1)$, however, the result will contain a remainder term as follows:

$$q = \frac{2v + 5}{v + 1} = 2 + \frac{3}{v + 1}$$

But, at any rate, this new expression for $q$ no longer has a numerator with $v$ in it. Noting that the remainder $3/(v + 1) \to 0$ as $v \to +\infty$, we can then conclude that $\lim_{v \to +\infty} q = 2$.

There also exist several useful theorems on the evaluation of limits. These will be discussed in Sec. 6.6.

† The division can be performed, as in the case of numbers, in the following manner:

$$\begin{array}{r|l} & 1 + v \\ \hline 1 - v & 1 \quad - v^2 \\ & \underline{1 - v} \\ & \quad v - v^2 \\ & \quad \underline{\underline{v - v^2}} \end{array}$$

Alternatively, we may resort to factoring as follows:

$$\frac{1 - v^2}{1 - v} = \frac{(1 + v)(1 - v)}{1 - v} = 1 + v \qquad (v \neq 1)$$

‡ Note that, unlike the $v \to 0$ case, where we want to take $v$ out of the *denominator* in order to avoid division by zero, the $v \to \infty$ case is better served by taking $v$ out of the *numerator*. As $v \to \infty$, an expression containing $v$ in the numerator will become infinite but an expression with $v$ in the denominator will, more conveniently for us, approach zero and quietly vanish from the scene.

## Formal View of the Limit Concept

The previous discussion should have conveyed some general ideas about the limit concept. Let us now give it a more precise definition. Since such a definition will make use of the concept of *neighborhood* of a point on a line (in particular, a specific number as a point on the line of real numbers), we shall first explain the latter term.

For a given number $L$, there can always be found a number $(L - a_1) < L$ and another number $(L + a_2) > L$, where $a_1$ and $a_2$ are some arbitrary positive numbers. The set of all numbers falling between $(L - a_1)$ and $(L + a_2)$ is called the *interval* between those two numbers. If the numbers $(L - a_1)$ and $(L + a_2)$ are included in the set, the set is a *closed interval;* if they are excluded, the set is an *open interval.* A closed interval between $(L - a_1)$ and $(L + a_2)$ is denoted by the bracketed expression

$$[L - a_1, L + a_2] \equiv \{q \mid L - a_1 \le q \le L + a_2\}$$

and the corresponding *open* interval is denoted with parentheses:

$$(L - a_1, L + a_2) \equiv \{q \mid L - a_1 < q < L + a_2\} \qquad \textbf{(6.4)}$$

Thus, [ ] relate to the weak inequality sign $\le$, whereas ( ) relate to the strict inequality sign $<$. But in both types of intervals, the smaller number $(L - a_1)$ is always listed first. Later on, we shall also have occasion to refer to *half-open and half-closed* intervals such as (3, 5] and [6, ∞), which have the following meanings:

$$(3, 5] \equiv \{x \mid 3 < x \le 5\} \qquad [6, \infty) \equiv \{x \mid 6 \le x < \infty\}$$

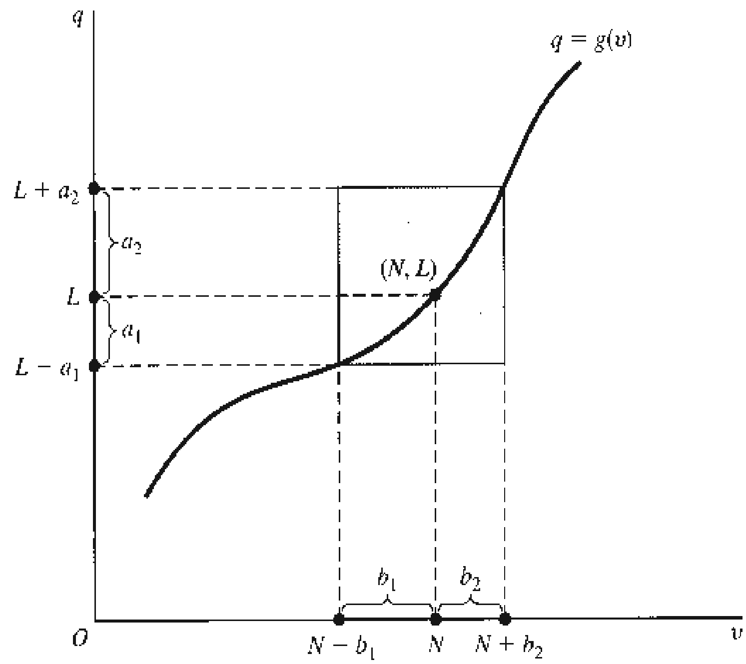
Now we may define a *neighborhood* of $L$ to be an open interval as defined in (6.4), which is an interval "covering" the number $L$.† Depending on the magnitudes of the arbitrary numbers $a_1$ and $a_2$, it is possible to construct various neighborhoods for the given number $L$. Using the concept of neighborhood, the limit of a function may then be defined as follows:

> As $v$ approaches a number $N$, the limit of $q = g(v)$ is the number $L$, if, for every neighborhood of $L$ that can be chosen, *however small,* there can be found a corresponding neighborhood of $N$ (excluding the point $v = N$) in the domain of the function such that, for every value of $v$ in that $N$-neighborhood, its image lies in the chosen $L$-neighborhood.

This statement can be clarified with the help of Fig. 6.3, which resembles Fig. 6.2*a*. From what was learned about Fig. 6.2*a*, we know that $\lim_{v \to N} q = L$ in Fig. 6.3. Let us show that $L$ does indeed fulfill the new definition of a limit. As the first step, select an arbitrary small neighborhood of $L$, say, $(L - a_1, L + a_2)$. (This should have been made even smaller, but we are keeping it relatively large to facilitate exposition.) Now construct a neighborhood of $N$, say, $(N - b_1, N + b_2)$, such that the two neighborhoods (when extended into quadrant I) will together define a rectangle (shaded in diagram) with two of its corners lying on the given curve. It can then be verified that, for every value of $v$ in this neighborhood of $N$ (not counting $v = N$), the corresponding value of $q = g(v)$ lies in the

† The identification of an open interval as the neighborhood of a point is valid only when we are considering a point on a line (one-dimensional space). In the case of a point in a plane (two-dimensional space), its neighborhood must be thought of as an area, say, a circular area that includes the point.

FIGURE 6.3




chosen neighborhood of $L$. In fact, no matter how *small* an $L$-neighborhood we choose, a (correspondingly small) $N$-neighborhood can be found with the property just cited. Thus $L$ fulfills the definition of a limit, as was to be demonstrated.

We can also apply the given definition to the step function of Fig. 6.2c in order to show that neither $L_1$ nor $L_2$ qualifies as $\lim_{v \to N} q$. If we choose a very small neighborhood of $L_1$—say, just a hair's width on each side of $L_1$—then, no matter what neighborhood we pick for $N$, the rectangle associated with the two neighborhoods cannot possibly enclose the lower step of the function. Consequently, for any value of $v > N$, the corresponding value of $q$ (located on the lower step) will not be in the neighborhood of $L_1$, and thus $L_1$ fails the test for a limit. By similar reasoning, $L_2$ must also be dismissed as a candidate for $\lim_{v \to N} q$. In fact, in this case no limit exists for $q$ as $v \to N$.

The fulfillment of the definition can also be checked algebraically rather than by graph. For instance, consider again the function

$$q = \frac{1 - v^2}{1 - v} = 1 + v \qquad (v \neq 1) \tag{6.5}$$

It has been found in Example 2 that $\lim_{v \to 1} q = 2$; thus, here we have $N = 1$ and $L = 2$. To verify that $L = 2$ is indeed the limit of $q$, we must demonstrate that, for every chosen neighborhood of $L$, $(2 - a_1, 2 + a_2)$, there exists a neighborhood of $N$, $(1 - b_1, 1 + b_2)$, such that, whenever $v$ is in this neighborhood of $N$, $q$ must be in the chosen neighborhood of $L$. This means essentially that, for given values of $a_1$ and $a_2$, however small, two numbers $b_1$ and $b_2$ must be found such that, whenever the inequality

$$1 - b_1 < v < 1 + b_2 \qquad (v \neq 1) \tag{6.6}$$

is satisfied, another inequality of the form

$$2 - a_1 < q < 2 + a_2 \tag{6.7}$$

must also be satisfied. To find such a pair of numbers $b_1$ and $b_2$, let us first rewrite (6.7) by substituting (6.5):

$$2 - a_1 < 1 + v < 2 + a_2 \tag{6.7'}$$

This, in turn, can be transformed (by subtracting 1 from each side) into the inequality

$$1 - a_1 < v < 1 + a_2 \tag{6.7''}$$

A comparison of (6.7″)—a variant of (6.7)—with (6.6) suggests that if we choose the two numbers $b_1$ and $b_2$ to be $b_1 = a_1$ and $b_2 = a_2$, the two inequalities (6.6) and (6.7) will always be satisfied simultaneously. Thus the neighborhood of $N$, $(1 - b_1, 1 + b_2)$, as required in the definition of a limit, can indeed be found for the case of $L = 2$, and this establishes $L = 2$ as the limit.

Let us now utilize the definition of a limit in the opposite way, to show that another value (say, 3) cannot qualify as $\lim_{v \to 1} q$ for the function in (6.5). If 3 were that limit, it would have to be true that, for every chosen neighborhood of 3, $(3 - a_1, 3 + a_2)$, there exists a neighborhood of 1, $(1 - b_1, 1 + b_2)$, such that, whenever $v$ is in the latter neighborhood, $q$ must be in the former neighborhood. That is, whenever the inequality

$$1 - b_1 < v < 1 + b_2$$

is satisfied, another inequality of the form

$$3 - a_1 < 1 + v < 3 + a_2$$

or

$$2 - a_1 < v < 2 + a_2$$

must also be satisfied. The *only* way to achieve this result is to choose $b_1 = a_1 - 1$ and $b_2 = a_2 + 1$. This would imply that the neighborhood of 1 is to be the open interval $(2 - a_1, 2 + a_2)$. According to the definition of a limit, however, $a_1$ and $a_2$ can be made arbitrarily small, say, $a_1 = a_2 = 0.1$. In that case, the last-mentioned interval will turn out to be $(1.9, 2.1)$ which lies entirely to the right of the point $v = 1$ on the horizontal axis and, hence, does not even qualify as a neighborhood of 1. Thus the definition of a limit cannot be satisfied by the number 3. A similar procedure can be employed to show that *any* number other than 2 will contradict the definition of a limit in the present case.

In general, if one number satisfies the definition of a limit of $q$ as $v \to N$, then no other number can. If a limit exists, it is unique.

## EXERCISE 6.4

1. Given the function $q = (v^2 + v - 56)/(v - 7)$, $(v \neq 7)$, find the left-side limit and the right-side limit of $q$ as $v$ approaches 7. Can we conclude from these answers that $q$ has a limit as $v$ approaches 7?
2. Given $q = [(v + 2)^3 - 8]/v$, $(v \neq 0)$, find:
   (*a*) $\lim_{v \to 0} q$ (*b*) $\lim_{v \to 2} q$ (*c*) $\lim_{v \to a} q$
3. Given $q = 5 - 1/v$, $(v \neq 0)$, find:
   (*a*) $\lim_{v \to +\infty} q$ (*b*) $\lim_{v \to -\infty} q$
4. Use Fig. 6.3 to show that we *cannot* consider the number $(L + a_2)$ as the limit of $q$ as $v$ tends to $N$.

## 6.5 Digression on Inequalities and Absolute Values

We have encountered inequality signs many times before. In the discussion of Sec. 6.4, we also applied mathematical operations to inequalities. In transforming (6.7′) into (6.7″), for example, we subtracted 1 from each side of the inequality. What rules of operations are generally applicable to inequalities (as opposed to equations)?

### Rules of Inequalities

To begin with, let us state an important property of inequalities: inequalities are *transitive*. This means that, if $a > b$ and if $b > c$, then $a > c$. Since equalities (equations) are also transitive, the transitivity property should apply to "weak" inequalities ($\geq$ or $\leq$) as well as to "strict" ones ($>$ or $<$). Thus we have

$$a > b, b > c \Rightarrow a > c$$
$$a \geq b, b \geq c \Rightarrow a \geq c$$

This property is what makes possible the writing of a *continued inequality*, such as $3 < a < b < 8$ or $7 \leq x \leq 24$. (In writing a continued inequality, the inequality signs are as a rule arranged in the same direction, usually with the smallest number on the left.)

The most important rules of inequalities are those governing the addition (subtraction) of a number to (from) an inequality, the multiplication or division of an inequality by a number, and the squaring of an inequality. Specifically, these rules are as follows.

**Rule I (addition and subtraction)** $a > b \Rightarrow a \pm k > b \pm k$

An inequality will continue to hold if an equal quantity is added to or subtracted from each side. This rule may be generalized thus: If $a > b > c$, then $a \pm k > b \pm k > c \pm k$.

**Rule II (multiplication and division)**

$$a > b \Rightarrow \begin{cases} ka > kb & (k > 0) \\ ka < kb & (k < 0) \end{cases}$$

The multiplication of both sides by a *positive* number preserves the inequality, but a *negative* multiplier will cause the *sense* (or *direction*) of the inequality to be reversed.

**Example 1**

Since $6 > 5$, multiplication by 3 will yield $3(6) > 3(5)$, or $18 > 15$; but multiplication by $-3$ will result in $(-3)6 < (-3)5$, or $-18 < -15$.

Division of an inequality by a number $n$ is equivalent to multiplication by the number $1/n$; therefore the rule on division is subsumed under the rule on multiplication.

**Rule III (squaring)** $a > b, (b \geq 0) \Rightarrow a^2 > b^2$

If its two sides are both nonnegative, the inequality will continue to hold when both sides are squared.

**Example 2**

Since $4 > 3$ and since both sides are positive, we have $4^2 > 3^2$, or $16 > 9$. Similarly, since $2 > 0$, it follows that $2^2 > 0^2$, or $4 > 0$.

Rules I through III have been stated in terms of strict inequalities, but their validity is unaffected if the $>$ signs are replaced by $\geq$ signs.

## Absolute Values and Inequalities

When the domain of a variable $x$ is an open interval $(a, b)$, the domain may be denoted by the set $\{x \mid a < x < b\}$ or, more simply, by the inequality $a < x < b$. Similarly, if it is a closed interval $[a, b]$, it may be expressed by the weak inequality $a \leq x \leq b$. In the special case of an interval of the form $(-a, a)$—say, $(-10, 10)$ it may be represented either by the inequality $-10 < x < 10$ or, alternatively, by the inequality

$$|x| < 10$$

where the symbol $|x|$ denotes the *absolute value* (or *numerical value*) of $x$.

For any real number $n$, the absolute value of $n$ is defined as follows:†

$$|n| \equiv \begin{cases} n & (\text{if } n > 0) \\ -n & (\text{if } n < 0) \\ 0 & (\text{if } n = 0) \end{cases} \tag{6.8}$$

Note that, if $n = 15$, then $|15| = 15$; but if $n = -15$, we find

$$|-15| = -(-15) = 15$$

also. In effect, therefore, the absolute value of any real number is simply its numerical value after the sign is removed. For this reason, we always have $|n| = |-n|$. The absolute value of $n$ is also called the *modulus* of $n$.

Given the expression $|x| = 10$, we may conclude from (6.8) that $x$ must be either 10 or $-10$. By the same token, the expression $|x| < 10$ means that (1) if $x > 0$, then $x \equiv |x| < 10$, so that $x$ must be less than 10; but also (2) if $x < 0$, then according to (6.8) we have $-x \equiv |x| < 10$, or $x > -10$, so that $x$ must be greater than $-10$. Hence, by combining the two parts of this result, we see that $x$ must lie within the open interval $(-10, 10)$. In general, we can write

$$|x| < n \Leftrightarrow -n < x < n \qquad (n > 0) \tag{6.9}$$

which can also be extended to weak inequalities as follows:

$$|x| \leq n \Leftrightarrow -n \leq x \leq n \qquad (n \geq 0) \tag{6.10}$$

Because they are themselves numbers, the absolute values of two numbers $m$ and $n$ can be added, subtracted, multiplied, and divided. The following properties characterize absolute values:

$$|m| + |n| \geq |m + n|$$

$$|m| \cdot |n| = |m \cdot n|$$

$$\frac{|m|}{|n|} = \left|\frac{m}{n}\right|$$

The first of these, interestingly, involves an inequality rather than an equation. The reason for this is easily seen: whereas the left-hand expression $|m| + |n|$ is definitely a *sum* of two

† We caution again that, although the absolute-value notation is similar to that of a first-order determinant, these two concepts are entirely different. The definition of a first-order determinant is $|a_{ij}| \equiv a_{ij}$, regardless of the sign of $a_{ij}$. In the definition of the absolute value $|n|$, on the other hand, the sign of $n$ will make a difference. The context of the discussion should normally make it clear whether an absolute value or a first-order determinant is under consideration.

numerical values (both taken as positive), the expression $|m + n|$ is the numerical value of *either* a sum (if $m$ and $n$ are, say, both positive) *or* a difference (if $m$ and $n$ have opposite signs). Thus the left side may exceed the right side.

**Example 3**

If $m = 5$ and $n = 3$, then $|m| + |n| = |m + n| = 8$. But if $m = 5$ and $n = -3$, then $|m| + |n| = 5 + 3 = 8$, whereas

$$|m + n| = |5 - 3| = 2$$

is a smaller number.

In the other two properties, on the other hand, it makes no difference whether $m$ and $n$ have identical or opposite signs, since, in taking the absolute value of the product or quotient on the right-hand side, the sign of the latter term will be removed in any case.

**Example 4**

If $m = 7$ and $n = 8$, then $|m| \cdot |n| = |m \cdot n| = 7(8) = 56$. But even if $m = -7$ and $n = 8$ (opposite signs), we still get the same result from

$$|m| \cdot |n| = |-7| \cdot |8| = 7(8) = 56$$

and

$$|m \cdot n| = |-7(8)| = 7(8) = 56$$

## Solution of an Inequality

Like an equation, an inequality containing a variable (say, $x$) may have a solution; the solution, if it exists, is a set of values of $x$ which make the inequality a true statement. Such a solution will itself usually be in the form of an inequality.

**Example 5**

Find the solution of the inequality

$$3x - 3 > x + 1$$

As in solving an equation, the variable terms should first be collected on one side of the inequality. By adding $(3 - x)$ to both sides, we obtain

$$3x - 3 + 3 - x > x + 1 + 3 - x$$

or

$$2x > 4$$

Multiplying both sides by $\frac{1}{2}$ (which does not reverse the sense of the inequality, because $\frac{1}{2} > 0$) will then yield the solution

$$x > 2$$

which is itself an inequality. This solution is not a single number, but a set of numbers. Therefore we may also express the solution as the set $\{x \mid x > 2\}$ or as the open interval $(2, \infty)$.

**Example 6**

Solve the inequality $|1 - x| \leq 3$. First, let us get rid of the absolute-value notation by utilizing (6.10). The given inequality is equivalent to the statement that

$$-3 \leq 1 - x \leq 3$$

or, after subtracting 1 from each side,

$$-4 \leq -x \leq 2$$

Multiplying each side by (−1), we then get

$$4 \geq x \geq -2$$

where the sense of inequality has been duly reversed. Writing the smaller number first, we may express the solution in the form of the inequality

$$-2 \leq x \leq 4$$

or in the form of the set $\{x \mid -2 \leq x \leq 4\}$ or the closed interval $[-2, 4]$.

Sometimes, a problem may call for the satisfaction of several inequalities in several variables simultaneously; then we must solve a system of simultaneous inequalities. This problem arises, for example, in nonlinear programming, which will be discussed in Chap. 13.

## EXERCISE 6.5

1. Solve the following inequalities:
   (a) $3x - 1 < 7x + 2$ (c) $5x + 1 < x + 3$
   (b) $2x + 5 < x - 4$ (d) $2x - 1 < 6x + 5$
2. If $8x - 3 < 0$ and $8x > 0$, express these in a continued inequality and find its solution.
3. Solve the following:
   (a) $|x + 1| < 6$ (b) $|4 - 3x| < 2$ (c) $|2x + 3| \leq 5$

# 6.6 Limit Theorems

Our interest in rates of change led us to the consideration of the concept of derivative, which, being in the nature of the limit of a difference quotient, in turn prompted us to study questions of the existence and evaluation of a limit. The basic process of limit evaluation, as illustrated in Sec. 6.4, involves letting the variable $v$ approach a particular number (say, $N$) and observing the value that $q$ approaches. When actually evaluating the limit of a function, however, we may draw upon certain established limit theorems, which can materially simplify the task, especially for complicated functions.

## Theorems Involving a Single Function

When a single function $q = g(v)$ is involved, the following theorems are applicable.

**Theorem I** If $q = av + b$, then $\lim_{v \to N} q = aN + b$ ($a$ and $b$ are constants).

**Example 1** Given $q = 5v + 7$, we have $\lim_{v \to 2} q = 5(2) + 7 = 17$. Similarly, $\lim_{v \to 0} q = 5(0) + 7 = 7$.

**Theorem II** If $q = g(v) = b$, then $\lim_{v \to N} q = b$.

This theorem, which says that the limit of a constant function is the constant in that function, is merely a special case of Theorem I, with $a = 0$. (You have already encountered an example of this case in Exercise 6.2-3.)

**Theorem III** If $q = v$, then $\lim_{v \to N} q = N$.
If $q = v^k$, then $\lim_{v \to N} q = N^k$.

**Example 2** Given $q = v^3$, we have $\lim_{v \to 2} q = (2)^3 = 8$.

You may have noted that, in Theorems I through III, what is done to find the limit of $q$ as $v \to N$ is indeed to let $v = N$. But these are special cases, and they do not vitiate the general rule that "$v \to N$" does not mean "$v = N$."

## Theorems Involving Two Functions

If we have two functions of the same independent variable $v$, $q_1 = g(v)$ and $q_2 = h(v)$, and if *both* functions possess limits as follows:

$$\lim_{v \to N} q_1 = L_1 \qquad \lim_{v \to N} q_2 = L_2$$

where $L_1$ and $L_2$ are two *finite* numbers, the following theorems are applicable.

**Theorem IV (sum-difference limit theorem)**

$$\lim_{v \to N}(q_1 \pm q_2) = L_1 \pm L_2$$

The limit of a sum (difference) of two functions is the sum (difference) of their respective limits.

In particular, we note that

$$\lim_{v \to N} 2q_1 = \lim_{v \to N}(q_1 + q_1) = L_1 + L_1 = 2L_1$$

which is in line with Theorem I.

**Theorem V (product limit theorem)**

$$\lim_{v \to N}(q_1 q_2) = L_1 L_2$$

The limit of a product of two functions is the product of their limits.

Applied to the square of a function, this gives

$$\lim_{v \to N}(q_1 q_1) = L_1 L_1 = L_1^2$$

which is in line with Theorem III.

**Theorem VI (quotient limit theorem)**

$$\lim_{v \to N} \frac{q_1}{q_2} = \frac{L_1}{L_2} \qquad (L_2 \neq 0)$$

The limit of a quotient of two functions is the quotient of their limits. Naturally, the limit $L_2$ is restricted to be nonzero; otherwise the quotient is undefined.

**Example 3** Find $\lim_{v \to 0}(1 + v)/(2 + v)$. Since we have here $\lim_{v \to 0}(1 + v) = 1$ and $\lim_{v \to 0}(2 + v) = 2$, the desired limit is $\frac{1}{2}$.

Remember that $L_1$ and $L_2$ represent finite numbers; otherwise these theorems do not apply. In the case of Theorem VI, furthermore, $L_2$ must be nonzero as well. If these restrictions are not satisfied, we must fall back on the method of limit evaluation illustrated in Examples 2 and 3 in Sec. 6.4, which relate to the cases, respectively, of $L_2$ being zero and of $L_2$ being infinite.

### Limit of a Polynomial Function

With the given limit theorems at our disposal, we can easily evaluate the limit of any polynomial function

$$q = g(v) = a_0 + a_1 v + a_2 v^2 + \cdots + a_n v^n \tag{6.11}$$

as $v$ tends to the number $N$. Since the limits of the separate terms are, respectively,

$$\lim_{v \to N} a_0 = a_0 \qquad \lim_{v \to N} a_1 v = a_1 N \qquad \lim_{v \to N} a_2 v^2 = a_2 N^2 \qquad \text{(etc.)}$$

the limit of the polynomial function is (by the sum limit theorem)

$$\lim_{v \to N} q = a_0 + a_1 N + a_2 N^2 + \cdots + a_n N^n \tag{6.12}$$

This limit is also, we note, actually equal to $g(N)$, that is, equal to the value of the function in (6.11) when $v = N$. This particular result will prove important in discussing the concept of *continuity* of the polynomial function.

## EXERCISE 6.6

1. Find the limits of the function $q = 7 - 9v + v^2$:
   (*a*) As $v \to 0$ (*b*) As $v \to 3$ (*c*) As $v \to -1$
2. Find the limits of $q = (v + 2)(v - 3)$:
   (*a*) As $v \to -1$ (*b*) As $v \to 0$ (*c*) As $v \to 5$
3. Find the limits of $q = (3v + 5)/(v + 2)$:
   (*a*) As $v \to 0$ (*b*) As $v \to 5$ (*c*) As $v \to -1$

## 6.7 Continuity and Differentiability of a Function

The preceding discussion of the concept of limit and its evaluation can now be used to define the continuity and differentiability of a function. These notions bear directly on the derivative of the function, which is what interests us.

### Continuity of a Function

When a function $q = g(v)$ possesses a limit as $v$ tends to the point $N$ in the domain, and when this limit is also equal to $g(N)$—that is, equal to the value of the function at $v = N$—the function is said to be *continuous* at $N$. As defined here, the term *continuity* involves no less than three requirements: (1) the point $N$ must be in the domain of the function; i.e., $g(N)$ is defined; (2) the function must have a limit as $v \to N$; i.e., $\lim_{v \to N} g(v)$ exists; and (3) that limit must be equal in value to $g(N)$; i.e., $\lim_{v \to N} g(v) = g(N)$.

It is important to note that while the point $(N, L)$ was excluded from consideration in discussing the limit of the curve in Fig. 6.3, we are no longer excluding it in the present context. Rather, as the third requirement specifically states, the point $(N, L)$ must be on the graph of the function before the function can be considered as continuous at point $N$.

Let us check whether the functions shown in Fig. 6.2 are continuous. In diagram *a*, all three requirements are met at point $N$. Point $N$ is in the domain; $q$ has the limit $L$ as $v \to N$; and the limit $L$ happens also to be the value of the function at $N$. Thus, the function represented by that curve is continuous at $N$. The same is true of the function depicted in Fig. 6.2*b*, since $L$ is the limit of the function as $v$ approaches the value $N$ in the domain, and since $L$ is also the value of the function at $N$. This last graphic example should suffice to establish that the continuity of a function at point $N$ does *not* necessarily imply that the graph of the function is "smooth" at $v = N$, for the point $(N, L)$ in Fig. 6.2*b* is actually a "sharp" point and yet the function is continuous at that value of $v$.

When a function $q = g(v)$ is continuous at all values of $v$ in the interval $(a, b)$, it is said to be continuous in that interval. If the function is continuous at all points in a subset $S$ of the domain (where the subset $S$ may be the union of several disjoint intervals), it is said to be continuous in $S$. And, finally, if the function is continuous at all points in its domain, we say that it is continuous in its domain. Even in this latter case, however, the graph of the function may nevertheless show a discontinuity (a gap) at some value of $v$, say, at $v = 5$, if that value of $v$ is *not* in its domain.

Again referring to Fig. 6.2, we see that in diagram *c* the function is *discontinuous* at $N$ because a limit does not exist at that point, in violation of the second requirement of continuity. Nevertheless, the function does satisfy the requirements of continuity in the interval $(0, N)$ of the domain, as well as in the interval $[N, \infty)$. Diagram *d* obviously is also discontinuous at $v = N$. This time, discontinuity emanates from the fact that $N$ is excluded from the domain, in violation of the first requirement of continuity.

On the basis of the graphs in Fig. 6.2, it appears that sharp points are consistent with continuity, as in diagram *b*, but that gaps are taboo, as in diagrams *c* and *d*. This is indeed the case. Roughly speaking, therefore, a function that is continuous in a particular interval is one whose graph can be drawn for the said interval without lifting the pencil or pen from the paper—a feat which is possible even if there are sharp points, but impossible when gaps occur.

## Polynomial and Rational Functions

Let us now consider the continuity of certain frequently encountered functions. For any polynomial function, such as $q = g(v)$ in (6.11), we have found from (6.12) that $\lim_{v \to N} q$ exists and is equal to the value of the function at $N$. Since $N$ is a point (any point) in the domain of the function, we can conclude that any polynomial function is continuous in its domain. This is a very useful piece of information, because polynomial functions will be encountered very often.

What about rational functions? Regarding continuity, there exists an interesting theorem (the continuity theorem) which states that the sum, difference, product, and quotient of any finite number of functions that are continuous in the domain are, respectively, also continuous in the domain. As a result, any rational function (a quotient of two polynomial functions) must also be continuous in its domain.

**Example 1** The rational function

$$q = g(v) = \frac{4v^2}{v^2 + 1}$$

is defined for all finite real numbers; thus its domain consists of the interval $(-\infty, \infty)$. For any number $N$ in the domain, the limit of $q$ is (by the quotient limit theorem)

$$\lim_{v \to N} q = \frac{\lim_{v \to N}(4v^2)}{\lim_{v \to N}(v^2+1)} = \frac{4N^2}{N^2+1}$$

which is equal to $g(N)$. Thus the three requirements of continuity are all met at $N$. Moreover, we note that $N$ can represent any point in the domain of this function; consequently, this function is continuous in its domain.

**Example 2** The rational function

$$q = \frac{v^3 + v^2 - 4v - 4}{v^2 - 4}$$

is not defined at $v = 2$ and at $v = -2$. Since those two values of $v$ are not in the domain, the function is discontinuous at $v = -2$ and $v = 2$, despite the fact that a limit of $q$ exists as $v \to -2$ or 2. Graphically, this function will display a gap at each of these two values of $v$. But for other values of $v$ (those which *are* in the domain), this function is continuous.

## Differentiability of a Function

The previous discussion has provided us with the tools for ascertaining whether any function has a limit as its independent variable approaches some specific value. Thus we can try to take the limit of any function $y = f(x)$ as $x$ approaches some chosen value, say, $x_0$. However, we can also apply the "limit" concept at a different level and take the limit of the difference quotient of that function, $\Delta y/\Delta x$, as $\Delta x$ approaches zero. The outcomes of limit-taking at these two different levels relate to two different, though related, properties of the function $f$.

Taking the limit of the function $y = f(x)$ itself, we can, in line with the discussion of the preceding subsection, examine whether the function $f$ is *continuous* at $x = x_0$. The conditions for continuity are (1) $x = x_0$ must be in the domain of the function $f$, (2) $y$ must have a limit as $x \to x_0$, and (3) the said limit must be equal to $f(x_0)$. When these are satisfied, we can write

$$\lim_{x \to x_0} f(x) = f(x_0) \qquad \text{[continuity condition]} \qquad \mathbf{(6.13)}$$

In contrast, when the "limit" concept is applied to the difference quotient $\Delta y/\Delta x$ as $\Delta x \to 0$, we deal instead with the question of whether the function $f$ is *differentiable* at $x = x_0$, i.e., whether the derivative $dy/dx$ exists at $x = x_0$, or whether $f'(x_0)$ exists. The term *differentiable* is used here because the process of obtaining the derivative $dy/dx$ is known as *differentiation* (also called *derivation*). Since $f'(x_0)$ exists if and only if the limit of $\Delta y/\Delta x$ exists at $x = x_0$ as $\Delta x \to 0$, the symbolic expression of the differentiability of $f$ is

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$\equiv \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \qquad \text{[differentiability condition]} \qquad \mathbf{(6.14)}$$

These two properties, continuity and differentiability, are very intimately related to each other—the continuity of $f$ is a *necessary* condition for its differentiability (although, as we shall see later, this condition is *not sufficient*). What this means is that, to be differentiable at $x = x_0$, the function must first pass the test of being continuous at $x = x_0$. To prove this, we shall demonstrate that, given a function $y = f(x)$, its continuity at $x = x_0$ follows from its differentiability at $x = x_0$; i.e., condition (6.13) follows from condition (6.14). Before doing this, however, let us simplify the notation somewhat by (1) replacing $x_0$ with the symbol $N$ and (2) replacing $(x_0 + \Delta x)$ with the symbol $x$. The latter is justifiable because the postchange value of $x$ can be any number (depending on the magnitude of the change) and hence is a variable denotable by $x$. The equivalence of the two notation systems is shown in Fig. 6.4, where the old notations appear (in brackets) alongside the new. Note that, with the notational change, $\Delta x$ now becomes $(x - N)$, so that the expression "$\Delta x \to 0$" becomes "$x \to N$," which is analogous to the expression $v \to N$ used before in connection with the function $q = g(v)$. Accordingly, (6.13) and (6.14) can now be rewritten, respectively, as

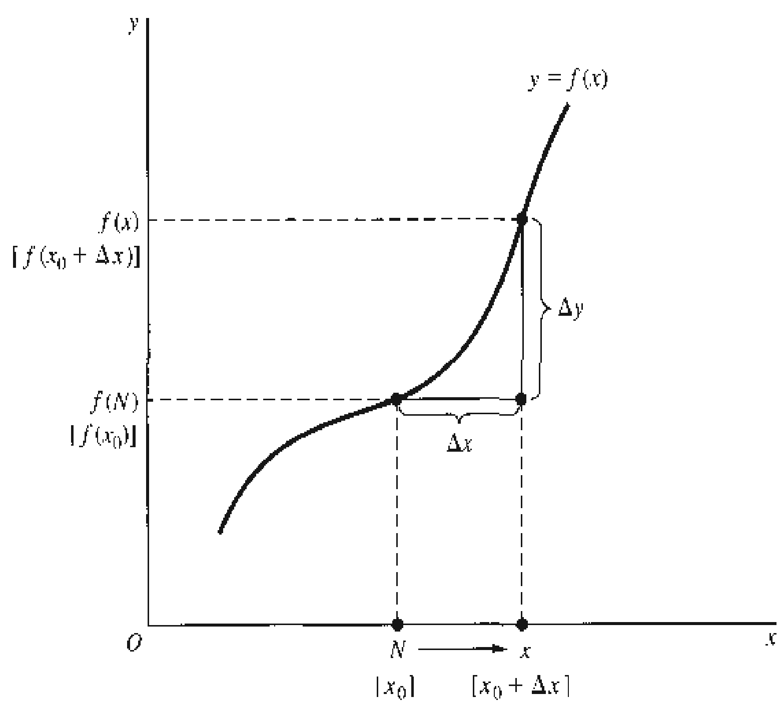
$$\lim_{x \to N} f(x) = f(N) \tag{6.13'}$$

$$f'(N) = \lim_{x \to N} \frac{f(x) - f(N)}{x - N} \tag{6.14'}$$

What we want to show is, therefore, that the continuity condition (6.13′) follows from the differentiability condition (6.14′). First, since the notation $x \to N$ implies that $x \neq N$, so that $x - N$ is a nonzero number, it is permissible to write the following identity:

$$f(x) - f(N) \equiv \frac{f(x) - f(N)}{x - N}(x - N) \tag{6.15}$$

**FIGURE 6.4**



Taking the limit of each side of (6.15) as $x \to N$ yields the following results:

$$\begin{aligned}
\text{Left side} &= \lim_{x\to N} f(x) - \lim_{x\to N} f(N) && \text{[difference limit theorem]}\\
&= \lim_{x\to N} f(x) - f(N) && [f(N) \text{ is a constant}]\\
\text{Right side} &= \lim_{x\to N} \frac{f(x) - f(N)}{x - N} \lim_{x\to N}(x - N) && \text{[product limit theorem]}\\
&= f'(N)(\lim_{x\to N} x - \lim_{x\to N} N) && \text{[by (6.14') and difference limit theorem]}\\
&= f'(N)(N - N) = 0
\end{aligned}$$

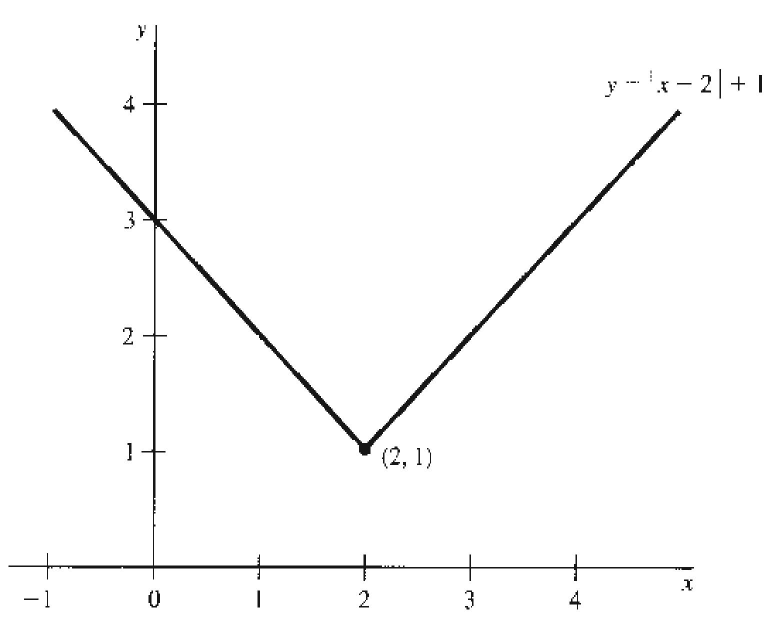
Note that we could not have written these results, if condition (6.14′) had not been granted, for if $f'(N)$ did not exist, then the right-side expression (and hence also the left-side expression) in (6.15) would not possess a limit. If $f'(N)$ does exist, however, the two sides will have limits as shown in the previous equations. Moreover, when the left-side result and the right-side result are equated, we get $\lim_{x\to N} f(x) - f(N) = 0$, which is identical with (6.13′). Thus we have proved that continuity, as shown in (6.13′), follows from differentiability, as shown in (6.14′). In general, if a function is differentiable at every point in its domain, we may conclude that it must be continuous in its domain.

Although differentiability implies continuity, the converse is not true. That is, continuity is a *necessary*, but *not* a *sufficient*, condition for differentiability. To demonstrate this, we merely have to produce a counterexample. Let us consider the function

$$y = f(x) = |x - 2| + 1 \qquad \textbf{(6.16)}$$

which is graphed in Fig. 6.5. As can be readily shown, this function is not differentiable, though continuous, when $x = 2$. That the function is continuous at $x = 2$ is easy to establish. First, $x = 2$ is in the domain of the function. Second, the limit of $y$ exists as $x$ tends to 2; to be specific, $\lim_{x\to 2^-} y = \lim_{x\to 2^-} y = 1$. Third, $f(2)$ is also found to be 1. Thus all three requirements of continuity are met. To show that the function $f$ is *not* differentiable at

**FIGURE 6.5**



$x = 2$, we must show that the limit of the difference quotient

$$\lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{|x-2| + 1 - 1}{x - 2} = \lim_{x \to 2} \frac{|x-2|}{x-2}$$

does *not* exist. This involves the demonstration of a disparity between the left-side and the right-side limits. Since, in considering the right-side limit, $x$ must exceed 2, according to the definition of absolute value in (6.8) we have $|x - 2| = x - 2$. Thus the right-side limit is

$$\lim_{x \to 2^+} \frac{|x-2|}{x-2} = \lim_{x \to 2^+} \frac{x-2}{x-2} = \lim_{x \to 2^-} 1 = 1$$

On the other hand, in considering the left-side limit, $x$ must be less than 2; thus, according to (6.8), $|x - 2| = -(x - 2)$. Consequently, the left-side limit is

$$\lim_{x \to 2^-} = \frac{|x-2|}{x-2} = \lim_{x \to 2^-} \frac{-(x-2)}{x-2} = \lim_{x \to 2} (-1) = -1$$

which is different from the right-side limit. This shows that continuity does not guarantee differentiability. In sum, all differentiable functions are continuous, but not all continuous functions are differentiable.

In Fig. 6.5, the nondifferentiability of the function at $x = 2$ is manifest in the fact that the point (2, 1) has no tangent line defined, and hence no definite slope can be assigned to the point. Specifically, to the left of that point, the curve has a slope of $-1$, but to the right it has a slope of $+1$, and the slopes on the two sides display no tendency to approach a common magnitude at $x = 2$. The point (2, 1) is, of course, a special point; it is the only sharp point on the curve. At other points on the curve, the derivative is defined and the function is differentiable. More specifically, the function in (6.16) can be divided into two linear functions as follows:

| | |
|---|---|
| Left part: | $y = -(x - 2) + 1 = 3 - x \quad (x \leq 2)$ |
| Right part: | $y = \quad (x - 2) + 1 = x - 1 \quad (x > 2)$ |

The left part is differentiable in the interval $(-\infty, 2)$, and the right part is differentiable in the interval $(2, \infty)$ in the domain.

In general, differentiability is a more restrictive condition than continuity, because it requires something beyond continuity. Continuity at a point only rules out the presence of a gap, whereas differentiability rules out "sharpness" as well. Therefore, differentiability calls for "smoothness" of the function (curve) as well as its continuity. Most of the *specific* functions employed in economics have the property that they are differentiable everywhere. When *general* functions are used, moreover, they are often assumed to be everywhere differentiable, as we shall in the subsequent discussion.

## EXERCISE 6.7

1. A function $y = f(x)$ is discontinuous at $x = x_0$ when *any* of the three requirements for continuity is violated at $x = x_0$. Construct three graphs to illustrate the violation of each of those requirements.

2. Taking the set of all finite real numbers as the domain of the function $q = g(v) = v^2 - 5v - 2$:
   (*a*) Find the limit of $q$ as $v$ tends to $N$ (a finite real number).
   (*b*) Check whether this limit is equal to $g(N)$.
   (*c*) Check whether the function is continuous at $N$ and continuous in its domain.
3. Given the function $q = g(v) = \dfrac{v+2}{v^2+2}$:
   (*a*) Use the limit theorems to find $\lim_{v \to N} q$, $N$ being a finite real number.
   (*b*) Check whether this limit is equal to $g(N)$.
   (*c*) Check the continuity of the function $g(v)$ at $N$ and in its domain $(-\infty, \infty)$.
4. Given $y = f(x) = \dfrac{x^2 - 9x + 20}{x - 4}$:
   (*a*) Is it possible to apply the quotient limit theorem to find the limit of this function as $x \to 4$?
   (*b*) Is this function continuous at $x = 4$? Why?
   (*c*) Find a function which, for $x \neq 4$, is equivalent to the given function, and obtain from the equivalent function the limit of $y$ as $x \to 4$.
5. In the rational function in Example 2, the numerator is evenly divisible by the denominator, and the quotient is $v + 1$. Can we for that reason replace that function outright by $q = v + 1$? Why or why not?
6. On the basis of the graphs of the six functions in Fig. 2.8, would you conclude that each such function is differentiable at every point in its domain? Explain.