polynomial equation ($n = 3$), we need to examine the signs of the first *three* determinants listed in the Routh theorem; for that purpose, we should set $a_4 = a_5 = 0$.

The relevance of this theorem to the convergence problem should become self-evident when we recall that, in order for the time path $y(t)$ to converge regardless of what the initial conditions happen to be, all the characteristic roots of the differential equation must have negative real parts. Since the characteristic equation (16.51′) is an nth-degree polynomial equation, with $a_0 = 1$, the Routh theorem can be of direct help in the testing of convergence. In fact, we note that the coefficients of the characteristic equation (16.51′) are wholly identical with those of the given differential equation (16.51), so it is perfectly acceptable to substitute the coefficients of (16.51) directly into the sequence of determinants shown in the Routh theorem for testing, provided that we always take $a_0 = 1$. Inasmuch as the condition cited in the theorem is given on the "if and only if" basis, it obviously constitutes a necessary-and-sufficient condition.

Example 2

Test by the Routh theorem whether the differential equation of Example 1 has a convergent time path. This equation is of the fourth order, so $n = 4$. The coefficient are $a_0 = 1$, $a_1 = 6$, $a_2 = 14$, $a_3 = 16$, $a_4 = 8$, and $a_5 = a_6 = a_7 = 0$. Substituting these into the first four determinants, we find their values to be 6, 68, 800, and 6,400, respectively. Because they are all positive, we can conclude that the time path is convergent.

EXERCISE 16.7

1. Find the particular integral of each of the following:
   (*a*) $y'''(t) + 2y''(t) + y'(t) + 2y = 8$
   (*b*) $y'''(t) + y''(t) + 3y'(t) = 1$
   (*c*) $3y'''(t) + 9y''(t) = 1$
   (*d*) $y^{(4)}(t) + y''(t) = 4$
2. Find the y_p and the y_c (and hence the general solution) of:
   (*a*) $y'''(t) - 2y''(t) - y'(t) + 2y = 4$
   [*Hint*: $r^3 - 2r^2 - r + 2 = (r-1)(r+1)(r-2)$]
   (*b*) $y'''(t) + 7y''(t) + 15y'(t) + 9y = 0$
   [*Hint*: $r^3 + 7r^2 + 15r + 9 = (r-1)(r^2 + 6r + 9)$]
   (*c*) $y'''(t) + 6y''(t) + 10y'(t) - 8y = 8$
   [*Hint*: $r^3 + 6r^2 + 10r + 8 = (r-4)(r^2 + 2r + 2)$]
3. On the basis of the signs of the characteristic roots obtained in Prob. 2, analyze the dynamic stability of equilibrium. Then check your answer by the Routh theorem.
4. Without finding their characteristic roots, determine whether the following differential equations will give rise to convergent time paths:
   (*a*) $y'''(t) - 10y''(t) + 27y'(t) - 18y = 3$
   (*b*) $y'''(t) - 11y''(t) + 34y'(t) + 24y = 5$
   (*c*) $y'''(t) + 4y''(t) + 5y'(t) - 2y = -2$
5. Deduce from the Routh theorem that, for the second-order linear differential equation $y''(t) + a_1 y'(t) + a_2 y = b$, the solution path will be convergent regardless of initial conditions if and only if the coefficients a_1 and a_2 are both positive.

Chapter 17

Discrete Time: First-Order Difference Equations

In the continuous-time context, the pattern of change of a variable y is embodied in the derivatives $y'(t)$, $y''(t)$, etc. The time change involved in these is occurring continuously. When time is, instead, taken to be a *discrete* variable, so that the variable t is allowed to take integer values only, the concept of the derivative obviously will no longer be appropriate. Then, as we shall see, the pattern of change of the variable y must be described by so-called differences, rather than by derivatives or differentials, of $y(t)$. Accordingly, the techniques of differential equations will give way to those of *difference equations*.

When we are dealing with discrete time, the value of variable y will change only when the variable t changes from one integer value to the next, such as from $t = 1$ to $t = 2$. Meanwhile, nothing is supposed to happen to y. In this light, it becomes more convenient to interpret the values of t as referring to *periods*—rather than *points*—of time, with $t = 1$ denoting period 1 and $t = 2$ denoting period 2, and so forth. Then we may simply regard y as having one unique value in each time period. In view of this interpretation, the discrete-time version of economic dynamics is often referred to as *period analysis*. It should be emphasized, however, that "period" is being used here not in the calendar sense but in the analytical sense. Hence, a period may involve one extent of calendar time in a particular economic model, but an altogether different one in another. Even in the same model, moreover, each successive period should not necessarily be construed as meaning equal calendar time. In the analytical sense, a period is merely a length of time that elapses before the variable y undergoes a change.

17.1 Discrete Time, Differences, and Difference Equations

The change from continuous time to discrete time produces no effect on the fundamental nature of dynamic analysis, although the formulation of the problem must be altered. Basically, our dynamic problem is still to find a time path from some given pattern of change of a variable y over time. But the pattern of change should now be represented by the difference quotient $\Delta y/\Delta t$, which is the discrete-time counterpart of the derivative dy/dt. Recall, however, that t can now take only integer values; thus, when we are comparing the values of y in two consecutive periods, we must have $\Delta t = 1$. For this reason, the difference quotient $\Delta y/\Delta t$ can be simplified to the expression Δy; this is called the *first difference* of y. The symbol Δ, meaning difference, can accordingly be interpreted as a directive to take the first difference of (y). As such, it constitutes the discrete-time counterpart of the operator symbol d/dt.

The expression Δy can take various values, of course, depending on which two consecutive time periods are involved in the difference-taking (or "differencing"). To avoid ambiguity, let us add a time subscript to y and define the first difference more specifically, as follows:

$$\Delta y_t \equiv y_{t+1} - y_t \tag{17.1}$$

where y_t means the value of y in the tth period, and y_{t+1} is its value in the period immediately following the tth period. With this symbology, we may describe the pattern of change of y by an equation such as

$$\Delta y_t = 2 \tag{17.2}$$

or

$$\Delta y_t = -0.1 y_t \tag{17.3}$$

Equations of this type are called *difference equations*. Note the striking resemblance between the last two equations, on the one hand, and the differential equations $dy/dt = 2$ and $dy/dt = -0.1y$ on the other.

Even though difference equations derive their name from difference expressions such as Δy_t, there are alternate equivalent forms of such equations which are completely free of Δ expressions and which are more convenient to use. By virtue of (17.1), we can rewrite (17.2) as

$$y_{t+1} - y_t = 2 \tag{17.2'}$$

or

$$y_{t+1} = y_t + 2 \tag{17.2''}$$

For (17.3), the corresponding alternate equivalent forms are

$$y_{t+1} - 0.9 y_t = 0 \tag{17.3'}$$

or

$$y_{t+1} = 0.9 y_t \tag{17.3''}$$

The double-prime-numbered versions will prove convenient when we are calculating a y value from a known y value of the preceding period. In later discussions, however, we shall employ mostly the single-prime-numbered versions, i.e., those of (17.2′) and (17.3′).

It is important to note that the choice of time subscripts in a difference equation is somewhat arbitrary. For instance, without any change in meaning, (17.2′) can be rewritten as $y_t - y_{t-1} = 2$, where $(t - 1)$ refers to the period which immediately precedes the tth. Or, we may express it equivalently as $y_{t+2} - y_{t+1} = 2$.

Also, it may be pointed out that, although we have consistently used subscripted y symbols, it is also acceptable to use $y(t)$, $y(t+1)$, and $y(t-1)$ in their stead. In order to avoid using the notation $y(t)$ for both continuous-time and discrete-time cases, however, we shall, in the discussion of period analysis, adhere to the subscript device.

Analogous to differential equations, difference equations can be either linear or nonlinear, homogeneous or nonhomogeneous, and of the first or second (or higher) orders. Take (17.2′) for instance. It can be classified as: (1) linear, for no y term (of any period) is raised to the second (or higher) power or is multiplied by a y term of another period; (2) nonhomogeneous, since the right-hand side (where there is no y term) is nonzero; and (3) of the first order, because there exists only a *first difference* Δy_t, involving a one-period time lag only. (In contrast, a second-order difference equation, to be discussed in Chap. 18, involves a two-period lag and thus entails three y terms: y_{t+2}, y_{t+1}, as well as y_t.)

Actually, (17.2′) can also be characterized as having constant coefficients and a constant term (= 2). Since the constant-coefficient case is the only one we shall consider, this characterization will henceforth be implicitly assumed. Throughout the present chapter, the constant-term feature will also be retained, although a method of dealing with the variable-term case will be discussed in Chap. 18.

Check that the equation (17.3′) is also linear and of the first order; but unlike (17.2′), it is homogeneous.

17.2 Solving a First-Order Difference Equation

In solving a differential equation, our objective was to find a time path $y(t)$. As we know, such a time path is a function of time which is totally free from any derivative (or differential) expressions and which is perfectly consistent with the given differential equation as well as with its initial conditions. The time path we seek from a difference equation is similar in nature. Again, it should be a function of t—a formula defining the values of y in every time period—which is consistent with the given difference equation as well as with its initial conditions. Besides, it must not contain any difference expressions such as Δy_t (or expressions like $y_{t+1} - y_t$).

Solving differential equations is, in the final analysis, a matter of integration. How do we solve a difference equation?

Iterative Method

Before developing a general method of attack, let us first explain a relatively pedestrian method, the *iterative method*—which, though crude, will prove immensely revealing of the essential nature of a so-called solution.

In this chapter we are concerned only with the first-order case; thus the difference equation describes the pattern of change of y between *two* consecutive periods only. Once such a pattern is specified, such as by (17.2″), and once we are given an initial value y_0, it is no problem to find y_1 from the equation. Similarly, once y_1 is found, y_2 will be immediately obtainable, and so forth, by repeated application (iteration) of the pattern of change specified in the difference equation. The results of iteration will then permit us to infer a time path.

Example 1

Find the solution of the difference equation (17.2), assuming an initial value of $y_0 = 15$. To carry out the iterative process, it is more convenient to use the alternative form of the difference equation (17.2″), namely, $y_{t+1} = y_t + 2$, with $y_0 = 15$. From this equation, we can deduce step-by-step that

$$y_1 = y_0 + 2$$
$$y_2 = y_1 + 2 = (y_0 + 2) + 2 = y_0 + 2(2)$$
$$y_3 = y_2 + 2 = [y_0 + 2(2)] + 2 = y_0 + 3(2)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

and, in general, for any period t,

$$y_t = y_0 + t(2) = 15 + 2t \tag{17.4}$$

This last equation indicates the y value of any time period (including the initial period $t = 0$); it therefore constitutes the solution of (17.2).

The process of iteration is crude— it corresponds roughly to solving simple differential equations by straight integration—but it serves to point out clearly the manner in which a time path is generated. In general, the value of y_t will depend in a specified way on the value of y in the immediately preceding period (y_{t-1}); thus a given initial value y_0 will successively lead to $y_1, y_2, \ldots$, via the prescribed pattern of change.

Example 2

Solve the difference equation (17.3); this time, let the initial value be unspecified and denoted simply by y_0. Again it is more convenient to work with the alternative version in (17.3″), namely, $y_{t+1} = 0.9y_t$. By iteration, we have

$$y_1 = 0.9y_0$$
$$y_2 = 0.9y_1 = 0.9(0.9y_0) = (0.9)^2 y_0$$
$$y_3 = 0.9y_2 = 0.9(0.9)^2 y_0 = (0.9)^3 y_0$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

These can be summarized into the solution

$$y_t = (0.9)^t y_0 \tag{17.5}$$

To heighten interest, we can lend some economic content to this example. In the simple multiplier analysis, a single investment expenditure in period 0 will call forth successive rounds of spending, which in turn will bring about varying amounts of income increment in succeeding time periods. Using y to denote *income increment,* we have y_0 = the amount of investment in period 0; but the subsequent income increments will depend on the marginal propensity to consume (MPC). If MPC = 0.9 and if the income of each period is consumed only in the next period, then 90 percent of y_0 will be consumed in period 1, resulting in an income increment in period 1 of $y_1 = 0.9y_0$. By similar reasoning, we can find $y_2 = 0.9y_1$, etc. These, we see, are precisely the results of the iterative process cited previously. In other words, the multiplier process of income generation can be described by a difference equation such as (17.3″), and a solution like (17.5) will tell us what the magnitude of income increment is to be in any time period t.

Example 3

Solve the homogeneous difference equation

$$my_{t+1} - ny_t = 0$$

Upon normalizing and transposing, this may be written as

$$y_{t+1} = \left(\frac{n}{m}\right) y_t$$

which is the same as (17.3″) in Example 2 except for the replacement of 0.9 by n/m. Hence, by analogy, the solution should be

$$y_t = \left(\frac{n}{m}\right)^t y_0$$

Watch the term $\left(\frac{n}{m}\right)^t$. It is through this term that various values of t will lead to their corresponding values of y. It therefore corresponds to the expression e^{rt} in the solutions to differential equations. If we write it more generally as b^t (b for base) and attach the more general multiplicative constant A (instead of y_0), we see that the solution of the general homogeneous difference equation of Example 3 will be in the form

$$y_t = Ab^t$$

We shall find that this expression Ab^t will play the same important role in difference equations as the expression Ae^{rt} did in differential equations.† However, even though both are exponential expressions, the former is to the base b, whereas the latter is to the base e. It stands to reason that, just as the type of the continuous-time path $y(t)$ depends heavily on the value of r, the discrete-time path y_t hinges principally on the value of b.

General Method

By this time, you must have become quite impressed with the various similarities between differential and difference equations. As might be conjectured, the general method of solution presently to be explained will parallel that for differential equations.

Suppose that we are seeking the solution to the first-order difference equation

$$y_{t+1} + ay_t = c \tag{17.6}$$

where a and c are two constants. The general solution will consist of the sum of two components: a *particular solution* y_p, which is *any* solution of the complete nonhomogeneous equation (17.6), and a *complementary function* y_c, which is the general solution of the reduced equation of (17.6):

$$y_{t+1} + ay_t = 0 \tag{17.7}$$

The y_p component again represents the intertemporal equilibrium level of y, and the y_c component, the deviations of the time path from that equilibrium. The sum of y_c and y_p constitutes the *general* solution, because of the presence of an arbitrary constant. As before, in order to definitize the solution, an initial condition is needed.

Let us first deal with the complementary function. Our experience with Example 3 suggests that we may try a solution of the form $y_t = Ab^t$ (with $Ab^t \neq 0$, for otherwise y_t will turn out simply to be a horizontal straight line lying on the t axis); in that case, we also

† You may object to this statement by pointing out that the solution (17.4) in Example 1 does not contain a term in the form of Ab^t. This latter fact, however, arises only because in Example 1 we have $b = n/m = 1/1 = 1$, so that the term Ab^t reduces to a constant.

have $y_{t+1} = Ab^{t+1}$. If these values of y_t and y_{t+1} hold, the homogeneous equation (17.7) will become

$$Ab^{t+1} + aAb^t = 0$$

which, upon canceling the nonzero common factor Ab^t, yields

$$b + a = 0 \qquad \text{or} \qquad b = -a$$

This means that, for the trial solution to work, we must set $b = -a$; then the complementary function should be written as

$$y_c(= Ab^t) = A(-a)^t$$

Now let us search for the particular solution, which has to do with the complete equation (17.6). In this regard, Example 3 is of no help at all, because that example relates only to a homogeneous equation. However, we note that for y_p we can choose *any* solution of (17.6); thus if a trial solution of the simplest form $y_t = k$ (a constant) can work out, no real difficulty will be encountered. Now, if $y_t = k$, then y will maintain the same constant value over time, and we must have $y_{t+1} = k$ also. Substitution of these values into (17.6) yields

$$k + ak = c \qquad \text{and} \qquad k = \frac{c}{1+a}$$

Since this particular k value satisfies the equation, the particular integral can be written as

$$y_p(= k) = \frac{c}{1+a} \qquad (a \neq -1)$$

This being a constant, a stationary equilibrium is indicated in this case.

If it happens that $a = -1$, as in Example 1, however, the particular solution $c/(1+a)$ is not defined, and some other solution of the nonhomogeneous equation (17.6) must be sought. In this event, we employ the now-familiar trick of trying a solution of the form $y_t = kt$. This implies, of course, that $y_{t+1} = k(t+1)$. Substituting these into (17.6), we find

$$k(t+1) + akt = c \qquad \text{and} \qquad k = \frac{c}{t+1+at} = c \qquad [\text{because } a = -1]$$

thus

$$y_p(= kt) = ct$$

This form of the particular solution is a nonconstant function of t; it therefore represents a moving equilibrium.

Adding y_c and y_p together, we may now write the general solution in one of the two following forms:

$$y_t = A(-a)^t + \frac{c}{1+a} \qquad [\text{general solution, case of } a \neq -1] \qquad \mathbf{(17.8)}$$

$$y_t = A(-a)^t + ct = A + ct \qquad [\text{general solution, case of } a = -1] \qquad \mathbf{(17.9)}$$

Neither of these is completely determinate, in view of the arbitrary constant A. To eliminate this arbitrary constant, we resort to the initial condition that $y_t = y_0$ when $t = 0$. Letting $t = 0$ in (17.8), we have

$$y_0 = A + \frac{c}{1+a} \qquad \text{and} \qquad A = y_0 - \frac{c}{1+a}$$

Consequently, the definite version of (17.8) is

$$y_t = \left(y_0 - \frac{c}{1+a}\right)(-a)^t + \frac{c}{1+a} \qquad \text{[definite solution, case of } a \neq -1] \qquad \textbf{(17.8')}$$

Letting $t = 0$ in (17.9), on the other hand, we find $y_0 = A$, so the definite version of (17.9) is

$$y_t = y_0 + ct \qquad \text{[definite solution, case of } a = -1] \qquad \textbf{(17.9')}$$

If this last result is applied to Example 1, the solution that emerges is exactly the same as the iterative solution (17.4).

You can check the validity of each of these solutions by the following two steps. First, by letting $t = 0$ in (17.8′), see that the latter equation reduces to the identity $y_0 = y_0$, signifying the satisfaction of the initial condition. Second, by substituting the y_t formula (17.8′) and a similar y_{t+1} formula—obtained by replacing t with $(t+1)$ in (17.8′)—into (17.6), see that the latter reduces to the identity $c = c$, signifying that the time path is consistent with the given difference equation. The check on the validity of solution (17.9′) is analogous.

Example 4 Solve the first-order difference equation

$$y_{t+1} - 5y_t = 1 \qquad \left(y_0 = \tfrac{7}{4}\right)$$

Following the procedure used in deriving (17.8′), we can find y_c by trying a solution $y_t = Ab^t$ (which implies $y_{t+1} = Ab^{t-1}$). Substituting these values into the homogeneous version $y_{t-1} - 5y_t = 0$ and canceling the common factor Ab^t, we get $b = 5$. Thus

$$y_c = A(5)^t$$

To find y_p, try the solution $y_t = k$, which implies $y_{t+1} = k$. Substituting these into the complete difference equation, we find $k = -\frac{1}{4}$. Hence

$$y_p = -\tfrac{1}{4}$$

It follows that the general solution is

$$y_t = y_c + y_p = A(5)^t - \tfrac{1}{4}$$

Letting $t = 0$ here and utilizing the initial condition $y_0 = \frac{7}{4}$, we obtain $A = 2$. Thus the definite solution may finally be written as

$$y_t = 2(5)^t - \tfrac{1}{4}$$

Since the given difference equation of this example is a special case of (17.6), with $a = -5$, $c = 1$, and $y_0 = \frac{7}{4}$, and since (17.8′) is the solution "formula" for this type of difference equation, we could have found our solution by inserting the specific parameter values into (17.8′), with the result that

$$y_t = \left(\frac{7}{4} - \frac{1}{1-5}\right)(5)^t + \frac{1}{1-5} = 2(5)^t - \frac{1}{4}$$

which checks perfectly with the earlier answer.

Note that the y_{t+1} term in (17.6) has a unit coefficient. If a given difference equation has a nonunit coefficient for this term, it must be normalized before using the solution formula (17.8′).

EXERCISE 17.2

1. Convert the following difference equations into the form of (17.2″):
   (*a*) $\Delta y_t = 7$
   (*b*) $\Delta y_t = 0.3y_t$
   (*c*) $\Delta y_t = 2y_t - 9$
2. Solve the following difference equations by iteration:
   (*a*) $y_{t+1} = y_t - 1 \qquad (y_0 = 10)$
   (*b*) $y_{t+1} = \alpha y_t \qquad (y_0 = \beta)$
   (*c*) $y_{t-1} = \alpha y_t - \beta \qquad (y_t = y_0 \text{ when } t = 0)$
3. Rewrite the equations in Prob. 2 in the form of (17.6), and solve by applying formula (17.8′) or (17.9′), whichever is appropriate. Do your answers check with those obtained by the iterative method?
4. For each of the following difference equations, use the procedure illustrated in the derivation of (17.8′) and (17.9′) to find y_c, y_p, and the definite solution:
   (*a*) $y_{t-1} + 3y_t = 4 \qquad (y_0 = 4)$
   (*b*) $2y_{t+1} - y_t = 6 \qquad (y_0 = 7)$
   (*c*) $y_{t-1} = 0.2y_t + 4 \qquad (y_0 = 4)$

17.3 The Dynamic Stability of Equilibrium

In the continuous-time case, the dynamic stability of equilibrium depends on the Ae^{rt} term in the complementary function. In period analysis, the corresponding role is played by the Ab^t term in the complementary function. Since its interpretation is somewhat more complicated than Ae^{rt}, let us try to clarify it before proceeding further.

The Significance of *b*

Whether the equilibrium is dynamically stable is a question of whether or not the complementary function will tend to zero as $t \to \infty$. Basically, we must analyze the path of the term Ab^t as t is increased indefinitely. Obviously, the value of b (the base of this exponential term) is of crucial importance in this regard. Let us first consider its significance alone, by disregarding the coefficient A (by assuming $A = 1$).

For analytical purposes, we can divide the range of possible values of b, $(-\infty, +\infty)$, into seven distinct regions, as set forth in the first two columns of Table 17.1, arranged in descending order of magnitude of b. These regions are also marked off in Fig. 17.1 on a vertical b scale, with the points $+1$, 0, and -1 as the demarcation points. In fact, these latter three points in themselves constitute the regions II, IV, and VI. Regions III and V, on the other hand, correspond to the set of all positive fractions and the set of all negative fractions, respectively. The remaining two regions, I and VII, are where the numerical value of b exceeds unity.

In each region, the exponential expression b^t generates a different type of time path. These are exemplified in Table 17.1 and illustrated in Fig. 17.1. In region I (where $b > 1$), b^t must increase with t at an increasing pace. The general configuration of the time path will therefore assume the shape of the top graph in Fig. 17.1. Note that this graph is shown

TABLE 17.1
A Classification of the Values of b

Region	Value of b		Value of b^t	Value of b^t in Different Time Periods $t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4 \cdots$
I	$b > 1$	$(\|b\| > 1)$	e.g., $(2)^t$	1	2	4	8	16
II	$b = 1$	$(\|b\| = 1)$	$(1)^t$	1	1	1	1	1
III	$0 < b < 1$	$(\|b\| < 1)$	e.g., $\left(\frac{1}{2}\right)^t$	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$
IV	$b = 0$	$(\|b\| = 0)$	$(0)^t$	0	0	0	0	0
V	$-1 < b < 0$	$(\|b\| < 1)$	e.g., $\left(-\frac{1}{2}\right)^t$	1	$-\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{8}$	$\frac{1}{16}$
VI	$b = -1$	$(\|b\| = 1)$	$(-1)^t$	1	-1	1	-1	1
VII	$b < -1$	$(\|b\| > 1)$	e.g., $(-2)^t$	1	-2	4	-8	16

as a step function rather than as a smooth curve; this is because we are dealing with period analysis. In region II ($b = 1$), b^t will remain at unity for all values of t. Its graph will thus be a horizontal straight line. Next, in region III, b^t represents a positive fraction raised to integer powers. As the power is increased, b^t must decrease, though it will always remain positive. The next case, that of $b = 0$ in region IV, is quite similar to the case of $b = 1$; but here we have $b^t = 0$ rather than $b^t = 1$, so its graph will coincide with the horizontal axis. However, this case is of peripheral interest only, since we have earlier adopted the assumption that $Ab^t \neq 0$.

When we move into the negative regions, an interesting new phenomenon occurs: The value of b^t will *alternate* between positive and negative values from period to period! This fact is clearly brought out in the last three rows of Table 17.1 and in the last three graphs of Fig. 17.1. In region V, where b is a negative fraction, the alternating time path tends to get closer and closer to the horizontal axis (cf. the positive-fraction region, III). In contrast, when $b = -1$ (region VI), a perpetual alternation between -1 and -1 results. And finally, when $b < -1$ (region VII), the alternating time path will deviate farther and farther from the horizontal axis.

What is striking is that, whereas the phenomenon of a fluctuating time path cannot possibly arise from a single Ae^{rt} term (the complex-root case of the second-order differential equation requires a *pair* of complex roots), fluctuation can be generated by a single b^t (or Ab^t) term. Note, however, that the character of the fluctuation is somewhat different; unlike the circular-function pattern, the fluctuation depicted in Fig. 17.1 is nonsmooth. For this reason, we shall employ the word *oscillation* to denote the new, nonsmooth type of fluctuation, even though many writers do use the terms fluctuation and oscillation interchangeably.

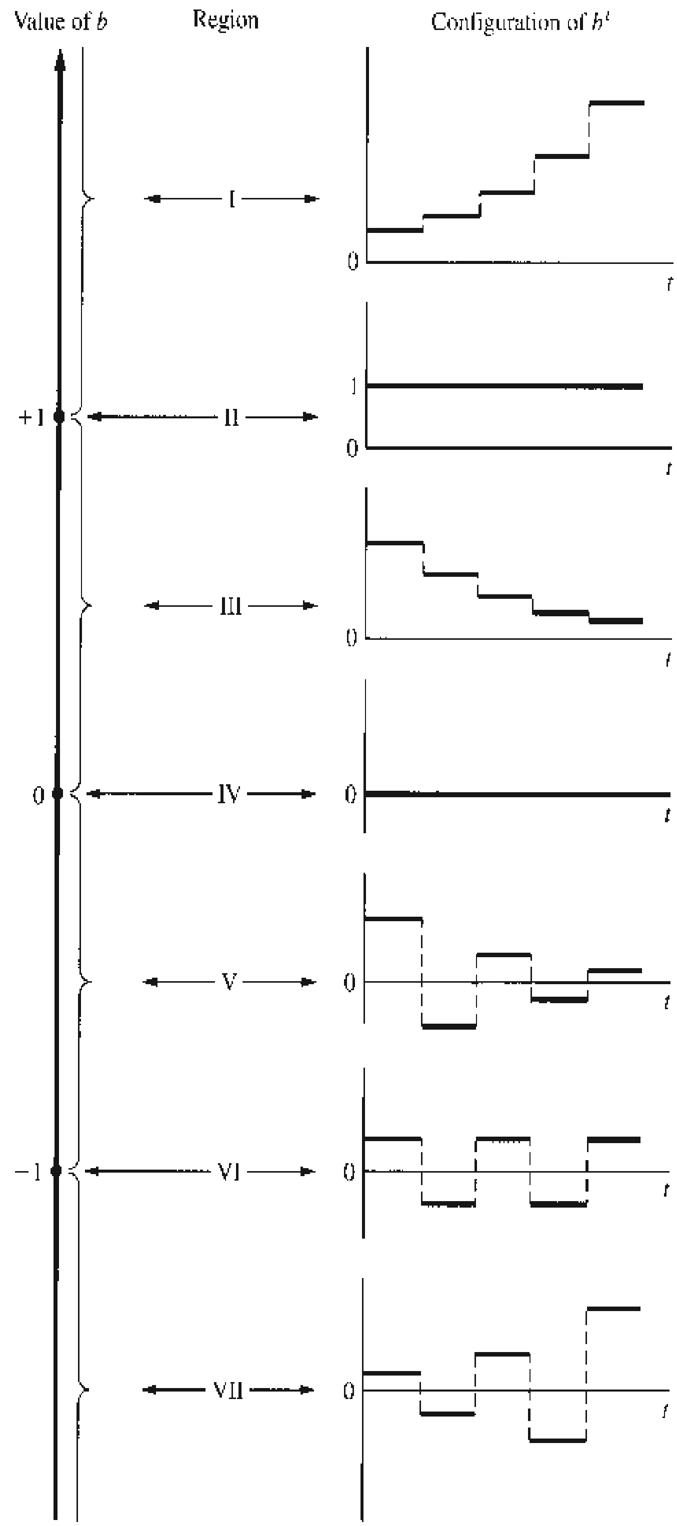
The essence of the preceding discussion can be conveyed in the following general statement: The time path of b^t $(b \neq 0)$ will be

$$\left.\begin{matrix}\text{Nonoscillatory} \\ \text{Oscillatory}\end{matrix}\right\} \quad \text{if} \quad \left\{\begin{matrix} b > 0 \\ b < 0 \end{matrix}\right.$$

$$\left.\begin{matrix}\text{Divergent} \\ \text{Convergent}\end{matrix}\right\} \quad \text{if} \quad \left\{\begin{matrix} |b| > 1 \\ |b| < 1 \end{matrix}\right.$$

It is important to note that, whereas the convergence of the expression e^{rt} depends on the *sign* of r, the convergence of the b^t expression hinges, instead, on the *absolute value* of b.

FIGURE 17.1



The Role of A

So far we have deliberately left out the multiplicative constant A. But its effects—of which there are two—are relatively easy to take into account. First, the *magnitude* of A can serve to "blow up" (if, say, $A = 3$) or "pare down" (if, say, $A = \frac{1}{5}$) the values of b^t. That is, it can produce a *scale effect* without changing the basic configuration of the time path. The *sign* of A, on the other hand, does materially affect the shape of the path because, if b^t is multiplied by $A = -1$, then each time path shown in Fig. 17.1 will be replaced by its own mirror image with reference to the horizontal axis. Thus, a negative A can produce a *mirror effect* as well as a scale effect.

Convergence to Equilibrium

The preceding discussion presents the interpretation of the Ab^t term in the complementary function, which, as we recall, represents the deviations from some intertemporal equilibrium level. If a term (say) $y_p = 5$ is added to the Ab^t term, the time path must be shifted up vertically by a constant value of 5. This will in no way affect the convergence or divergence of the time path, but it will alter the level with reference to which convergence or divergence in gauged. What Fig. 17.1 pictures is the convergence (or lack of it) of the Ab^t expression to zero. When the y_p is included, it becomes a question of the convergence of the time path $y_t = y_c + y_p$ to the equilibrium level y_p.

In this connection, let us add a word of explanation for the special case of $b = 1$ (region II). A time path such as

$$y_t = A(1)^t + y_p = A + y_p$$

gives the impression that it converges, because the multiplicative term $(1)^t = 1$ produces no explosive effect. Observe, however, that y_t will now take the value $(A + y_p)$ rather than the equilibrium value y_p; in fact, it can never reach y_p (unless $A = 0$). As an illustration of this type of situation, we can cite the time path in (17.9), in which a moving equilibrium $y_p = ct$ is involved. This time path is to be considered divergent, not because of the appearance of t in the particular solution but because, with a nonzero A, there will be a constant deviation from the moving equilibrium. Thus, in stipulating the condition for convergence of time path y_t to the equilibrium y_p, we must rule out the case of $b = 1$.

In sum, the solution

$$y_t = Ab^t + y_p$$

is a convergent path if and only if $|b| < 1$.

Example 1 What kind of time path is represented by $y_t = 2(-\frac{4}{5})^t + 9$? Since $b = -\frac{4}{5} < 0$, the time path is oscillatory. But since $|b| = \frac{4}{5} < 1$, the oscillation is damped, and the time path converges to the equilibrium level of 9.

You should exercise care not to confuse $2(-\frac{4}{5})^t$ with $-2(\frac{4}{5})^t$; they represent entirely different time-path configurations.

Example 2 How do you characterize the time path $y_t = 3(2)^t + 4$? Since $b = 2 > 0$, no oscillation will occur. But since $|b| = 2 > 1$, the time path will diverge from the equilibrium level of 4.

EXERCISE 17.3

1. Discuss the nature of the following time paths:

   (a) $y_t = 3^t + 1$ (c) $y_t = 5\left(-\frac{1}{10}\right)^t + 3$

   (b) $y_t = 2\left(\frac{1}{3}\right)^t$ (d) $y_t = -3\left(\frac{1}{4}\right)^t + 2$

2. What is the nature of the time path obtained from each of the difference equations in Exercise 17.2-4?
3. Find the solutions of the following, and determine whether the time paths are oscillatory and convergent:
   (a) $y_{t+1} - \frac{1}{3}y_t = 6 \qquad (y_0 = 1)$
   (b) $y_{t+1} + 2y_t = 9 \qquad (y_0 = 4)$
   (c) $y_{t+1} + \frac{1}{4}y_t = 5 \qquad (y_0 = 2)$
   (d) $y_{t+1} - y_t = 3 \qquad (y_0 = 5)$

17.4 The Cobweb Model

To illustrate the use of first-order difference equations in economic analysis, we shall cite two variants of the market model for a single commodity. The first variant, known as the *cobweb model,* differs from our earlier market models in that it treats Q_s as a function not of the current price but of the price of the preceding time period.

The Model

Consider a situation in which the producer's output decision must be made one period in advance of the actual sale—such as in agricultural production, where planting must precede by an appreciable length of time the harvesting and sale of the output. Let us assume that the output decision in period t is based on the then-prevailing price P_t. Since this output will not be available for the sale until period $(t + 1)$, however, P_t will determine not Q_{st}, but $Q_{s,t+1}$. Thus we now have a "lagged" supply function.[†]

$$Q_{s,t+1} = S(P_t)$$

or, equivalently, by shifting back the time subscripts by one period,

$$Q_{st} = S(P_{t-1})$$

When such a supply function interacts with a demand function of the form

$$Q_{dt} = D(P_t)$$

interesting dynamic price patterns will result.

Taking the linear versions of these (lagged) supply and (unlagged) demand functions, and assuming that in each time period the market price is always set at a level which clears the market, we have a market model with the following three equations:

$$\begin{aligned} Q_{dt} &= Q_{st} \\ Q_{dt} &= \alpha - \beta P_t \qquad (\alpha, \beta > 0) \\ Q_{st} &= -\gamma + \delta P_{t-1} \qquad (\gamma, \delta > 0) \end{aligned} \tag{17.10}$$

[†] We are making the implicit assumption here that the entire output of a period will be placed on the market, with no part of it held in storage. Such an assumption is appropriate when the commodity in question is perishable or when no inventory is ever kept. A model with inventory will be considered in Sec. 17.5.