

Sources of Error

Two sources of numerical error

- 1) Round off error
- 2) Truncation error

Round-off Error

Computational Physics

Round off Error

- Caused by representing a number approximately

$$\frac{1}{3} \cong 0.333333$$

$$\sqrt{2} \cong 1.4142\dots$$

Problems created by round off error

- 28 Americans were killed on February 25, 1991 by an Iraqi Scud missile in Dhahran, Saudi Arabia.
- The patriot defense system failed to track and intercept the Scud. Why?

Problem with Patriot missile



- Clock cycle of 1/10 seconds was represented in 24-bit fixed point register created an error of 9.5×10^{-8} seconds.
- The battery was on for 100 consecutive hours, thus causing an inaccuracy of

$$\begin{aligned} &= 9.5 \times 10^{-8} \frac{\text{s}}{0.1\text{s}} \times 100\text{hr} \times \frac{3600\text{s}}{1\text{hr}} \\ &= 0.342\text{s} \end{aligned}$$

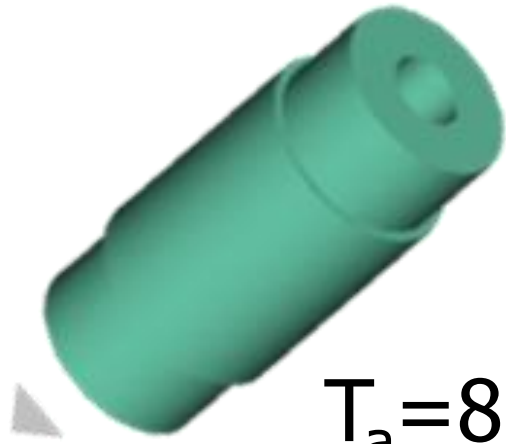
Problem (cont.)

- The shift calculated in the ranging system of the missile was 687 meters.
- The target was considered to be out of range at a distance greater than 137 meters.

Effect of Carrying Significant Digits in Calculations

Computational Physics

Find the contraction in the diameter

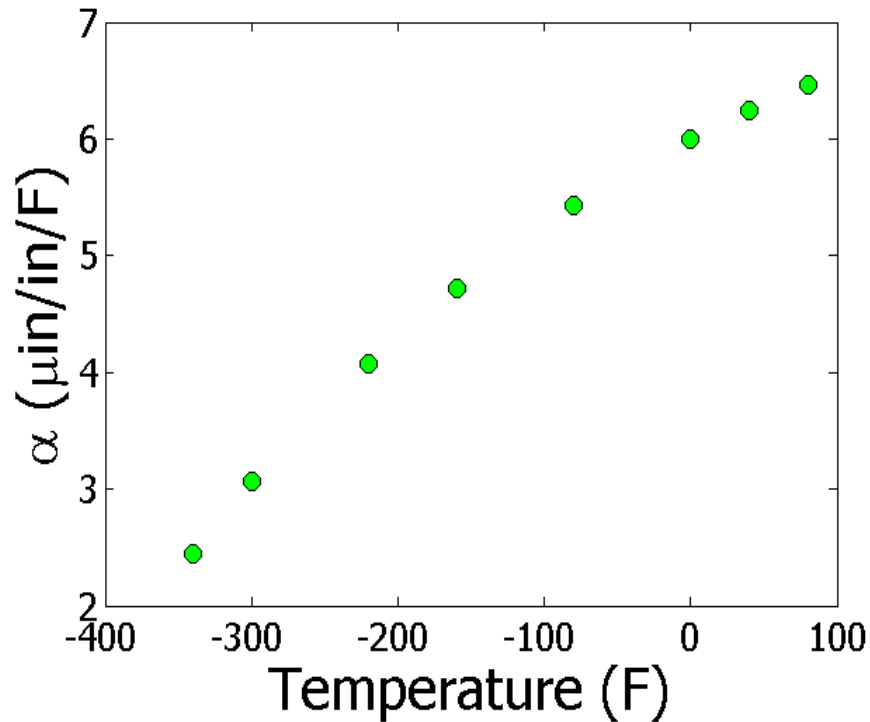


$$\Delta D = D \int_{T_a}^{T_c} \alpha(T) dT$$

$$T_a = 80^\circ\text{F}; T_c = -108^\circ\text{F}; D = 12.363''$$

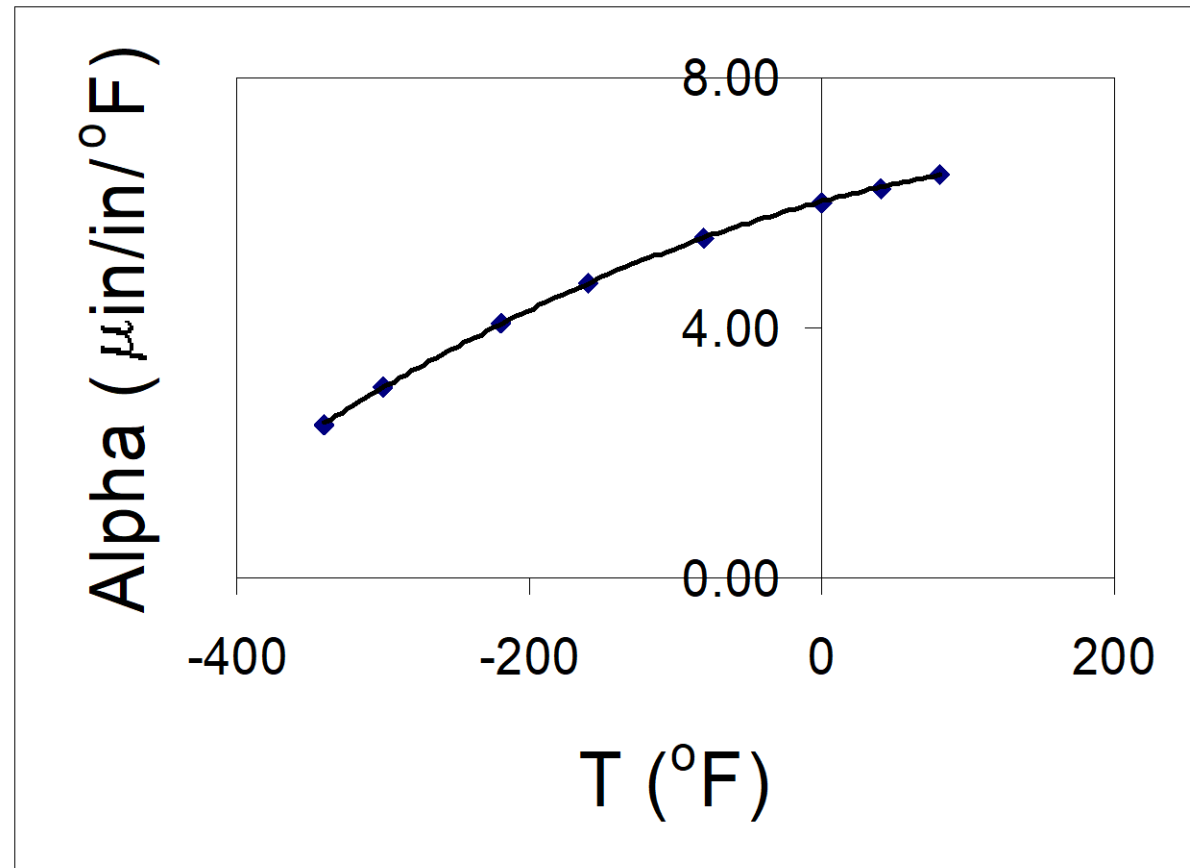
$$\alpha = a_0 + a_1 T + a_2 T^2$$

Thermal Expansion Coefficient vs Temperature



T($^{\circ}$ F)	α ($\mu\text{in/in}/^{\circ}$ F)
-340	2.45
-300	3.07
-220	4.08
-160	4.72
-80	5.43
0	6.00
40	6.24
80	6.47

Regressing Data in Excel (general format)



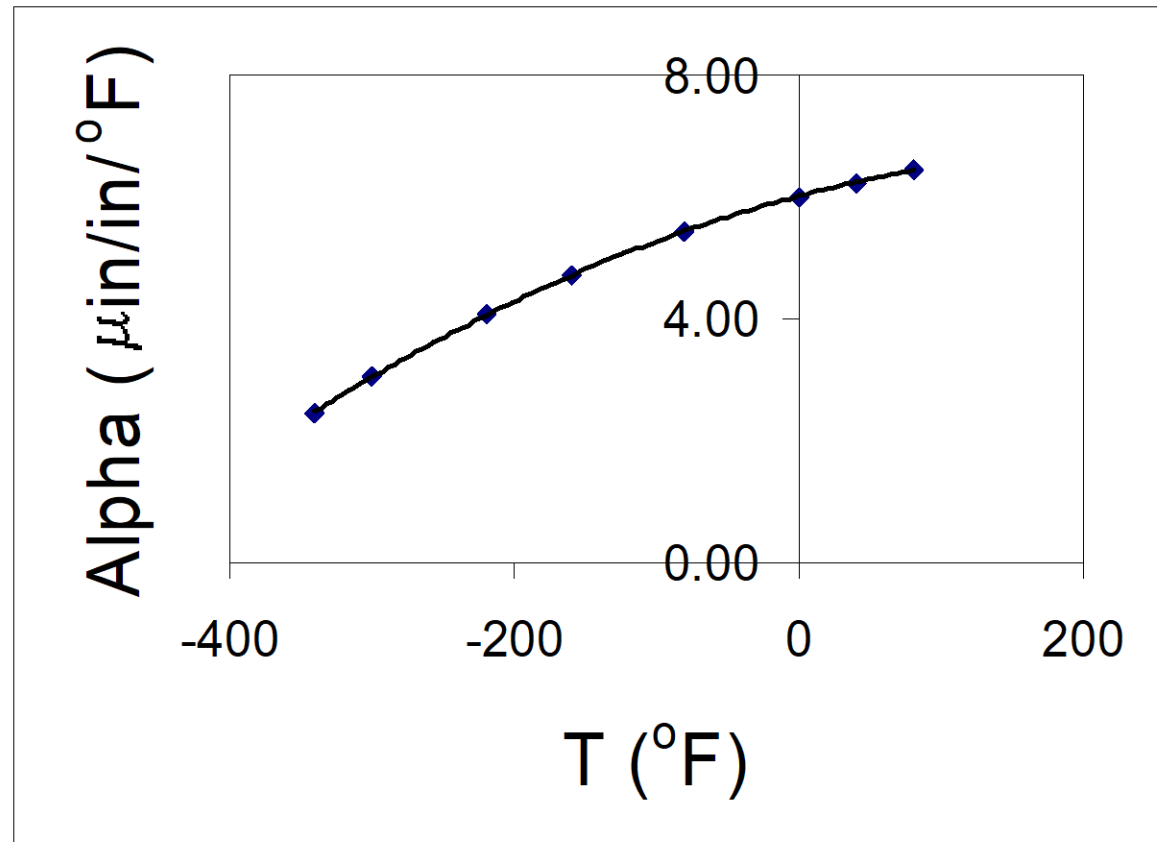
$$\alpha = -1\text{E-}05T^2 + 0.0062T + 6.0234$$

Observed and Predicted Values

$$\alpha = -1\text{E-}05T^2 + 0.0062T + 6.0234$$

T(°F)	α ($\mu\text{in/in/}^\circ\text{F}$) Given	α ($\mu\text{in/in/}^\circ\text{F}$) Predicted
-340	2.45	2.76
-300	3.07	3.26
-220	4.08	4.18
-160	4.72	4.78
-80	5.43	5.46
0	6.00	6.02
40	6.24	6.26
80	6.47	6.46

Regressing Data in Excel (scientific format)



$$\alpha = -1.2360\text{E-}05T^2 + 6.2714\text{E-}03T + 6.0234$$

Observed and Predicted Values

$$\alpha = -1.2360\text{E-}05T^2 + 6.2714\text{E-}03T + 6.0234$$

T(°F)	α ($\mu\text{in/in/}^\circ\text{F}$) Given	α ($\mu\text{in/in/}^\circ\text{F}$) Predicted
-340	2.45	2.46
-300	3.07	3.03
-220	4.08	4.05
-160	4.72	4.70
-80	5.43	5.44
0	6.00	6.02
40	6.24	6.25
80	6.47	6.45

Observed and Predicted Values

$$\alpha = -1.2360\text{E-}05T^2 + 6.2714\text{E-}03T + 6.0234$$

$$\alpha = -1\text{E-}05T^2 + 0.0062T + 6.0234$$

T(°F)	α ($\mu\text{in/in/}^\circ\text{F}$) Given	α ($\mu\text{in/in/}^\circ\text{F}$) Predicted	α ($\mu\text{in/in/}^\circ\text{F}$) Predicted
-340	2.45	2.46	2.76
-300	3.07	3.03	3.26
-220	4.08	4.05	4.18
-160	4.72	4.70	4.78
-80	5.43	5.44	5.46
0	6.00	6.02	6.02
40	6.24	6.25	6.26
80	6.47	6.45	6.46

THE END

Truncation Error

Computational Physics

Truncation error

- Error caused by truncating or approximating a mathematical procedure.

Example of Truncation Error

Taking only a few terms of a Maclaurin series to approximate e^x

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

If only 3 terms are used,

$$\text{Truncation Error} = e^x - \left(1 + x + \frac{x^2}{2!} \right)$$

Another Example of Truncation Error

Using a finite Δx to approximate $f'(x)$

$$f'(x) \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

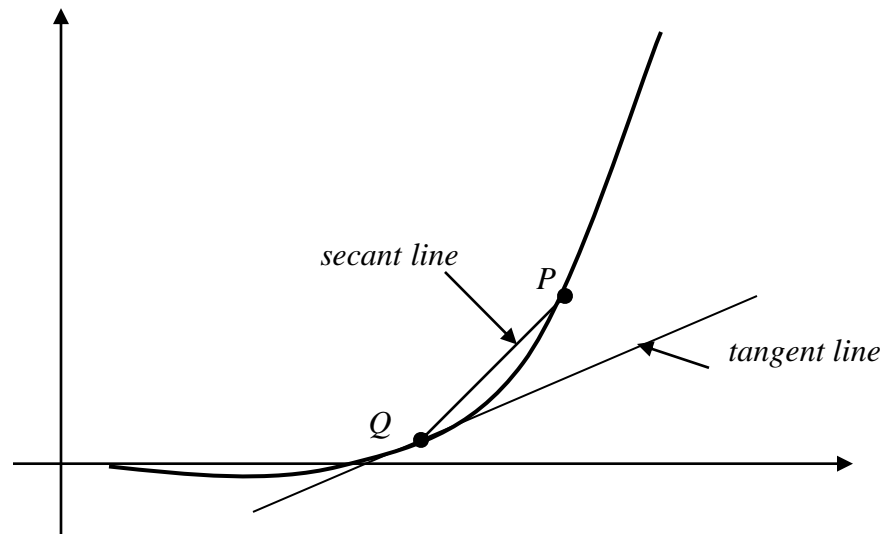
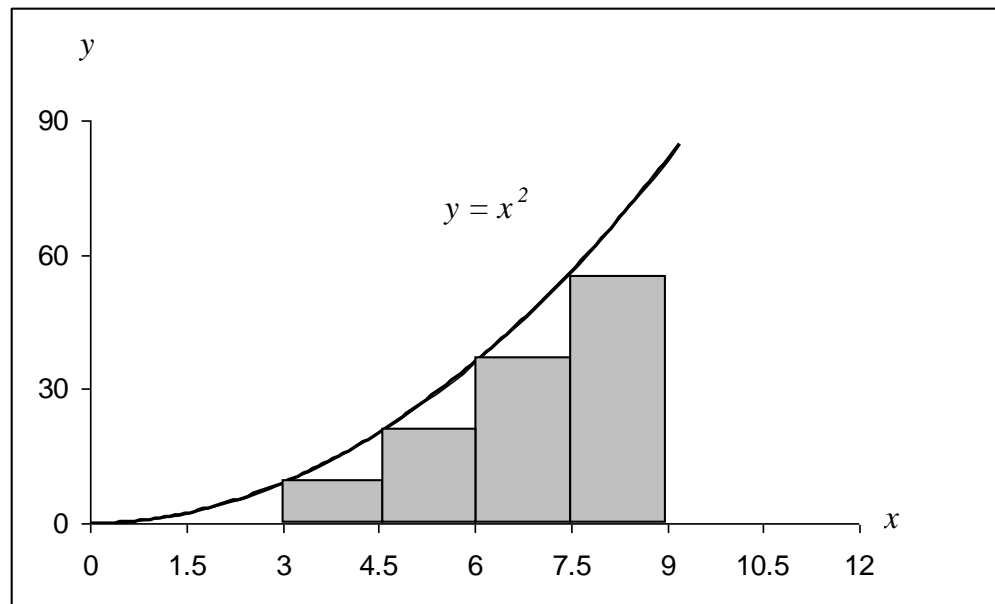


Figure 1. Approximate derivative using finite Δx

Another Example of Truncation Error

Using finite rectangles to approximate an integral.



Example 1 —Maclaurin series

Calculate the value of $e^{1.2}$ with an absolute relative approximate error of less than 1%.

$$e^{1.2} = 1 + 1.2 + \frac{1.2^2}{2!} + \frac{1.2^3}{3!} + \dots$$

n	$e^{1.2}$	E_a	$ \epsilon_a \%$
1	1	—	—
2	2.2	1.2	54.545
3	2.92	0.72	24.658
4	3.208	0.288	8.9776
5	3.2944	0.0864	2.6226
6	3.3151	0.020736	0.62550

6 terms are required. How many are required to get at least 1 significant digit correct in your answer?

Example 2 —Differentiation

Find $f'(3)$ for $f(x) = x^2$ using $f'(x) \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}$
and $\Delta x = 0.2$

$$\begin{aligned} f'(3) &= \frac{f(3 + 0.2) - f(3)}{0.2} \\ &= \frac{f(3.2) - f(3)}{0.2} = \frac{3.2^2 - 3^2}{0.2} = \frac{10.24 - 9}{0.2} = \frac{1.24}{0.2} = 6.2 \end{aligned}$$

The actual value is

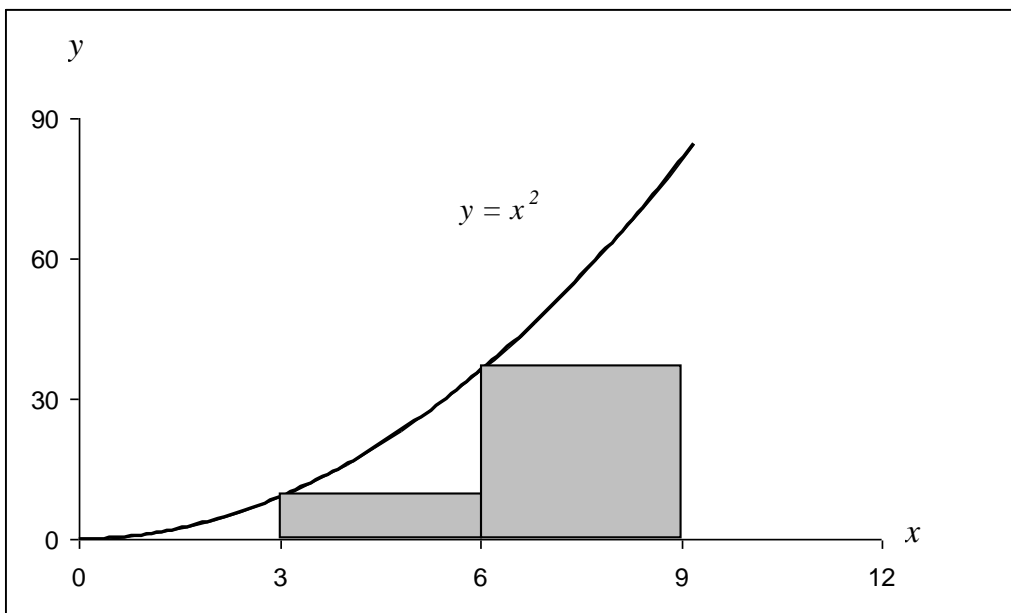
$$f'(x) = 2x, \quad f'(3) = 2 \times 3 = 6$$

Truncation error is then, $6 - 6.2 = -0.2$

Can you find the truncation error with $\Delta x = 0.1$ ²³

Example 3 — Integration

Use two rectangles of equal width to approximate the area under the curve for $f(x) = x^2$ over the interval $[3,9]$



$$\int_3^9 x^2 dx$$

Integration example (cont.)

Choosing a width of 3, we have

$$\begin{aligned}\int_3^9 x^2 dx &= (x^2)\Big|_{x=3}^{6-3} + (x^2)\Big|_{x=6}^{9-6} \\ &= (3^2)3 + (6^2)3 \\ &= 27 + 108 = 135\end{aligned}$$

Actual value is given by

$$\int_3^9 x^2 dx = \left[\frac{x^3}{3} \right]_3^9 = \left[\frac{9^3 - 3^3}{3} \right] = 234$$

Truncation error is then

$$234 - 135 = 99$$

Can you find the truncation error with 4 rectangles?

THE END

Computational Physics