The one-Quarter Fraction of the 2^k Design

For a moderately large number of factors, smaller fraction of the 2^k design are frequently useful. Consider a one-quarter fraction of the 2^k design:

one-quarter fraction $\Rightarrow \frac{2^k}{4} = \frac{2^k}{2^2} = 2^k \cdot 2^{-2} = 2^{k-2}$

This design contains 2^{k-2} runs and its usually called a 2^{k-2} fractional factorial.

The 2^{k-2} design may be constructed by first writing down a basic design consisting of the runs associated with a full factorial in $k-2$ factors and then associating the two additional columns with appropriately chosen interactions involving the first $k-2$ factors. Thus, a one-quarter fraction of the 2^k design has two generators.

If P and Q represent the generators chosen, then $I = P$ and $I = Q$ are called the generating relations for this design. The sign of P and Q (either $+$ or $-$) determine which one of the one-quarter fractions is produced. All four fractions associated with the choice of generators $\pm P$ and $\pm Q$ are members of the same family.

1. $I = +P$ and $I = +Q$
2. $I = +P$ and $I = -Q$
3. $I = -P$ and $I = +Q$
4. $I = -P$ and $I = -Q$

The fraction for which both P and Q are positive is the Principal fraction

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The complete defining relation for the design consist of P, Q and their generalized interaction PQ; that is, the defining relation is:

$$I = P = Q = PQ$$

The aliases of any effect are produced by the multiplication of the column for that effect by each word in the defining relation. Clearly, each effect has three aliases. The experimenter should be careful in choosing the generators so that potentially important effects are not aliased with each other.

As an example, consider the one-quarter fraction of the 2^4 design. Suppose we choose $I = ABCD$ and $I = BCD$ as the design generators.

Treatment combination of full factorial design.

$$2^4 = 16$$

One-quarter fraction of the 2^4 design.

$$\frac{2^4}{4} = \frac{2^4}{2^2} = 2^4 \cdot 2^{-2} = 2^{4-2} = 2^2$$

Now the treatment combination: $2^2 = 4$

Generators $I = ABCD$ and $I = BCD$

We know complete defining relation consist P, Q and their generalized interaction PQ.

So generalized interaction

$$ABCD \times BCD = AB^2C^2D^2 = A$$

Hence complete defining relation is.

$$I = ABCD = BCD = A$$

⇒ Selecting the treatment combination we this complete defining relation. All these three words are positive so select that treatment combination which have plus sign in the effect ABCD, BCD and A rows in sign table.

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By using the sign table, the selected treatment combinations are:

ab, ac, ad, abcd.

$\Rightarrow$ Effects:

$\boxed{2^{k-2}-1} \Rightarrow 2^{4-2}-1 = 2^2-1 = 4-1 = \boxed{3}$

:OR:

Effect for full factorial design $= 2^4 - 1 = 15$

two effects are generators $= 15 - 2 = 13$

One is a generalized interation $= 13 - 1 = 12$ (B, AB, C, AC, BC, ABC, D, AD, BD, CD, ACD, ABCD)

one-quarter effect $= \frac{12}{4} = \boxed{3}$

Aliases:

B x (ABCD, BCD, A) = ACD, CD, AB
C x (ABCD, BCD, A) = ABD, BD, AC
D x (ABCD, BCD, A) = ABC, BC, AD

Thus when we estimate B, we are really estimating B + ACD + CD + AB.

So

[B] $\rightarrow$ B + ACD + CD + AB
[C] $\rightarrow$ C + ABD + BD + AC
[D] $\rightarrow$ D + ABC + BC + AD

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Consider the one-quarter fraction of the 2^5 design. Suppose we chose $I = -ABCDE$ and $I = BCD$ as the design generators.

⇒ Treatment combination:

$$2^5 = 32$$

One-quarter fraction:

$$2^{5-2} = 2^3 = 8$$

Generators $I = -ABCDE$ & $I = BCD$

generalized interaction

$$-ABCDE \times BCD = -AE$$

So complete defining relation.

$$I = -ABCDE = BCD = -AE$$

Use sign table, select those treatment combinations which have minus sign in ABCDE & AE rows and plus sign in BCD row.

ab, ac, ad, abcd, be, ce, de, bcde

⇒ Effect

$$2^{5-2} - 1 = 2^3 - 1 = 7$$

Aliases

$$A \times (-ABCDE, BCD, -AE) = -BCDE, ABCD, -E$$
$$B \times (-ABCDE, BCD, -AE) = -ACDE, CD, -ABE$$
$$C \times (-ABCDE, BCD, -AE) = -ABDE, BD, -ACE$$
$$D \times (-ABCDE, BCD, -AE) = -ABCE, BC, -ADE$$
$$E \times (-ABCDE, BCD, -AE) = -ABCD, BCDE, -A$$
$$AB \times (-ABCDE, BCD, -AE) = -CDE, CD, -BE$$
$$AC \times (-ABCDE, BCD, -AE) = -BDE, ABD, -CE$$

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[A] → A − BCDE + ABCD − E
[B] → B − ACDE + CD − ABE
[C] → C − ABDE + BD − ACE
[D] → D − ABCE + BC − ADE
[E] → E − ABCD + BCDE − A
[AB] → AB − CDE + CD − BE
[AC] → AC − BDE − ABD − CE

Numerical:

An example was run in a semiconductor fabrication plant in an affort to increase yield. Five factors, each at two levels, were studied. The factors (and levels) were A = aperture setting (small, large), B = exposure time (20% below nominal and 20% above nominal), C = development time (30 and 45 s), D = mask dimension (small, large), and E = etch time (14.5 and 15.5 min). The unreplicated 2^5 design shown below was run.

(1) = 7	d = 8	e = 8	de = 6
a = 9	ad = 10	ae = 12	ade = 10
b = 34	bd = 32	be = 35	bde = 30
ab = 55	abd = 50	abe = 52	abde = 53
c = 16	cd = 18	ce = 15	cde = 15
ac = 20	acd = 21	ace = 22	acde = 20
bc = 40	bcd = 44	bce = 45	bcde = 41
abc = 60	abcd = 61	abce = 65	abcde = 63

Suppose that only eight runs could be made in this process. Set up an appropriate (2^{5-2}) design. Analyse the results. (use I = ABD and I = ACE).

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⇒ Treatment Combinations:

$$2^{5-2} = 2^3 = 8$$

Generators: $I = ABD$ & $I = ACE$

generalized interaction $= ABD \times ACE = BCDE$

So defining relation:

$$I = ABD = ACE = BCDE$$

The treatment combination by using sign table:

a, bc, abd, cd, be, ace, de, abcde :

⇒ effect:

$$2^{5-2} - 1 = 8 - 1 = 7$$

$A \times (ABD, ACE, BCDE) = BD, CE, BCDE$

$B \times (ABD, ACE, BCDE) = AD, ABCE, CDE$

$C \times (ABD, ACE, BCDE) = ABCD, AE, BDE$

$D \times (ABD, ACE, BCDE) = AB, ACDE, BCE$

$E \times (ABD, ACE, BCDE) = ABDE, AC, BCD$

$AB \times (ABD, ACE, BCDE) = D, BCE, ACDE$

$AC \times (ABD, ACE, BCDE) = BCD, E, BDE$

Its a resolution III design, so we can write it as 2^{5-2}_{III}

A 2^{5-2}_{III} design:

Runs	A	B	C	D = AB	E = AC	Treatment combination	Yeild
	Basic design(2^3)						
1	+	−	−	−	−	a	9
2	−	+	+	−	−	bc	40
3	+	+	−	+	−	abd	50
4	−	−	+	+	−	cd	18
5	−	+	−	−	+	be	35
6	+	−	+	−	+	ace	22
7	−	−	−	+	+	de	6
8	+	+	+	+	+	abcde	63
							243

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effect of A = $+9-40+50-18-35+22-6+63$
$= 45$
effect of B = $-9+40+50-18+35-22-6+63$
$= 133$
effect of C = $-9+40-50+18-35+22-6+63$
$= 43$
effect of D = $-9-40+50+18-35-22+6+63$
$= 31$
effect of E = $-9-40-50-18+35+22+6+63$
$= 9$
effect of AB = $-9-40+50+18-35-22+6+63$
$= 31$
effect of AC = $-9-40-50-18+35+22+6+63$
$= 9$

estimated effect of A = $\dfrac{\text{effect of } A}{2^{5-2}/2\,(8)}$

$= \dfrac{45}{2^3/2\,(1)} = \dfrac{45}{4} = 11.25$

$\Downarrow$

$A + BD + CE + BCDE$

estimated effect of B = $\dfrac{133}{4}$

$= 33.25 \rightarrow B + AD + ABCE + CDE$

estimated effect of C = $\dfrac{43}{4}$

$= 10.75 \rightarrow C + ABCD + AE + BDE$

estimated effect of D = $\dfrac{31}{4}$

$= 7.75 \rightarrow D + AB + ACDE + BCE$

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estimated effect of $E = \frac{9}{4}$

$= 2.25 \rightarrow E + ABDE + AC + BCD$

estimated effect of $AB = \frac{31}{4}$

$= 7.75 \rightarrow ABD + BCE + ACDE$

estimated effect of $AC = \frac{9}{4}$

$= 2.25 \rightarrow AC + BCD + E + BDE$

$$SSA = \frac{(\text{effect of } A)^2}{2^{5-2}(r)} = \frac{(45)^2}{2^3(1)} = \frac{(45)^2}{8} = 253.125$$

$$SSB = \frac{(133)^2}{8} = 2211.125$$

$$SSC = \frac{(43)^2}{8} = 231.125$$

$$SSD = \frac{(31)^2}{8} = 120.125$$

$$SSE = \frac{(9)^2}{8} = 10.125$$

$$SSAB = \frac{(31)^2}{8} = 120.125$$

$$SSAC = \frac{(9)^2}{8} = 10.125$$

$$\text{percentage Contribution} = \frac{\text{SS of factor}}{\text{Total SS}} \times 100$$

$$TSS = \sum y^2 - \frac{(T..)^2}{2^{5-2}(r)}$$

$$= (9)^2 + (40)^2 + \cdots (63)^2 - \frac{(243)^2}{8(1)}$$

$$= 10219 - 7381.125 = 2837.88$$

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$$PC \text{ of } A = \frac{253.125}{2837.88} \times 100 = 8.92\%$$

$$PC \text{ of } B = \frac{2211.125}{2837.88} \times 100 = 77.91\%$$

$$PC \text{ of } C = \frac{231.125}{2837.88} \times 100 = 8.14\%$$

$$PC \text{ of } D = \frac{120.125}{2837.88} \times 100 = 4.23\%$$

$$PC \text{ of } E = \frac{10.125}{2837.88} \times 100 = 0.36\%$$

$$PC \text{ of } AB = \frac{120.125}{2837.88} \times 100 = 4.23\%$$

$$PC \text{ of } AC = \frac{10.125}{2837.88} \times 100 = 0.36\%$$

Model Term	estimated effects	Percentage Contribution
A	11.25	8.92%
B	33.25	77.91%
C	10.75	8.14%
D	7.75	4.23%
E	2.25	0.36%
AB	7.75	4.23%
AC	2.25	0.36%

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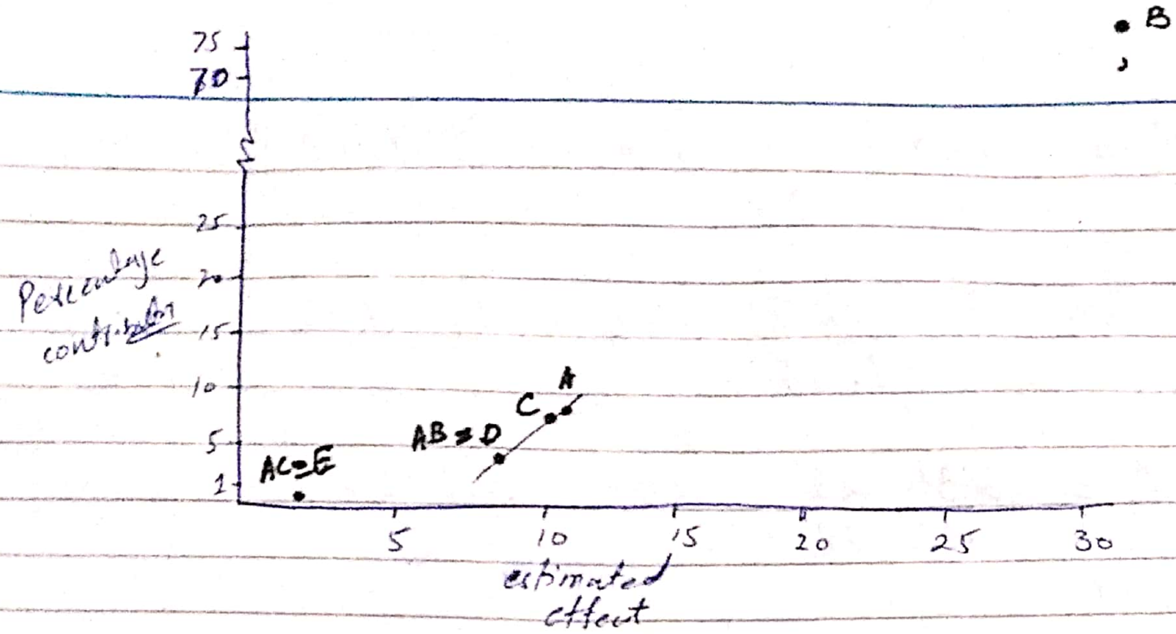

So the effect of A, C, D and AB are neglect and combine these as an estimate of error.
Thus the model contain the effect of B, AC and E

① (i) H_0: The main effect of exposure time (B) is insignificant.
H_1: " " " " significant

(ii) H_0: The main effect of etch time (E) is insignificant
H_1: " " " " significant.

(iii) H_0: There is no interaction between aperture setting (A) and development time (C).
H_1: There is interaction " " " "

② $\alpha = 0.05$

③ (i) $F_1 = \frac{MSB}{MSE_r}$ (ii) $F_2 = \frac{MSE}{MSE_r}$ (iii) $F_3 = \frac{MSAC}{MSE_r}$

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④

SoV	d.f.	SS	MS	F
B	2-1=1	2211.125	2211.125	14.58 > 7.71
E	2-1=1	10.125	10.125	0.07 < 7.71
AC	2-1=1	10.125	10.125	0.07 < 7.71
Error	4	606.51	151.63	
Total	$2^{5-2}(1)-1=7$	2837.88		

⑤

$$F_{\alpha(v_1, v_2)} = F_{0.05(1, 4)} = 7.71$$

If $F_{cal} \geq 7.71$, then reject H_0 otherwise don't reject H_0.

⑥

As $F_1 > 7.71$ so we reject H_0 and concluded that the effect of exposure time (B) is significant.

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