Two-Level Fractional Factorial Designs

As the number of factors in a 2^k factorial design increases, the number of runs required for a complete replicate of the design rapidly outgrows the resourses of most experimenters. Then we use fractional factorial Designs.

Fractional factorial Designs:

If the experimenter can reasonably assume that high-order interactions are negligible, information on the main effects and low-order interactions may be obtained by running only a fraction of the complete factorial experiment.

Uses:

These fractional factorial designs are among the most widely used types of designs for product and process design and for process improvement.

A major use of fractional factorials is in screening experiments (experiments in which many factors are considered and the objective is to identify those factors that have large effect).

The successful use of fractional factorial designs is based on three Key ideas:

1- The sparsity of effects principle:

When there are several variables, the system or process is likely to be driven primarily by some of the main effects and low-order interactions.

2. The projection property:
Fractional factorial designs can be projected into stronger (larger) designs in the subset of significant factors.

3- Sequential experimentation:
It is possible to combine the run of two (or more) fractional factorials to assemble sequentially a larger design to estimate the factor effects and interactions of interest.

The One-Half Fraction of the 2^k Design:

Consider a situation in which three factors, each at two levels, are of interest but the experimenter cannot affort to all run ($2^3 = 8$ treatment combination). They can, however, afford for runs - This suggests a one-half fraction of a 2^3 design. Because the design contains

$$\frac{2^3}{2} = 2^3 \times 2^{-1} = 2^{3-1} = 2^2 = 4 \text{ (treatment combination)}$$

a one-half fraction of the 2^3 design is often called a 2^{3-1} design.

⇒ In general one-half fraction of the 2^k design is called a 2^{k-1} design.

Now how to select 4 treatment combinations out of 8 treatment combinations?

* For selecting half treatment combinations we use high order interaction term of the factorial design. This is called generator of this particular fraction. For the above example the generator will be ABC. We also call it as

$$I = ABC$$

the defining relation.

* Then we use sign table. If $I = ABC$ then we select the treatment combination that have plus signs in ABC column. And if $I = -ABC$

then we select the treatment combinations that have minus signs in ABC column.
Suppose $I = ABC$ then

	1	a	b	ab	c	ac	bc	abc
A	−	+	−	+	−	+	−	+
B	−	−	+	+	−	−	+	+
AB	+	−	−	+	+	−	−	+
C	−	−	−	−	+	+	+	+
AC	+	−	+	−	−	+	−	+
BC	+	+	−	−	−	−	+	+
ABC	−	+	+	−	+	−	−	+

∴ The four treatment combinations a, b, c, and abc are selected as our one-half fraction.
If $I = -ABC$ then 1, ab, ac, bc are selected as our one-half fraction.

→ Main or estimated effects:
For 2^3 factorial design → $2^3 - 1 = 7$ (A, B, AB, C, AC, BC, ABC)
One effect is a generator → $7 - 1 = 6$ (A, B, AB, C, AC, BC)
half effect → $\frac{6}{2} = 3$
Now which 3 factors have to estimate?
So if $I = ABC$ (a, b, c, abc treatment combinations)
then
effect of $A = +a - b - c + abc$

$$[A] = \text{estimated effect of } A = \frac{1}{\frac{2^{3-1}}{2}(r)}\,[+a - b - c + abc]$$

effect of $B = -a + b - c + abc$

$$[B] = \text{estimated effect of } B = \frac{1}{\frac{2^{3-1}}{2}(r)}\,[-a + b - c + abc]$$

• effect of C $= -a - b + c + abc$

$[C]$ = estimated effect of C $= \frac{1}{2^{3-1}/2\,(r)}\,[-a-b+c+abc]$

where the notation $[A], [B], [C]$ is used to indicate the linear combinations.
Now the linear combinations of two-factor interactions:

effect of BC $= a - b - c + abc$

$[BC]$ = estimated effect of BC $= \frac{1}{2^{3-1}/2\,(r)}\,[a-b-c+abc]$

effect of AC $= -a + b - c + abc$

$[AC]$ = estimated effect of AC $= \frac{1}{2^{3-1}/2\,(r)}\,[-a+b-c+abc]$

effect of AB $= -a - b + c + abc$

$[AB]$ = estimated effect of AB $= \frac{1}{2^{3-1}/2\,(r)}\,[-a-b+c+ab]$

Thus $[A] = [BC]$, $[B] = [AC]$ and $[C] = [AB]$.
In fact, when we estimate A, B, C we are really estimating $A + BC$, $B + AC$ and $C + AB$. Two or more effects that have this property are called aliases.

⇒ Method for finding aliases.

The alias structure for this design may be easily determined by using the definning relation $I = ABC$. Multiplying any effect by the defining relation yields the aliases for that effect. For the above example, the alias of A yields as:

$$A \cdot I = A \cdot ABC = A^2BC = BC$$

because the square of A column sign will Positive (is just the identity I).

Similarly

$$B \cdot I = B \cdot ABC = AB^2C = AC$$

$$C \cdot I = C \cdot ABC = ABC^2 = AB$$

So you can use either A, B, C or BC, AC, AB. Mostly we prefer that alias which contain main effects like A, B, C.

This one-half fraction, with I = ABC is usally called the principle fraction.

Suppose we had chosen the other one-half fraction that is the treatment combinations associated with minus in the ABC column. then $I = -ABC$ (consisting of the run 1, ab, ac, bc).

$$[A]' = -1 + ab + ac - bc$$
$$[B]' = -1 + ab - ac + bc$$
$$[C]' = -1 - ab + ac + bc$$
$$[AB]' = +1 + ab - ac - bc$$
$$[AC]' = +1 - ab + ac - bc$$
$$[BC]' = +1 - ab - ac + bc$$

Thus when we estimate A, B and C we are really estimating A−BC, B−AC and C−AB.

This one-half fraction, with $I = -ABC$ is called alternate or complementary.

We indicate this by the notation $[A]' \rightarrow A - BC$, $[B]' \rightarrow B - AC$ and $[C]' \rightarrow C - AB$.

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The one-half fraction of the 2^4 design.

$2^4 = 16$

one-half fraction

$\frac{2^4}{2} = 2^4 . 2^{-1} = 2^{4-1} = 2^3 = 8$

$\Rightarrow$ The defining relation $I = ABCD$

So we select the treatment combinations, that have plus sign in ABCD row.

(1), ab, ac, bc, ad, bd, cd, abcd.

$\Rightarrow$ Effect

$2^4 - 1 = 15$

one is a generator so $\Rightarrow 15 - 1 = 14$

one-half effect $\Rightarrow \frac{14}{2} = 7$

Aliases

$A . ABCD = A^2BCD = BCD$

$B . ABCD = AB^2CD = ACD$

$C . ABCD = ABC^2D = ABD$

$D . ABCD = ABCD^2 = ABC$

$AB . ABCD = A^2B^2CD = CD$

$AC . ABCD = A^2BC^2D = BD$

$AD . ABCD = A^2BCD^2 = BC$

$[A] \rightarrow A + BCD$

$[B] \rightarrow B + ACD$

$[C] \rightarrow C + ABD$

$[D] \rightarrow D + ABC$

$[AB] \rightarrow AB + CD$

$[AC] \rightarrow AC + BD$

$[AD] \rightarrow AD + BC$

The one-half fraction for 2^5 design.

$2^5 = 32$

one-half

$2^{5-1} = 2^4 = 16$

$\Rightarrow$ let's take ABCDE as a genator.

$I = ABCDE$

So we use sign table and select that treatmet combinations which have positive sign in ABCDE row:

$a, b, c, abc, d, abd, acd, bcd, e, abe, ace, bce, ade, bde, cde, abcde.$

$\Rightarrow$ effect

$2^5 - 1 = 31$

one is generator $\Rightarrow 31 - 1 = 30$

one-half effects $\Rightarrow 30/2 = 15$

Aliases

$A \cdot ABCDE = A^2BCDE = BCDE$	$[A] \rightarrow A + BCDE$
$B \cdot ABCDE = AB^2CDE = ACDE$	$[B] \rightarrow B + ACDE$
$C \cdot ABCDE = ABC^2DE = ABDE$	$[C] \rightarrow C + ABDE$
$D \cdot ABCDE = ABCD^2E = ABCE$	$[D] \rightarrow D + ABCE$
$E \cdot ABCDE = ABCDE^2 = ABCD$	$[E] \rightarrow E + ABCD$
$AB \cdot ABCDE = A^2B^2CDE = CDE$	$[AB] \rightarrow AB + CDE$
$AC \cdot ABCDE = A^2BC^2DE = BDE$	$[AC] \rightarrow AC + BDE$
$AD \cdot ABCDE = A^2BCD^2E = BCE$	$[AD] \rightarrow AD + BCE$
$AE \cdot ABCDE = A^2BCDE^2 = BCD$	$[AE] \rightarrow AE + BCD$
$BC \cdot ABCDE = AB^2C^2DE = ADE$	$[BC] \rightarrow BC + ADE$
$BD \cdot ABCDE = AB^2CD^2E = ACE$	$[BD] \rightarrow BD + ACE$
$BE \cdot ABCDE = AB^2CDE^2 = ACD$	$[BE] \rightarrow BE + ACD$
$CD \cdot ABCDE = ABC^2D^2E = ABE$	$[CD] \rightarrow CD + ABE$
$CE \cdot ABCDE = ABC^2DE^2 = ABD$	$[CE] \rightarrow CE + ABD$
$DE \cdot ABCDE = ABCD^2E^2 = ABC$	$[DE] \rightarrow DE + ABC$

Design Resolution:

A design is of resolution (R) if no P-factor effect is aliased with another effect containing less than (R − P)-factors. We usually employ a Roman numeral subscript to denote design resolution.

Designs of resolution III, IV and V are particularly important. The definitions of these designs and an example of each follow:

1- Resolution III designs:

These are designs in which no main effects are aliased with any other main effect, but main effects are aliased with two-factor interactions and some two-factor interactions may be aliased with each other. A 2^{3-1} design with $I = ABC$ (or $I = -ABC$) is a resolution III design (2^{3-1}_{III}) because the main effects are aliased with two-factor interactions.

2. Resolution IV designs:

These are designs in which no main effect is alised with any other main effect or with any two-factor interaction, but two-factor interactions are alised with each other. A 2^{4-1} design with $I = ABCD$ (or $I = -ABCD$) is a resolution IV design (2^{4-1}_{IV})

3- Resolution V designs:

These are designs in which no main effect or two factor interaction is alised with any other main effect or two-factor interaction, but two-factor interactions are aliased with three-factor interactions. A 2^{5-1} design with $I = ABCDE$ (or $I = -ABCDE$) is a resolution V design (2^{5-1}_{V}).

Example of one-half fractional factorial Design.

A 2^3 factorial design was used to develop a nitride etch process on a single-wafer plasma etching tool. The design factors are the gap between the electrodes (A), the gas flow (B) and RF power applied to the cathode (C). Each factor is run at two levels and the design is replicated twice. The response variable is the etch rate for silicon nitride. The etch rate data are shown in the following table.

Treatment combination	Replicate 1	Replicate 2
1	550	604
a	669	650
b	633	601
ab	642	635
c	1037	1052
ac	749	868
bc	1075	1063
abc	729	860

Suppose that only a one-half fraction of the design could be run. Set up the design and analyze the data.

Solution:

one-half fraction of the 2^3 design $= 2^{3-1} = 2^2$

Now possible treatment combinations $= 2^2 = 4$

The defining relation

$$I = ABC$$

So the four treatment combinations a, b, c and abc are selected as our one half fraction because these treatment combinations have positive sign in ABC effect in sign table.

effect of one-half fraction of 2^3 design.

$$2^{3-1} - 1 = 2^2 - 1 = 4 - 1 = 3$$

Aliases:

$A \cdot ABC = BC$ — So $[A] \rightarrow A + BC$

$B \cdot ABC = AC$ — $[B] \rightarrow B + AC$

$C \cdot ABC = AB$ — $[C] \rightarrow C + AB$

The 2^{3-1}_{III} Design with defining relation $I = ABC$.

Treatment Combination	Etch Rate Replicate 1	Replicate 2	Total	Basic Design 2^2 A	B	$C = AB$
a	669	650	1319	+	−	−
b	633	601	1234	−	+	−
c	1037	1052	2089	−	−	+
abc	729	860	1589	+	+	+
			6231			

$$\text{effect of } A = +1319 - 1234 - 2089 + 1589 = -415$$

$$\text{estimated effect of } A = [A] = \frac{\text{effect of } A}{\frac{2^{k-1}}{2}(r)}$$

$$= \frac{-415}{\frac{2^{3-1}}{2}(2)} = \frac{-415}{2(2)}$$

$$= \frac{-415}{4} = -103.75 \rightarrow A + BC$$

$$\text{effect of } B = -1319 + 1234 - 2089 + 1589 = 2053$$

$$\text{estimated effect of } B = [B] = \frac{2053}{\frac{2^{3-1}}{2}(2)} = \frac{2053}{4}$$

$$= 513.25 \rightarrow B + AC$$

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effect of $C = -1319 - 1234 + 2089 + 1589$
$= 3763$

estimated effect of $C = [C] = \frac{3763}{2^{3-1}/2\,(2)} = \frac{3763}{4}$

$= 940.75 \rightarrow C + AB$

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