behavioral equations can be used to describe the general institutional setting of a model, including the technological (e.g., production function) and legal (e.g., tax structure) aspects. Before a behavioral equation can be written, however, it is always necessary to adopt definite assumptions regarding the behavior pattern of the variable in question. Consider the two cost functions

$$C = 75 + 10Q \tag{2.1}$$

$$C = 110 + Q^2 \tag{2.2}$$

where Q denotes the quantity of output. Since the two equations have different forms, the production condition assumed in each is obviously different from the other. In (2.1), the fixed cost (the value of C when $Q = 0$) is 75, whereas in (2.2) it is 110. The variation in cost is also different. In (2.1), for each unit increase in Q, there is a constant increase of 10 in C. But in (2.2), as Q increases unit after unit, C will increase by progressively larger amounts. Clearly, it is primarily through the specification of the form of the behavioral equations that we give mathematical expression to the assumptions adopted for a model.

As the third type, a *conditional equation* states a requirement to be satisfied. For example, in a model involving the notion of equilibrium, we must set up an *equilibrium condition,* which describes the prerequisite for the attainment of equilibrium. Two of the most familiar equilibrium conditions in economics are

$$Q_d = Q_s \qquad \text{[quantity demanded = quantity supplied]}$$

and

$$S = I \qquad \text{[intended saving = intended investment]}$$

which pertain, respectively, to the equilibrium of a market model and the equilibrium of the national-income model in its simplest form. Similarly, an optimization model either derives or applies one or more *optimization conditions.* One such condition that comes easily to mind is the condition

$$\text{MC} = \text{MR} \qquad \text{[marginal cost = marginal revenue]}$$

in the theory of the firm. Because equations of this type are neither definitional nor behavioral, they constitute a class by themselves.

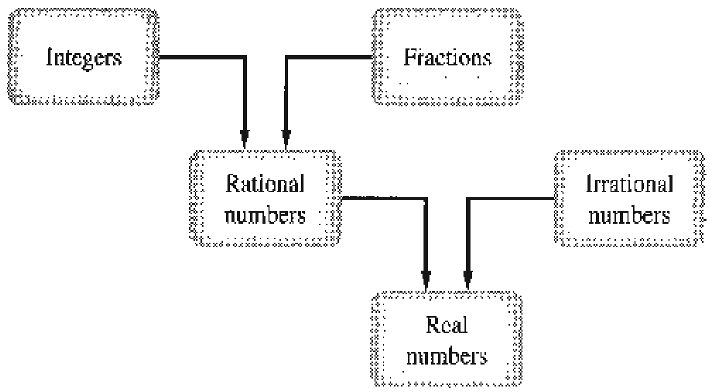
2.2 The Real-Number System

Equations and variables are the essential ingredients of a mathematical model. But since the values that an economic variable takes are usually numerical, a few words should be said about the number system. Here, we shall deal only with so-called real numbers.

Whole numbers such as 1, 2, 3, . . . are called *positive integers;* these are the numbers most frequently used in counting. Their negative counterparts −1, −2, −3, . . . are called *negative integers;* these can be employed, for example, to indicate subzero temperatures (in degrees). The number 0 (zero), on the other hand, is neither positive nor negative, and is in that sense unique. Let us lump all the positive and negative integers and the number zero into a single category, referring to them collectively as the *set of all integers.*

Integers, of course, do not exhaust all the possible numbers, for we have *fractions,* such as $\frac{2}{3}$, $\frac{5}{4}$, and $\frac{7}{3}$, which—if placed on a ruler—would fall between the integers. Also, we have negative fractions, such as $-\frac{1}{2}$ and $-\frac{2}{5}$. Together, these make up the *set of all fractions.*

FIGURE 2.1



The common property of all fractional numbers is that each is expressible as a ratio of two integers. Any number that can be expressed as a ratio of two integers is called a *rational number.* But integers themselves are also rational, because any integer n can be considered as the ratio $n/1$. The set of all integers and the set of all fractions together form the *set of all rational numbers.* An alternative defining characteristic of a rational number is that it is expressible as either a terminating decimal (e.g., $\frac{1}{4} = 0.25$) or a repeating decimal (e.g., $\frac{1}{3} = 0.3333\ldots$), where some number or series of numbers to the right of the decimal point is repeated indefinitely.

Once the notion of rational numbers is used, there naturally arises the concept of *irrational numbers*—numbers that *cannot* be expressed as ratios of a pair of integers. One example is the number $\sqrt{2} = 1.4142\ldots$, which is a nonrepeating, nonterminating decimal. Another is the special constant $\pi = 3.1415\ldots$ (representing the ratio of the circumference of any circle to its diameter), which is again a nonrepeating, nonterminating decimal, as is characteristic of all irrational numbers.

Each irrational number, if placed on a ruler, would fall between two rational numbers, so that, just as the fractions fill in the gaps between the integers on a ruler, the irrational numbers fill in the gaps between rational numbers. The result of this filling-in process is a continuum of numbers, all of which are so-called real numbers. This continuum constitutes the *set of all real numbers,* which is often denoted by the symbol R. When the set R is displayed on a straight line (an extended ruler), we refer to the line as the *real line.*

In Fig. 2.1 are listed (in the order discussed) all the number sets, arranged in relationship to one another. If we read from bottom to top, however, we find in effect a classificatory scheme in which the set of real numbers is broken down into its component and subcomponent number sets. This figure therefore is a summary of the structure of the real-number system.

Real numbers are all we need for the first 15 chapters of this book, but they are not the only numbers used in mathematics. In fact, the reason for the term *real* is that there are also "imaginary" numbers, which have to do with the square roots of negative numbers. That concept will be discussed later, in Chap. 16.

2.3 The Concept of Sets

We have already employed the word *set* several times. Inasmuch as the concept of sets underlies every branch of modern mathematics, it is desirable to familiarize ourselves at least with its more basic aspects.

Set Notation

A *set* is simply a collection of distinct objects. These objects may be a group of (distinct) numbers, persons, food items, or something else. Thus, all the students enrolled in a particular economics course can be considered a set, just as the three integers 2, 3, and 4 can form a set. The objects in a set are called the *elements* of the set.

There are two alternative ways of writing a set: by *enumeration* and by *description*. If we let S represent the set of three numbers 2, 3, and 4, we can write, by enumeration of the elements,

$$S = \{2, 3, 4\}$$

But if we let I denote the set of *all* positive integers, enumeration becomes difficult, and we may instead simply describe the elements and write

$$I = \{x \mid x \text{ a positive integer}\}$$

which is read as follows: "I is the set of all (numbers) x, such that x is a positive integer." Note that a pair of braces is used to enclose the set in either case. In the descriptive approach, a vertical bar (or a colon) is always inserted to separate the general designating symbol for the elements from the description of the elements. As another example, the set of all real numbers greater than 2 but less than 5 (call it J) can be expressed symbolically as

$$J = \{x \mid 2 < x < 5\}$$

Here, even the descriptive statement is symbolically expressed.

A set with a finite number of elements, exemplified by the previously given set S, is called a *finite set*. Set I and set J, each with an infinite number of elements, are, on the other hand, examples of an *infinite set*. Finite sets are always *denumerable* (or *countable*), i.e., their elements can be counted one by one in the sequence 1, 2, 3, ... Infinite sets may, however, be either denumerable (set I), or *nondenumerable* (set J). In the latter case, there is no way to associate the elements of the set with the natural counting numbers 1, 2, 3, . . . , and thus the set is not countable.

Membership in a set is indicated by the symbol $\in$ (a variant of the Greek letter epsilon ϵ for "element"), which is read as follows: "is an element of." Thus, for the two sets S and I defined previously, we may write

$$2 \in S \qquad 3 \in S \qquad 8 \in I \qquad 9 \in I \qquad \text{(etc.)}$$

but obviously $8 \notin S$ (read: "8 is not an element of set S"). If we use the symbol R to denote the set of all real numbers, then the statement "x is some real number" can be simply expressed by

$$x \in R$$

Relationships between Sets

When two sets are compared with each other, several possible kinds of relationship may be observed. If two sets S_1 and S_2 happen to contain identical elements,

$$S_1 = \{2, 7, a, f\} \qquad \text{and} \qquad S_2 = \{2, a, 7, f\}$$

then S_1 and S_2 are said to be *equal* ($S_1 = S_2$). Note that the order of appearance of the elements in a set is immaterial. Whenever we find even one element to be different in any two sets, however, those two sets are not equal.

Another kind of set relationship is that one set may be a *subset* of another set. If we have two sets

$$S = \{1, 3, 5, 7, 9\} \qquad \text{and} \qquad T = \{3, 7\}$$

then T is a subset of S, because every element of T is also an element of S. A more formal statement of this is: T is a subset of S if and only if $x \in T$ implies $x \in S$. Using the set inclusion symbols $\subset$ (is contained in) and $\supset$ (includes), we may then write

$$T \subset S \qquad \text{or} \qquad S \supset T$$

It is possible that two given sets happen to be subsets of each other. When this occurs, however, we can be sure that these two sets are equal. To state this formally: we can have $S_1 \subset S_2$ and $S_2 \subset S_1$ if and only if $S_1 = S_2$.

Note that, whereas the $\in$ symbol relates an individual *element* to a *set*, the $\subset$ symbol relates a *subset* to a *set*. As an application of this idea, we may state on the basis of Fig. 2.1 that the set of all integers is a subset of the set of all rational numbers. Similarly, the set of all rational numbers is a subset of the set of all real numbers.

How many subsets can be formed from the five elements in the set $S = \{1, 3, 5, 7, 9\}$? First of all, each individual element of S can count as a distinct subset of S, such as $\{1\}$ and $\{3\}$. But so can any pair, triple, or quadruple of these elements, such as $\{1, 3\}$, $\{1, 5\}$, and $\{3, 7, 9\}$. Any subset that does *not* contain *all* the elements of S is called a *proper subset* of S. But the set S itself (with all its five elements) can also be considered as one of its own subsets—every element of S is an element of S, and thus the set S itself fulfills the definition of a subset. This is, of course, a limiting case, that from which we get the largest possible subset of S, namely, S itself.

At the other extreme, the smallest possible subset of S is a set that contains no element at all. Such a set is called the *null set*, or *empty set*, denoted by the symbol $\varnothing$ or $\{\ \}$. The reason for considering the null set as a subset of S is quite interesting: If the null set is not a subset of S ($\varnothing \not\subset S$), then $\varnothing$ must contain at least one element x such that $x \notin S$. But since by definition the null set has no element whatsoever, we cannot say that $\varnothing \not\subset S$; hence the null set is a subset of S.

It is extremely important to distinguish the symbol $\varnothing$ or $\{\ \}$ clearly from the notation $\{0\}$; the former is devoid of elements, but the latter does contain an element, zero. The null set is unique; there is only one such set in the whole world, and it is considered a subset of *any* set that can be conceived.

Counting all the subsets of S, including the two limiting cases S and $\varnothing$, we find a total of $2^5 = 32$ subsets. In general, if a set has n elements, a total of 2^n subsets can be formed from those elements.†

† Given a set with n elements $\{a, b, c, \ldots, n\}$ we may first classify its subsets into two categories: one with the element a in it, and one without. Each of these two can be further classified into two subcategories: one with the element b in it, and one without. Note that by considering the second element b, we double the number of categories in the classification from 2 to 4 ($= 2^2$). By the same token, the consideration of the element c will increase the total number of categories to 8 ($= 2^3$). When all n elements are considered, the total number of categories will become the total number of subsets, and that number is 2^n.

As a third possible type of set relationship, two sets may have no elements in common at all. In that case, the two sets are said to be *disjoint*. For example, the set of all positive integers and the set of all negative integers are mutually exclusive; thus they are disjoint sets.

A fourth type of relationship occurs when two sets have some elements in common but some elements peculiar to each. In that event, the two sets are neither equal nor disjoint; also, neither set is a subset of the other.

Operations on Sets

When we add, subtract, multiply, divide, or take the square root of some numbers, we are performing mathematical operations. Although sets are different from numbers, one can similarly perform certain mathematical operations on them. Three principal operations to be discussed here involve the union, intersection, and complement of sets.

To take the *union* of two sets A and B means to form a new set containing those elements (and only those elements) belonging to A, or to B, or to both A and B. The union set is symbolized by $A \cup B$ (read: "A union B").

Example 1 If $A = \{3, 5, 7\}$ and $B = \{2, 3, 4, 8\}$, then

$$A \cup B = \{2, 3, 4, 5, 7, 8\}$$

This example, incidentally, illustrates the case in which two sets A and B are neither equal nor disjoint and in which neither is a subset of the other.

Example 2 Again referring to Fig. 2.1, we see that the union of the set of all integers and the set of all fractions is the set of all rational numbers. Similarly, the union of the rational-number set and the irrational-number set yields the set of all real numbers.

The *intersection* of two sets A and B, on the other hand, is a new set which contains those elements (and only those elements) belonging to *both A and B*. The intersection set is symbolized by $A \cap B$ (read: "A intersection B").

Example 3 From the sets A and B in Example 1, we can write

$$A \cap B = \{3\}$$

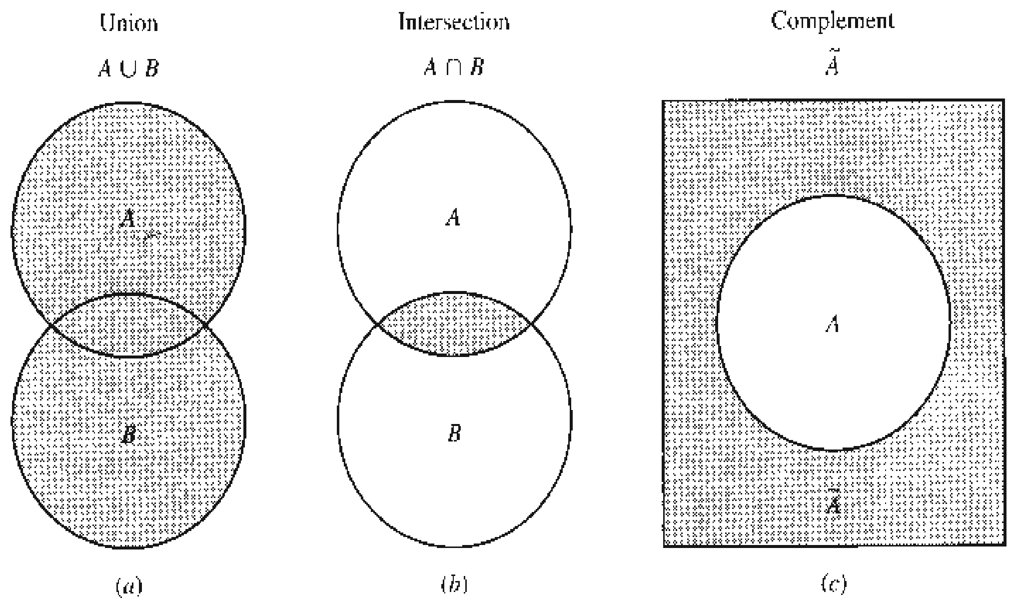
Example 4 If $A = \{-3, 6, 10\}$ and $B = \{9, 2, 7, 4\}$, then $A \cap B = \emptyset$. Set A and set B are disjoint; therefore their intersection is the empty set—no element is common to A and B.

It is obvious that intersection is a more restrictive concept than union. In the former, only the elements *common to A and B* are acceptable, whereas in the latter, membership in *either A or B* is sufficient to establish membership in the union set. The operator symbols $\cap$ and $\cup$—which, incidentally, have the same kind of general status as the symbols $\sqrt{\ }$, $+$, $\div$, etc.—therefore have the connotations "and" and "or," respectively. This point can be better appreciated by comparing the following formal definitions of intersection and union:

$$\text{Intersection:} \quad A \cap B = \{x \mid x \in A \quad \text{and} \quad x \in B\}$$

$$\text{Union:} \quad A \cup B = \{x \mid x \in A \quad \text{or} \quad x \in B\}$$

FIGURE 2.2



What about the *complement* of a set? To explain this, let us first introduce the concept of the *universal set*. In a particular context of discussion, if the only numbers used are the set of the first seven positive integers, we may refer to it as the universal set U. Then, with a given set, say, $A = \{3, 6, 7\}$, we can define another set $\tilde{A}$ (read: "the complement of A") as the set that contains all the numbers in the universal set U that are not in the set A. That is,

$$\tilde{A} = \{x \mid x \in U \quad \text{and} \quad x \notin A\} = \{1, 2, 4, 5\}$$

Note that, whereas the symbol $\cup$ has the connotation "or" and the symbol $\cap$ means "and," the complement symbol $\sim$ carries the implication of "not."

Example 5 If $U = \{5, 6, 7, 8, 9\}$ and $A = \{5, 6\}$, then $\tilde{A} = \{7, 8, 9\}$.

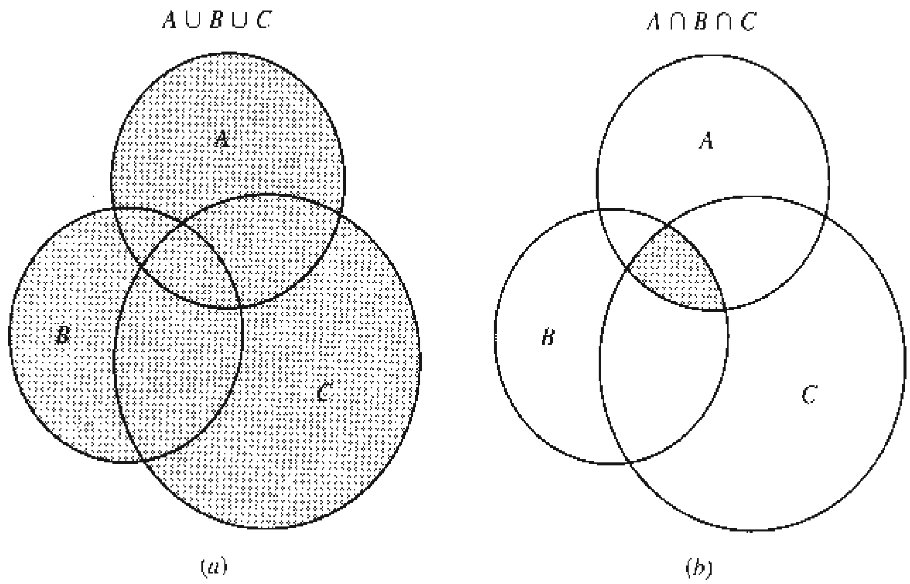
Example 6 What is the complement of U? Since every object (number) under consideration is included in the universal set, the complement of U must be empty. Thus $\tilde{U} = \varnothing$.

The three types of set operation can be visualized in the three diagrams of Fig. 2.2, known as *Venn diagrams*. In diagram *a*, the points in the upper circle form a set A, and the points in the lower circle form a set B. The union of A and B then consists of the shaded area covering both circles. In diagram *b* are shown the same two sets (circles). Since their intersection should comprise only the points common to both sets, only the (shaded) overlapping portion of the two circles satisfies the definition. In diagram *c*, let the points in the rectangle be the universal set and let A be the set of points in the circle; then the complement set $\tilde{A}$ will be the (shaded) area outside the circle.

Laws of Set Operations

From Fig. 2.2, it may be noted that the shaded area in diagram *a* represents not only $A \cup B$ but also $B \cup A$. Analogously, in diagram *b* the small shaded area is the visual

FIGURE 2.3



representation not only of $A \cap B$ but also of $B \cap A$. When formalized, this result is known as the *commutative law* (of unions and intersections):

$$A \cup B = B \cup A \qquad A \cap B = B \cap A$$

These relations are very similar to the algebraic laws $a + b = b + a$ and $a \times b = b \times a$.

To take the union of three sets A, B, and C, we first take the union of any two sets and then "union" the resulting set with the third; a similar procedure is applicable to the intersection operation. The results of such operations are illustrated in Fig. 2.3. It is interesting that the order in which the sets are selected for the operation is immaterial. This fact gives rise to the *associative law* (of unions and intersections):

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

These equations are strongly reminiscent of the algebraic laws $a + (b + c) = (a + b) + c$ and $a \times (b \times c) = (a \times b) \times c$.

There is also a law of operation that applies when unions and intersections are used in combination. This is the *distributive law* (of unions and intersections):

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

These resemble the algebraic law $a \times (b + c) = (a \times b) + (a \times c)$.

Example 7 Verify the distributive law, given $A = \{4, 5\}$, $B = \{3, 6, 7\}$, and $C = \{2, 3\}$. To verify the first part of the law, we find the left- and right-hand expressions separately:

Left: $A \cup (B \cap C) = \{4, 5\} \cup \{3\} = \{3, 4, 5\}$

Right: $(A \cup B) \cap (A \cup C) = \{3, 4, 5, 6, 7\} \cap \{2, 3, 4, 5\} = \{3, 4, 5\}$

Since the two sides yield the same result, the law is verified. Repeating the procedure for the second part of the law, we have

$$\text{Left:} \qquad A \cap (B \cup C) = \{4, 5\} \cap \{2, 3, 6, 7\} = \varnothing$$

$$\text{Right:} \qquad (A \cap B) \cup (A \cap C) = \varnothing \cup \varnothing = \varnothing$$

Thus the law is again verified.

To verify a law means to check by a specific example whether the law actually works out. If the law is valid, then any specific example ought indeed to work out. This implies that if the law does not check out in as many as one single example, then the law is invalidated. On the other hand, the successful verification by specific examples (however many) does not in itself prove the law. To *prove* a law, it is necessary to demonstrate that the law is valid for all possible cases. The procedure involved in such a demonstration will be illustrated later (see, e.g., Sec. 2.5).

EXERCISE 2.3

1. Write the following in set notation:
   (*a*) The set of all real numbers greater than 34.
   (*b*) The set of all real numbers greater than 8 but less than 65.
2. Given the sets $S_1 = \{2, 4, 6\}$, $S_2 = \{7, 2, 6\}$, $S_3 = \{4, 2, 6\}$, and $S_4 = \{2, 4\}$, which of the following statements are true?
   (*a*) $S_1 = S_3$
   (*b*) $S_1 = R$ (set of real numbers)
   (*c*) $8 \in S_2$
   (*d*) $3 \notin S_2$
   (*e*) $4 \notin S_3$
   (*f*) $S_4 \subset R$
   (*g*) $S_1 \supset S_4$
   (*h*) $\varnothing \subset S_2$
   (*i*) $S_3 \supset \{1, 2\}$
3. Referring to the four sets given in Prob. 2, find:
   (*a*) $S_1 \cup S_2$
   (*b*) $S_1 \cup S_3$
   (*c*) $S_2 \cap S_3$
   (*d*) $S_2 \cap S_4$
   (*e*) $S_4 \cap S_2 \cap S_1$
   (*f*) $S_3 \cup S_1 \cup S_4$
4. Which of the following statements are valid?
   (*a*) $A \cup A = A$
   (*b*) $A \cap A = A$
   (*c*) $A \cup \varnothing = A$
   (*d*) $A \cup U = U$
   (*e*) $A \cap \varnothing = \varnothing$
   (*f*) $A \cap U = A$
   (*g*) The complement of $\tilde{A}$ is A.
5. Given $A = \{4, 5, 6\}$, $B = \{3, 4, 6, 7\}$, and $C = \{2, 3, 6\}$, verify the distributive law.
6. Verify the distributive law by means of Venn diagrams, with different orders of successive shading.
7. Enumerate all the subsets of the set $\{5, 6, 7\}$.
8. Enumerate all the subsets of the set $S = \{a, b, c, d\}$. How many subsets are there altogether?
9. Example 6 shows that $\varnothing$ is the complement of U. But since the null set is a subset of *any* set, $\varnothing$ must be a subset of U. Inasmuch as the term "complement of U" implies the notion of being *not in* U, whereas the term "subset of U" implies the notion of being *in* U, it seems paradoxical for $\varnothing$ to be both of these. How do you resolve this paradox?

2.4 Relations and Functions

Our discussion of sets was prompted by the usage of that term in connection with the various kinds of numbers in our number system. However, sets can refer as well to objects other than numbers. In particular, we can speak of sets of "ordered pairs"—to be defined presently—which will lead us to the important concepts of relations and functions.

Ordered Pairs

In writing a set $\{a, b\}$, we do not care about the order in which the elements a and b appear, because by definition $\{a, b\} = \{b, a\}$. The pair of elements a and b is in this case an *unordered pair.* When the ordering of a and b does carry a significance, however, we can write two different *ordered pairs* denoted by (a, b) and (b, a), which have the property that $(a, b) \neq (b, a)$ unless $a = b$. Similar concepts apply to a set with more than two elements, in which case we can distinguish between ordered and unordered triples, quadruples, quintuples, and so forth. Ordered pairs, triples, etc., collectively can be called *ordered sets;* they are enclosed with parentheses rather than braces.

Example 1

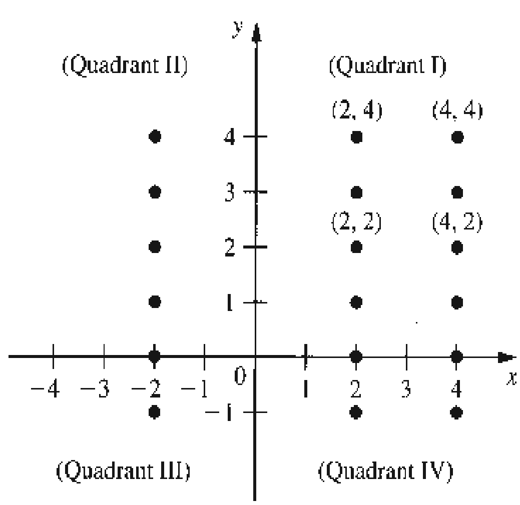
To show the age and weight of each student in a class, we can form ordered pairs (a, w), in which the first element indicates the age (in years) and the second element indicates the weight (in pounds). Then (19, 127) and (127, 19) would obviously mean different things. Moreover, the latter ordered pair would hardly fit any student anywhere.

Example 2

When we speak of the set of all contestants in an Olympic game, the order in which they are listed is of no consequence and we have an unordered set. But the set {gold-medalist, silver-medalist, bronze-medalist} is an ordered triple.

Ordered pairs, like other objects, can be elements of a set. Consider the rectangular (Cartesian) coordinate plane in Fig. 2.4, where an x axis and a y axis cross each other at a right angle, dividing the plane into four quadrants. This xy plane is an infinite set of points, each of which represents an ordered pair whose first element is an x value and the second element a y value. Clearly, the point labeled (4, 2) is different from the point (2, 4); thus ordering is significant here.

FIGURE 2.4



With this visual understanding, we are ready to consider the process of generation of ordered pairs. Suppose, from two given sets, $x = \{1, 2\}$ and $y = \{3, 4\}$, we wish to form all the possible ordered pairs with the first element taken from set x and the second element taken from set y. The result will, of course, be the set of four ordered pairs (1, 3), (1, 4), (2, 3), and (2, 4). This set is called the *Cartesian product* (named after Descartes), or *direct product*, of the sets x and y and is denoted by $x \times y$ (read: "x cross y"). It is important to remember that, while x and y are sets of numbers, the Cartesian product turns out to be a set of ordered pairs. By enumeration, or by description, we may express this Cartesian product alternatively as

$$x \times y = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$$

or

$$x \times y = \{(a, b) \mid a \in x \text{ and } b \in y\}$$

The latter expression may in fact be taken as the general definition of Cartesian product for any given sets x and y.

To broaden our horizon, now let both x and y include all the real numbers. Then the resulting Cartesian product

$$x \times y = \{(a, b) \mid a \in R \text{ and } b \in R\} \tag{2.3}$$

will represent the set of all ordered pairs with real-valued elements. Besides, each ordered pair corresponds to a *unique* point in the Cartesian coordinate plane of Fig. 2.4, and, conversely, each point in the coordinate plane also corresponds to a *unique* ordered pair in the set $x \times y$. In view of this double uniqueness, a *one-to-one correspondence* is said to exist between the set of ordered pairs in the Cartesian product (2.3) and the set of points in the rectangular coordinate plane. The rationale for the notation $x \times y$ is now easy to perceive; we may associate it with the crossing of the x axis and the y axis in Fig. 2.4. A simpler way of expressing the set $x \times y$ in (2.3) is to write it directly as $R \times R$; this is also commonly denoted by R^2.

Extending this idea, we may also define the Cartesian product of three sets x, y, and z as follows:

$$x \times y \times z = \{(a, b, c) \mid a \in x, b \in y, c \in z\}$$

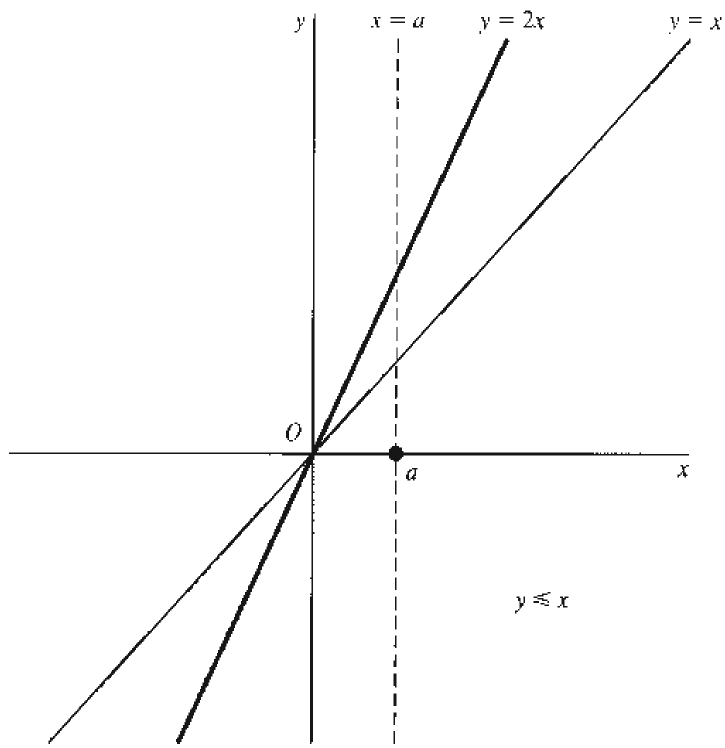
which is a set of ordered triples. Furthermore, if the sets x, y, and z each consist of all the real numbers, the Cartesian product will correspond to the set of all points in a three-dimensional space. This may be denoted by $R \times R \times R$, or more simply, R^3. In the present discussion, all the variables are taken to be real-valued; thus the framework will generally be R^2, or R^3, . . . , or R^n.

Relations and Functions

Since any ordered pair associates a y value with an x value, any collection of ordered pairs—any subset of the Cartesian product (2.3)—will constitute a *relation* between y and x. Given an x value, one or more y values will be specified by that relation. For convenience, we shall now write the elements of $x \times y$ generally as (x, y) rather than as (a, b), as was done in (2.3)—where both x and y are variables.

Example 3 The set $\{(x, y) \mid y = 2x\}$ is a set of ordered pairs including, for example, (1, 2), (0, 0), and (−1, −2). It constitutes a relation, and its graphical counterpart is the set of points lying on the straight line $y = 2x$, as seen in Fig. 2.5.

FIGURE 2.5



Example 4

The set $\{(x, y) \mid y \leq x\}$, which consists of such ordered pairs as (1, 0), (1, 1), and (1, −4), constitutes another relation. In Fig. 2.5, this set corresponds to the set of all points in the shaded area which satisfy the inequality $y \leq x$.

Observe that, when the x value is given, it may not always be possible to determine a *unique* y value from a relation. In Example 4, the three exemplary ordered pairs show that if $x = 1$, y can take various values, such as 0, 1, or −4, and yet in each case satisfy the stated relation. Graphically, two or more points of a relation may fall on a single vertical line in the xy plane. This is exemplified in Fig. 2.5, where many points in the shaded area (representing the relation $y \leq x$) fall on the broken vertical line labeled $x = a$.

As a special case, however, a relation may be such that for each x value there exists only *one* corresponding y value. The relation in Example 3 is a case in point. In such a case, y is said to be a *function* of x, and this is denoted by $y = f(x)$, which is read as "y equals f of x." [*Note:* $f(x)$ does *not* mean f times x.] A function is therefore a set of ordered pairs with the property that any x value *uniquely* determines a y value.† It should be clear that a function must be a relation, but a relation may not be a function.

Although the definition of a function stipulates a unique y for each x, the converse is not required. In other words, more than one x value may legitimately be associated with the same y value. This possibility is illustrated in Fig. 2.6, where the values x_1 and x_2 in the x set are both associated with the same value (y_0) in the y set by the function $y = f(x)$.

A function is also called a *mapping*, or *transformation;* both words connote the action of associating one thing with another. In the statement $y = f(x)$, the functional notation f

† This definition of *function* corresponds to what would be called a *single-valued function* in the older terminology. What was formerly called a *multivalued function* is now referred to as a *relation* or *correspondence*.