CHAPTER 6

CURRENT AND VOLTAGE RELATIONS ON A TRANSMISSION LINE

We have examined the parameters of a transmission line and are ready to consider the line as an element of a power system. Figure 6.1 shows a 500-kV line having bundled conductors. In overhead lines the conductors are suspended from the tower and insulated from it and from each other by insulators, the number of which is determined by the voltage of the line. Each insulator string in Fig. 6.1 has 22 insulators. The two shorter arms above the phase conductors support wires usually made of steel. These wires, much smaller in diameter than the phase conductors, are not visible in the picture, but they are electrically connected to the tower and are therefore at ground potential. These wires are referred to as *shield* or *ground wires* and shield the phase conductors from lightning strokes.

A very important problem in the design and operation of a power system is the maintenance of the voltage within specified limits at various points in the system. In this chapter we develop formulas by which we can calculate the voltage, current, and power at any point on a transmission line, provided we know these values at one point, usually at one end of the line.

The purpose of this chapter, however, is not merely to develop the pertinent equations, but also to provide an opportunity to understand the effects of the parameters of the line on bus voltages and the flow of power. In

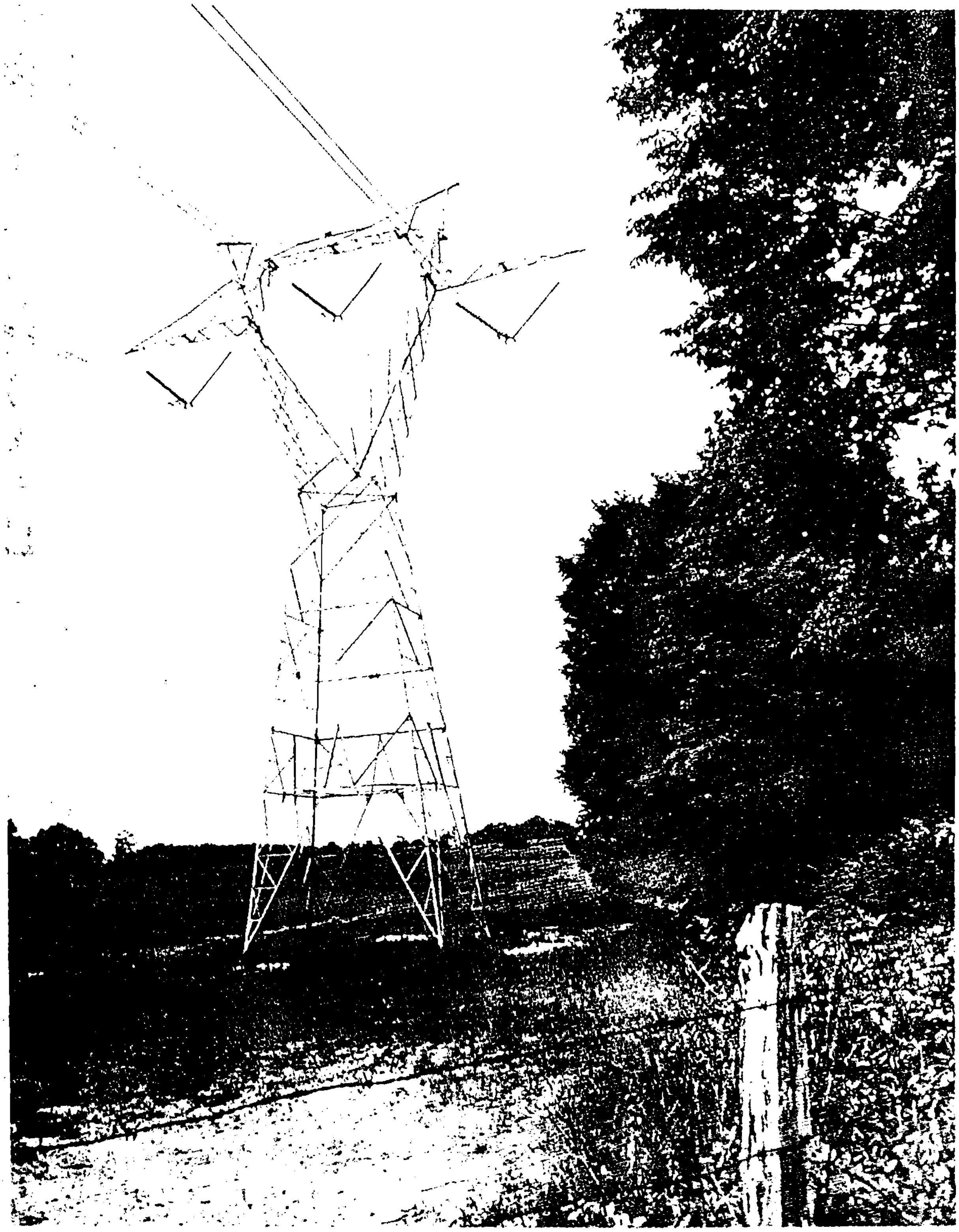


FIGURE 6.1
A 500-kV transmission line. Conductors are 76/19 ACSR with aluminum cross section of 2,515,000 cmil. Spacing between phases is 30 ft 3 in and the two conductors per bundle are 18 in apart. (*Courtesy Carolina Power and Light Company.*)

this way we can see the importance of the design of the line and better understand the developments to come in later chapters. This chapter also provides an introduction to the study of transients on lossless lines in order to indicate how problems arise due to surges caused by lightning and switching.

In the modern power system data from all over the system are being fed continuously into on-line computers for control and information purposes. Power-flow studies performed by a computer readily supply answers to questions concerning the effect of switching lines into and out of the system or of changes in line parameters. Equations derived in this chapter remain important, however, in developing an overall understanding of what is occurring on a system and in calculating efficiency of transmission, losses, and limits of power flow over a line for both steady-state and transient conditions.

6.1 REPRESENTATION OF LINES

The general equations relating voltage and current on a transmission line recognize the fact that all four of the parameters of a transmission line discussed in the two preceding chapters are uniformly distributed along the line. We derive these general equations later, but first we use lumped parameters which give good accuracy for short lines and for lines of medium length. If an overhead line is classified as short, shunt capacitance is so small that it can be omitted entirely with little loss of accuracy, and we need to consider only the series resistance R and the series inductance L for the total length of the line.

A medium-length line can be represented sufficiently well by R and L as lumped parameters, as shown in Fig. 6.2, with half the capacitance to neutral of the line lumped at each end of the equivalent circuit. Shunt conductance G, as mentioned previously, is usually neglected in overhead power transmission lines when calculating voltage and current. The same circuit represents the short line if capacitors are omitted.

Insofar as the handling of capacitance is concerned, open-wire 60-Hz lines less than about 80 km (50 mi) long are short lines. Medium-length lines are roughly between 80 km (50 mi) and 240 km (150 mi) long. Lines longer than 240 km (150 mi) require calculations in terms of distributed constants if a high degree of accuracy is required, although for some purposes a lumped-parameter representation can be used for lines up to 320 km (200 mi) long.

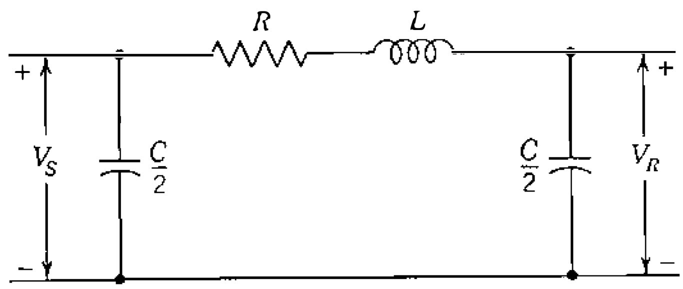


FIGURE 6.2
Single-phase equivalent of a medium-length line. The capacitors are omitted for a short line.

Normally, transmission lines are operated with balanced three-phase loads. Although the lines are not spaced equilaterally and not transposed, the resulting dissymmetry is slight and the phases are considered to be balanced.

In order to distinguish between the total series impedance of a line and the series impedance per unit length, the following nomenclature is adopted:

z = series impedance per unit length per phase

y = shunt admittance per unit length per phase to neutral

l = length of line

$Z = zl$ = total series impedance per phase

$Y = yl$ = total shunt admittance per phase to neutral

6.2 THE SHORT TRANSMISSION LINE

The equivalent circuit of a short transmission line is shown in Fig. 6.3, where I_S and I_R are the sending- and receiving-end currents, respectively, and V_S and V_R are the sending- and receiving-end line-to-neutral voltages.

The circuit is solved as a simple series ac circuit. So,

$$I_S = I_R \tag{6.1}$$

$$V_S = V_R + I_R Z \tag{6.2}$$

where Z is zl, the total series impedance of the line.

The effect of the variation of the power factor of the load on the voltage regulation of a line is most easily understood for the short line and therefore will be considered at this time. Voltage regulation of a transmission line is the rise in voltage at the receiving end, expressed in percent of full-load voltage, when full load at a specified power factor is removed while the sending-end

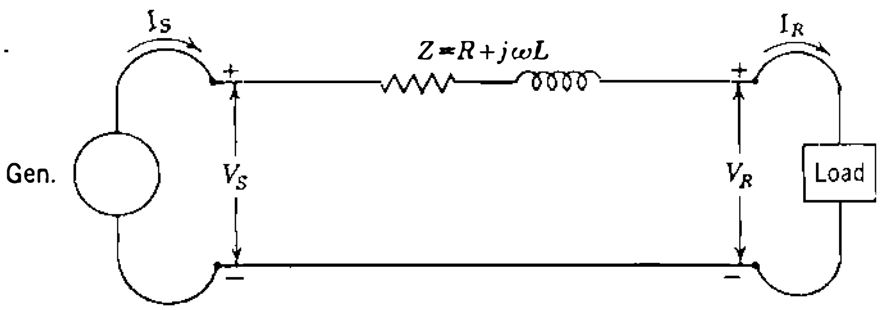
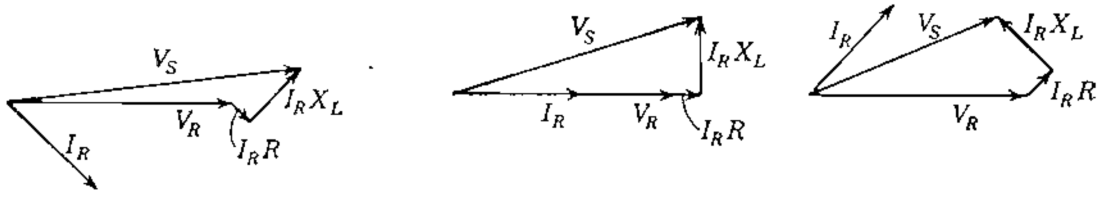


FIGURE 6.3
Equivalent circuit of a short transmission line where the resistance R and inductance L are values for the entire length of the line.



(*a*) Load p.f. =70% lag (*b*) Load p.f. =100% (*c*) Load p.f.=70% lead

FIGURE 6.4
Phasor diagrams of a short transmission line. All diagrams are drawn for the same magnitudes of V_R and I_R.

voltage is held constant. Corresponding to Eq. (2.33) we can write

$$\text{Percent regulation} = \frac{|V_{R,NL}| - |V_{R,FL}|}{|V_{R,FL}|} \times 100 \tag{6.3}$$

where $|V_{R,NL}|$ is the magnitude of receiving-end voltage at no load and $|V_{R,FL}|$ is the magnitude of receiving-end voltage at full load with $|V_S|$ constant. After the load on a short transmission line, represented by the circuit of Fig. 6.3, is removed, the voltage at the receiving end is equal to the voltage at the sending end. In Fig. 6.3, with the load connected, the receiving-end voltage is designated by V_R, and $|V_R| = |V_{R,FL}|$. The sending-end voltage is V_S, and $|V_S| = |V_{R,NL}|$. The phasor diagrams of Fig. 6.4 are drawn for the same magnitudes of the receiving-end voltage and current and show that a larger value of the sending-end voltage is required to maintain a given receiving-end voltage when the receiving-end current is lagging the voltage than when the same current and voltage are in phase. A still smaller sending-end voltage is required to maintain the given receiving-end voltage when the receiving-end current leads the voltage. The voltage drop is the same in the series impedance of the line in all cases; because of the different power factors, however, the voltage drop is added to the receiving-end voltage at a different angle in each case. The regulation is greatest for lagging power factors and least, or even negative, for leading power factors. The inductive reactance of a transmission line is larger than the resistance, and the principle of regulation illustrated in Fig. 6.4 is true for any load supplied by a predominantly inductive circuit. The magnitudes of the voltage drops $I_R R$ and $I_R X_L$ for a short line have been exaggerated with respect to V_R in drawing the phasor diagrams in order to illustrate the point more clearly. The relation between power factor and regulation for longer lines is similar to that for short lines but is not visualized so easily.

Example 6.1. A 300-MVA 20-kV three-phase generator has a subtransient reactance of 20%. The generator supplies a number of synchronous motors over a 64-km transmission line having transformers at both ends, as shown on the one-line diagram of Fig. 6.5. The motors, all rated 13.2 kV, are represented by just

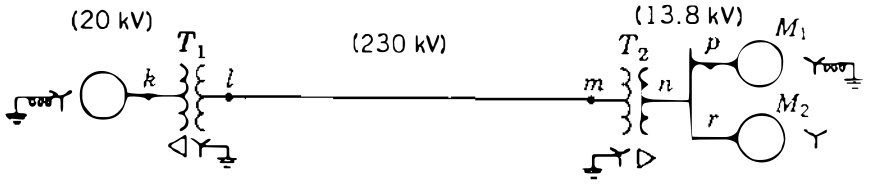


FIGURE 6.5
One-line diagram for Example 6.1.

two equivalent motors. The neutral of one motor M_1 is grounded through reactance. The neutral of the second motor M_2 is not connected to ground (an unusual condition). Rated inputs to the motors are 200 MVA and 100 kVA for M_1 and M_2, respectively. For both motors $X_d'' = 20\%$. The three-phase transformer T_1 is rated 350 MVA, 230/20 kV with leakage reactance of 10%. Transformer T_2 is composed of three single-phase transformers, each rated 127/13.2 kV, 100 MVA with leakage reactance of 10%. Series reactance of the transmission line is 0.5 Ω/km. Draw the reactance diagram with all reactances marked in per unit. Select the generator rating as base in the generator circuit.

Solution. The three-phase rating of transformer T_2 is

$$3 \times 100 = 300 \text{ kVA}$$

and its line-to-line voltage ratio is

$$\sqrt{3} \times \frac{127}{13.2} = \frac{220}{13.2} \text{ kV}$$

A base of 300 MVA, 20 kV in the generator circuit requires a 300-MVA base in all parts of the system and the following voltage bases:

In the transmission line: 230 kV (since T_1 is rated 230/20 kV)

In the motor circuit: $$230\frac{13.2}{220} = 13.8 \text{ kV}$$

These bases are shown in parentheses on the one-line diagram of Fig. 6.5. The reactances of the transformers converted to the proper base are

Transformer T_1: $$X = 0.1 \times \frac{300}{350} = 0.0857 \text{ per unit}$$

Transformer T_2: $$X = 0.1\left(\frac{13.2}{13.8}\right)^2 = 0.0915 \text{ per unit}$$

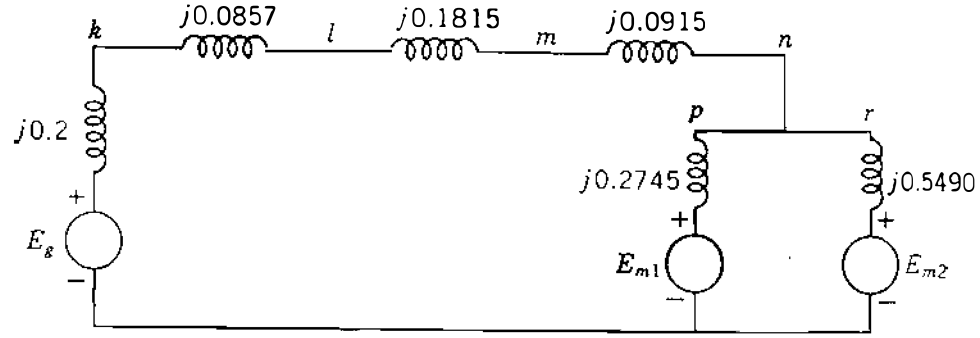


FIGURE 6.6
Reactance diagram for Example 6.1. Reactances are in per unit on the specified base.

The base impedance of the transmission line is

$$\frac{(230)^2}{300} = 176.3\Omega$$

and the reactance of the line is

$$\frac{0.5 \times 64}{176.3} = 0.1815 \text{ per unit}$$

$$\text{Reactance } X_d'' \text{ of motor } M_1 = 0.2\left(\frac{300}{200}\right)\left(\frac{13.2}{13.8}\right)^2 = 0.2745 \text{ per unit}$$

$$\text{Reactance } X_d'' \text{ of motor } M_2 = 0.2\left(\frac{300}{100}\right)\left(\frac{13.2}{13.8}\right)^2 = 0.5490 \text{ per unit}$$

Figure 6.6 is the required reactance diagram when transformer phase shifts are omitted.

Example 6.2. If the motors M_1 and M_2 of Example 6.1 have inputs of 120 and 60 MW, respectively, at 13.2 kV, and both operate at unity power factor, find the voltage at the terminals of the generator and the voltage regulation of the line.

Solution. Together the motors take 180 MW, or

$$\frac{180}{300} = 0.6 \text{ per unit}$$

Therefore, with V and I at the motors in per unit,

$$|V| \times |I| = 0.6 \text{ per unit}$$

With phase-a voltage at the motor terminals as reference, we have

$$V = \frac{13.2}{13.8} = 0.9565\underline{/0^\circ} \text{ per unit}$$

$$I = \frac{0.6}{0.9565} = 0.6273\underline{/0^\circ} \text{ per unit}$$

Phase-a per-unit voltages at other points of Fig. 6.6 are

At m: $\quad V = 0.9565 + 0.6273(j0.0915)$

$$0.9565 + j0.0574 = 0.9582\underline{/3.434^\circ} \text{ per unit}$$

At l: $\quad V = 0.9565 + 0.6273(j0.0915 + j0.1815)$

$$0.9565 + j0.1713 = 0.9717\underline{/10.154^\circ} \text{ per unit}$$

At k: $\quad V = 0.9565 + 0.6273(j0.0915 + j0.1815 + j0.0857)$

$$0.9565 + j0.2250 = 0.9826\underline{/13.237^\circ} \text{ per unit}$$

The voltage regulation of the line is

$$\text{Percent regulation} = \frac{0.9826 - 0.9582}{0.9582} \times 100 = 2.55\%$$

and the magnitude of the voltage at the generator terminals is

$$0.9826 \times 20 = 19.652 \text{ kV}$$

If it is desired to show the phase shifts due to the $Y-\Delta$ transformers, the angles of the phase-a voltages at m and l should be increased by 30°. Then the angle of the phase-a current in the line should also be increased by 30° from 0°.

6.3 THE MEDIUM-LENGTH LINE

The shunt admittance, usually pure capacitance, is included in the calculations for a line of medium length. If the total shunt admittance of the line is divided into two equal parts placed at the sending and receiving ends of the line, the circuit is called a *nominal* π. We refer to Fig. 6.7 to derive equations. To obtain an expression for V_S, we note that the current in the capacitance at the

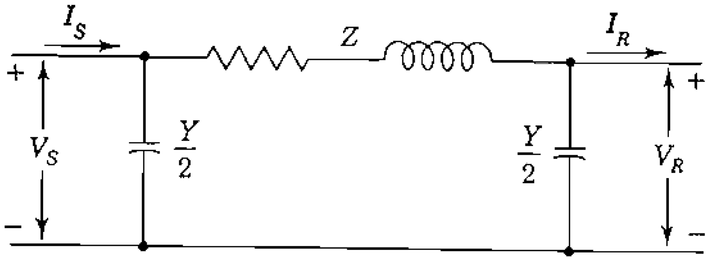


FIGURE 6.7
Nominal-π circuit of a medium-length transmission line.

receiving end is $V_R Y/2$ and the current in the series arm is $I_R + V_R Y/2$. Then,

$$V_S = \left(V_R \frac{Y}{2} + I_R\right) Z + V_R \tag{6.4}$$

$$V_S = \left(\frac{ZY}{2} + 1\right) V_R + Z I_R \tag{6.5}$$

To derive I_S, we note that the current in the shunt capacitance at the sending end is $V_S Y/2$, which added to the current in the series arm gives

$$I_S = V_S \frac{Y}{2} + V_R \frac{Y}{2} + I_R \tag{6.6}$$

Substituting V_S, as given by Eq. (6.5), in Eq. (6.6) yields

$$I_S = V_R Y\left(1 + \frac{ZY}{4}\right) + \left(\frac{ZY}{2} + 1\right) I_R \tag{6.7}$$

Equations (6.5) and (6.7) may be expressed in the general form

$$V_S = A V_R + B I_R \tag{6.8}$$

$$I_S = C V_R + D I_R \tag{6.9}$$

where

$$\begin{aligned} A = D &= \frac{ZY}{2} + 1 \\ B = Z \qquad C &= Y\left(1 + \frac{ZY}{4}\right) \end{aligned} \tag{6.10}$$

These $ABCD$ constants are sometimes called the *generalized circuit constants* of the transmission line. In general, they are complex numbers. A and D are dimensionless and equal each other if the line is the same when viewed from either end. The dimensions of B and C are ohms and mhos or siemens, respectively. The constants apply to any linear, passive, and bilateral four-terminal network having two pairs of terminals. Such a network is called a *two-port network*.

A physical meaning is easily assigned to the constants. By letting I_R be zero in Eq. (6.8), we see that A is the ratio V_S/V_R at no load. Similarly, B is the ratio V_S/I_R when the receiving end is short-circuited. The constant A is useful in computing regulation. If $V_{R,FL}$ is the receiving-end voltage at full load for a sending-end voltage of V_S, Eq. (6.3) becomes

$$\text{Percent regulation} = \frac{|V_S|/|A| - |V_{R,FL}|}{|V_{R,FL}|} \times 100 \tag{6.11}$$

Table A.6 in the Appendix lists $ABCD$ constants for various networks and combinations of networks.

6.4 THE LONG TRANSMISSION LINE: SOLUTION OF THE DIFFERENTIAL EQUATIONS

The exact solution of any transmission line and the one required for a high degree of accuracy in calculating 60-Hz lines more than approximately 150 mi long must consider the fact that the parameters of the lines are not lumped but, rather, are distributed uniformly throughout the length of the line.

Figure 6.8 shows one phase and the neutral connection of a three-phase line. Lumped parameters are not shown because we are ready to consider the solution of the line with the impedance and admittance uniformly distributed. In Fig. 6.8 we consider a differential element of length dx in the line at a distance x from the receiving end of the line. Then $z\,dx$ and $y\,dx$ are, respectively, the series impedance and shunt admittance of the elemental section. V and I are phasors which vary with x.

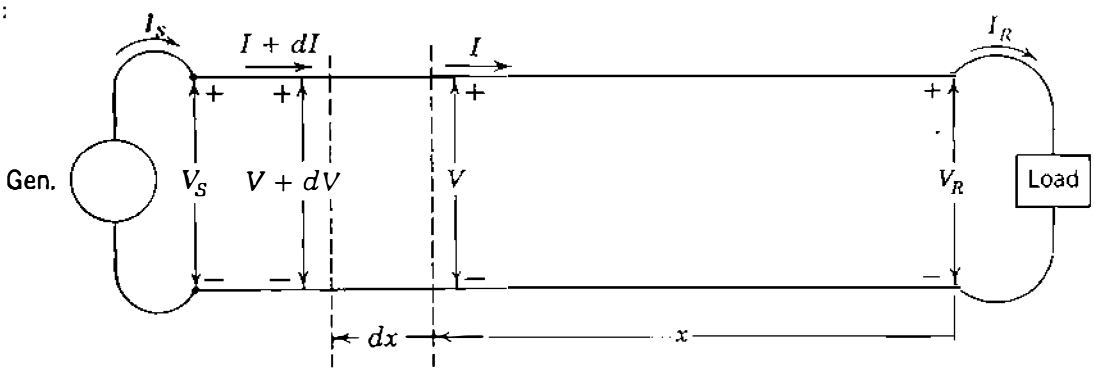


FIGURE 6.8
Schematic diagram of a transmission line showing one phase and the neutral return. Nomenclature for the line and the elemental length are indicated.

Average line current in the element is $(I + I + dI)/2$, and the increase of V in the distance dx is quite accurately expressed as

$$dV = \frac{I + I + dI}{2} z\,dx = Iz\,dx \tag{6.12}$$

when products of the differential quantities are neglected. Similarly,

$$dI = \frac{V + V + dV}{2} y\,dx = Vy\,dx \tag{6.13}$$

Then, from Eqs. (6.12) and (6.13) we have

$$\frac{dV}{dx} = Iz \tag{6.14}$$

and

$$\frac{dI}{dx} = Vy \tag{6.15}$$

Let us differentiate Eqs. (6.14) and (6.15) with respect to x, and we obtain

$$\frac{d^2V}{dx^2} = z\frac{dI}{dx} \tag{6.16}$$

and

$$\frac{d^2I}{dx^2} = y\frac{dV}{dx} \tag{6.17}$$

If we substitute the values of dI/dx and dV/dx from Eqs. (6.15) and (6.14) in Eqs. (6.16) and (6.17), respectively, we obtain

$$\frac{d^2V}{dx^2} = yzV \tag{6.18}$$

and

$$\frac{d^2I}{dx^2} = yzI \tag{6.19}$$

Now we have Eq. (6.18) in which the only variables are V and x and Eq. (6.19) in which the only variables are I and x. The solutions of those equations for V and I, respectively, must be expressions which when differentiated twice with respect to x yield the original expression times the constant yz. For instance, the solution for V when differentiated twice with respect to x must yield yzV.

This suggests an exponential form of solution. Assume that the solution of Eq. (6.18) is

$$V = A_1 \varepsilon^{\sqrt{yz}x} + A_2 \varepsilon^{-\sqrt{yz}x} \tag{6.20}$$

Taking the second derivative of V with respect to x in Eq. (6.20) yields

$$\frac{d^2V}{dx^2} = yz\left[A_1 \varepsilon^{\sqrt{yz}x} + A_2 \varepsilon^{-\sqrt{yz}x}\right] \tag{6.21}$$

which is yz times the assumed solution for V. Therefore, Eq. (6.20) is the solution of Eq. (6.18). When we substitute the value given by Eq. (6.20) for V in Eq. (6.14), we obtain

$$I = \frac{1}{\sqrt{z/y}} A_1 \varepsilon^{\sqrt{yz}x} - \frac{1}{\sqrt{z/y}} A_2 \varepsilon^{-\sqrt{yz}x} \tag{6.22}$$

The constants A_1 and A_2 can be evaluated by using the conditions at the receiving end of the line; namely, when $x = 0$, $V = V_R$ and $I = I_R$. Substitution of these values in Eqs. (6.20) and (6.22) yields

$$V_R = A_1 + A_2 \qquad \text{and} \qquad I_R = \frac{1}{\sqrt{z/y}}(A_1 - A_2)$$

Substituting $Z_c = \sqrt{z/y}$ and solving for A_1 give

$$A_1 = \frac{V_R + I_R Z_c}{2} \qquad \text{and} \qquad A_2 = \frac{V_R - I_R Z_c}{2}$$

Then, substituting the values found for A_1 and A_2 in Eqs. (6.20) and (6.22) and letting $\gamma = \sqrt{yz}$, we obtain

$$V = \frac{V_R + I_R Z_c}{2} \varepsilon^{\gamma x} + \frac{V_R - I_R Z_c}{2} \varepsilon^{-\gamma x} \tag{6.23}$$

$$I = \frac{V_R/Z_c + I_R}{2} \varepsilon^{\gamma x} - \frac{V_R/Z_c - I_R}{2} \varepsilon^{-\gamma x} \tag{6.24}$$

where $Z_c = \sqrt{z/y}$ and is called the *characteristic impedance* of the line, and $\gamma = \sqrt{zy}$ and is called the *propagation constant*.

Equations (6.23) and (6.24) give the rms values of V and I and their phase angles at any specified point along the line in terms of the distance x from the receiving end to the specified point, provided V_R, I_R, and the parameters of the line are known.

6.5 THE LONG TRANSMISSION LINE: INTERPRETATION OF THE EQUATIONS

Both γ and Z_c are complex quantities. The real part of the propagation constant γ is called the *attenuation constant* α and is measured in nepers per unit length. The quadrature part of γ is called the *phase constant* β and is measured in radians per unit length. Thus,

$$\gamma = \alpha + j\beta \tag{6.25}$$

and Eqs. (6.23) and (6.24) become

$$V = \frac{V_R + I_R Z_c}{2}\varepsilon^{\alpha x}\varepsilon^{j\beta x} + \frac{V_R - I_R Z_c}{2}\varepsilon^{-\alpha x}\varepsilon^{-j\beta x} \tag{6.26}$$

and
$$I = \frac{V_R/Z_c + I_R}{2}\varepsilon^{\alpha x}\varepsilon^{j\beta x} - \frac{V_R/Z_c - I_R}{2}\varepsilon^{-\alpha x}\varepsilon^{-j\beta x} \tag{6.27}$$

The properties of $\varepsilon^{\alpha x}$ and $\varepsilon^{j\beta x}$ help to explain the variation of the phasor values of voltage and current as a function of distance along the line. The term $\varepsilon^{\alpha x}$ changes in magnitude as x changes, but $\varepsilon^{j\beta x}$ (identical to $\cos \beta x + j \sin \beta x$) always has a magnitude of 1 and causes a shift in phase of β radians per unit length of line.

The first term in Eq. (6.26), $[(V_R + I_R Z_c)/2]\varepsilon^{\alpha x}\varepsilon^{j\beta x}$, increases in magnitude and advances in phase as distance x from the receiving end increases. Conversely, as progress along the line from the sending end toward the receiving end is considered, the term diminishes in magnitude and is retarded in phase. This is the characteristic of a traveling wave and is similar to the behavior of a wave in water, which varies in magnitude with time at any point, whereas its phase is retarded and its maximum value diminishes with distance from the origin. The variation in instantaneous value is not expressed in the term but is understood since V_R and I_R are phasors. The first term in Eq. (6.26) is called the *incident voltage*.

The second term in Eq. (6.26), $[(V_R - I_R Z_c)/2]\varepsilon^{-\alpha x}\varepsilon^{-j\beta x}$, diminishes in magnitude and is retarded in phase from the receiving end toward the sending end. It is called the *reflected voltage*. At any point along the line the voltage is the sum of the component incident and reflected voltages at that point.

Since the equation for current is similar to the equation for voltage, the current may be considered to be composed of incident and reflected currents.

If a line is terminated in its characteristic impedance Z_c, receiving-end voltage V_R is equal to $I_R Z_c$ and there is no reflected wave of either voltage or current, as may be seen by substituting $I_R Z_c$ for V_R in Eqs. (6.26) and (6.27). A line terminated in its characteristic impedance is called a *flat line* or an *infinite line*. The latter term arises from the fact that a line of infinite length cannot have a reflected wave. Usually, power lines are not terminated in their characteristic impedance, but communication lines are frequently so terminated in order to eliminate the reflected wave. A typical value of Z_c is 400 Ω for a single-circuit overhead line and 200 Ω for two circuits in parallel. The phase angle of Z_c is usually between 0 and $-15°$. Bundled-conductor lines have lower values of Z_c since such lines have lower L and higher C than lines with a single conductor per phase.

In power system work characteristic impedance is sometimes called *surge impedance*. The term "surge impedance," however, is usually reserved for the special case of a lossless line. If a line is lossless, its series resistance and shunt conductance are zero and the characteristic impedance reduces to the real number $\sqrt{L/C}$, which has the dimensions of ohms when L is the series inductance of the line in henrys and C is the shunt capacitance in farads. Also, the propagation constant $\gamma = \sqrt{zy}$ for the line of length l reduces to the imaginary number $j\beta = j\omega\sqrt{LC}/l$ since the attenuation constant α resulting from line losses is zero. When dealing with high frequencies or with surges due to lightning, losses are often neglected and the surge impedance becomes important. Surge-impedance loading (SIL) of a line is the power delivered by a line to a purely resistive load equal to its surge impedance. When so loaded, the line supplies a current of

$$|I_L| = \frac{|V_L|}{\sqrt{3} \times \sqrt{L/C}} \text{ A}$$

where $|V_L|$ is the line-to-line voltage at the load. Since the load is pure resistance,

$$\text{SIL} = \sqrt{3}|V_L|\frac{|V_L|}{\sqrt{3} \times \sqrt{L/C}} \text{ W}$$

or with $|V_L|$ in kilovolts,

$$\text{SIL} = \frac{|V_L|^2}{\sqrt{L/C}} \text{ MW} \tag{6.28}$$

Power system engineers sometimes find it convenient to express the power transmitted by a line in terms of per unit of SIL, that is, as the ratio of the power transmitted to the surge-impedance loading. For instance, the permissible loading of a transmission line may be expressed as a fraction of its SIL, and SIL provides a comparison of load-carrying capabilities of lines.[1]

A *wavelength* λ is the distance along a line between two points of a wave which differ in phase by 360°, or 2π rad. If β is the phase shift in radians per

[1]See R. D. Dunlop, R. Gutman, and P. P. Marchenko, "Analytical Development of Loadability Characteristics for EHV and UHV Transmission Lines," *IEEE Transactions on Power Apparatus and Systems*, vol. PAS-98, no. 2, 1979, pp. 606–617.

mile, the wavelength in miles is

$$\lambda = \frac{2\pi}{\beta} \tag{6.29}$$

The velocity of propagation of a wave in miles per second is the product of the wavelength in miles and the frequency in hertz, or

$$\text{Velocity} = \lambda f = \frac{2\pi f}{\beta} \tag{6.30}$$

For the lossless line of length l meters $\beta = 2\pi f\sqrt{LC}/l$ and Eqs. (6.29) and (6.30) become

$$\lambda = \frac{l}{f\sqrt{LC}}\ \text{m} \qquad \text{velocity} = \frac{l}{\sqrt{LC}}\ \text{m/s}$$

When values of L and C for low-loss overhead lines are substituted in these equations, it is found that the wavelength is approximately 3000 mi at a frequency of 60 Hz and the velocity of propagation is very nearly the speed of light in air (approximately 186,000 mi/s or 3×10^8 m/s).

If there is no load on a line, I_R is equal to zero, and as determined by Eqs. (6.26) and (6.27), the incident and reflected voltages are equal in magnitude and in phase at the receiving end. In this case the incident and reflected currents are equal in magnitude but are 180° out of phase at the receiving end. Thus, the incident and reflected currents cancel each other at the receiving end of an open line but not at any other point unless the line is entirely lossless so that the attenuation α is zero.

6.6 THE LONG TRANSMISSION LINE: HYPERBOLIC FORM OF THE EQUATIONS

The incident and reflected waves of voltage are seldom found when calculating the voltage of a power line. The reason for discussing the voltage and the current of a line in terms of the incident and reflected components is that such an analysis is helpful in obtaining a better understanding of some of the phenomena of transmission lines. A more convenient form of the equations for computing current and voltage of a power line is found by introducing hyperbolic functions. Hyperbolic functions are defined in exponential form

$$\sinh\theta = \frac{\varepsilon^{\theta} - \varepsilon^{-\theta}}{2} \tag{6.31}$$

$$\cosh\theta = \frac{\varepsilon^{\theta} + \varepsilon^{-\theta}}{2} \tag{6.32}$$

By rearranging Eqs. (6.23) and (6.24) and substituting hyperbolic functions for the exponential terms, we find a new set of equations. The new equations, giving voltage and current anywhere along the line, are

$$V = V_R \cosh \gamma x + I_R Z_c \sinh \gamma x \tag{6.33}$$

$$I = I_R \cosh \gamma x + \frac{V_R}{Z_c} \sinh \gamma x \tag{6.34}$$

Letting $x = l$ to obtain the voltage and the current at the sending end, we have

$$V_S = V_R \cosh \gamma l + I_R Z_c \sinh \gamma l \tag{6.35}$$

$$I_S = I_R \cosh \gamma l + \frac{V_R}{Z_c} \sinh \gamma l \tag{6.36}$$

From examination of these equations we see that the generalized circuit constants for a long line are

$$A = \cosh \gamma l \qquad C = \frac{\sinh \gamma l}{Z_c}$$
$$B = Z_c \sinh \gamma l \qquad D = \cosh \gamma l \tag{6.37}$$

Solving Eqs. (6.35) and (6.36) for V_R and I_R in terms of V_S and I_S, we obtain

$$V_R = V_S \cosh \gamma l - I_S Z_c \sinh \gamma l \tag{6.38}$$

$$I_R = I_S \cosh \gamma l - \frac{V_S}{Z_c} \sinh \gamma l \tag{6.39}$$

For balanced three-phase lines the currents in the above equations are line currents and the voltages are line-to-neutral voltages, that is, line voltages divided by $\sqrt{3}$. In order to solve the equations, the hyperbolic functions must be evaluated. Since γl is usually complex, the hyperbolic functions are also complex and can be evaluated with the assistance of a calculator or computer.

For solving an occasional problem without resorting to a computer there are several choices. The following equations give the expansions of hyperbolic sines and cosines of complex arguments in terms of circular and hyperbolic functions of real arguments:

$$\cosh(\alpha l + j\beta l) = \cosh \alpha l \cos \beta l + j \sinh \alpha l \sin \beta l \tag{6.40}$$

$$\sinh(\alpha l + j\beta l) = \sinh \alpha l \cos \beta l + j \cosh \alpha l \sin \beta l \tag{6.41}$$

Equations (6.40) and (6.41) make possible the computation of hyperbolic functions of complex arguments. The correct mathematical unit for βl is the **radian,** and the radian is the unit found for βl by computing the quadrature component of γl. Equations (6.40) and (6.41) can be verified by substituting in them the exponential forms of the hyperbolic functions and the similar exponential forms of the circular functions.

Another method of evaluating complex hyperbolic functions is suggested by Eqs. (6.31) and (6.32). Substituting $\alpha + j\beta$ for θ, we obtain

$$\cosh(\alpha + j\beta) = \frac{\varepsilon^{\alpha}\varepsilon^{j\beta} + \varepsilon^{-\alpha}\varepsilon^{-j\beta}}{2} = \frac{1}{2}\left(\varepsilon^{\alpha}\underline{/\beta} + \varepsilon^{-\alpha}\underline{/-\beta}\right) \tag{6.42}$$

$$\sinh(\alpha + j\beta) = \frac{\varepsilon^{\alpha}\varepsilon^{j\beta} - \varepsilon^{-\alpha}\varepsilon^{-j\beta}}{2} = \frac{1}{2}\left(\varepsilon^{\alpha}\underline{/\beta} - \varepsilon^{-\alpha}\underline{/-\beta}\right) \tag{6.43}$$

Example 6.3. A single-circuit 60-Hz transmission line is 370 km (230 mi) long. The conductors are *Rook* with flat horizontal spacing and 7.25 m (23.8 ft) between conductors. The load on the line is 125 MW at 215 kV with 100% power factor. Find the voltage, current, and power at the sending end and the voltage regulation of the line. Also, determine the wavelength and velocity of propagation of the line.

Solution. Feet and miles rather than meters and kilometers are chosen for the calculations in order to use Tables A.3 through A.5 in the Appendix:

$$D_{eq} = \sqrt[3]{23.8 \times 23.8 \times 47.6} \cong 30.0 \text{ ft}$$

and from the tables for *Rook*

$$z = 0.1603 + j(0.415 + 0.4127) = 0.8431\underline{/79.04^\circ}\ \Omega/\text{mi}$$

$$y = j[1/(0.0950 + 0.1009)] \times 10^{-6} = 5.105 \times 10^{-6}\underline{/90^\circ}\ \text{S/mi}$$

$$\gamma l = \sqrt{yz}\, l = 230\sqrt{0.8431 \times 5.105 \times 10^{-6}}\ \underline{\Big/\frac{79.04^\circ + 90^\circ}{2}}$$

$$= 0.4772\underline{/84.52^\circ} = 0.0456 + j0.4750$$

$$Z_c = \sqrt{\frac{z}{y}} = \sqrt{\frac{0.8431}{5.105 \times 10^{-6}}}\ \underline{\Big/\frac{79.04^\circ - 90^\circ}{2}} = 406.4\underline{/-5.48^\circ}\ \Omega$$

$$V_R = \frac{215{,}000}{\sqrt{3}} = 124{,}130\underline{/0^\circ}\ \text{V to neutral}$$

$$I_R = \frac{125{,}000{,}000}{\sqrt{3} \times 215{,}000} = 335.7\underline{/0^\circ}\ \text{A}$$

From Eqs. (6.42) and (6.43) and noting that 0.4750 rad = 27.22°

$$\cosh \gamma l = \frac{1}{2}\varepsilon^{0.0456}\angle 27.22^\circ + \frac{1}{2}\varepsilon^{-0.0456}\angle -27.22^\circ$$

$$= 0.4654 + j0.2394 + 0.4248 - j0.2185$$

$$= 0.8902 + j0.0209 = 0.8904\angle 1.34^\circ$$

$$\sinh \gamma l = 0.4654 + j0.2394 - 0.4248 + j0.2185$$

$$= 0.0406 + j0.4579 = 0.4597\angle 84.93^\circ$$

Then, from Eq. (6.35)

$$V_S = 124{,}130 \times 0.8904\angle 1.34^\circ + 335.7 \times 406.4\angle -5.48^\circ \times 0.4597\angle 84.93^\circ$$

$$= 110{,}495 + j2{,}585 + 11{,}483 + j61{,}656$$

$$= 137{,}860\angle 27.77^\circ \text{ V}$$

and from Eq. (6.36)

$$I_S = 335.7 \times 0.8904\angle 1.34^\circ + \frac{124{,}130}{406.4\angle -5.48^\circ} \times 0.4597\angle 84.93^\circ$$

$$= 298.83 + j6.99 - 1.00 + j140.41$$

$$= 332.31\angle 26.33^\circ \text{ A}$$

At the sending end

$$\text{Line voltage} = \sqrt{3} \times 137.86 = 238.8 \text{ kV}$$

$$\text{Line current} = 332.3 \text{ A}$$

$$\text{Power factor} = \cos(27.77^\circ - 26.33^\circ) = 0.9997 \cong 1.0$$

$$\text{Power} = \sqrt{3} \times 238.8 \times 332.3 \times 1.0 = 137{,}443 \text{ kW}$$

From Eq. (6.35) we see that at no load ($I_R = 0$)

$$V_R = \frac{V_S}{\cosh \gamma l}$$

So, the voltage regulation is

$$\frac{137.86/0.8904 - 124.13}{124.13} \times 100 = 24.7\%$$

The wavelength and velocity of propagation are computed as follows:

$$\beta = \frac{0.4750}{230} = 0.002065 \text{ rad/mi}$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{0.002065} = 3043 \text{ mi}$$

$$\text{Velocity} = f\lambda = 60 \times 3043 = 182{,}580 \text{ mi/s}$$

We note particularly in this example that in the equations for V_S and I_S the value of voltage must be expressed in volts and must be the line-to-neutral voltage.

Example 6.4. Solve for the sending-end voltage and the current found in Example 6.3 using per-unit calculations.

Solution. We choose a base of 125 MVA, 215 kV to achieve the simplest per-unit values and to compute base impedance and base current as follows:

$$\text{Base impedance} = \frac{215^2}{125} = 370\ \Omega$$

$$\text{Base current} = \frac{125{,}000}{\sqrt{3} \times 215} = 335.7 \text{ A}$$

So,

$$Z_c = \frac{406.4\angle -5.48^\circ}{370} = 1.098\angle -5.48^\circ \text{ per unit}$$

$$V_R = \frac{215}{215} = \frac{215/\sqrt{3}}{215/\sqrt{3}} = 1.0 \text{ per unit}$$

For use in Eq. (6.35) we chose V_R as the reference voltage. So,

$$V_R = 1.0\angle 0^\circ \text{ per unit (as a line-to-neutral voltage)}$$

and since the load is at unity power factor,

$$I_R = \frac{337.5\angle 0^\circ}{337.5} = 1.0\angle 0^\circ$$

If the power factor had been less than 100%, I_R would have been greater than 1.0 and would have been at an angle determined by the power factor. By Eq. (6.35)

$$V_S = 1.0 \times 0.8904 + 1.0 \times 1.098\angle -5.48^\circ \times 0.4597\angle 84.93^\circ$$

$$= 0.8902 + j0.0208 + 0.0923 + j0.4961$$

$$= 1.1102\angle 27.75^\circ \text{ per unit}$$

and by Eq. (6.36)

$$I_S = 1.0 \times 0.8904\angle 1.34^\circ + \frac{1.0\angle 0^\circ}{1.098\angle -5.48^\circ} \times 0.4597\angle 84.93^\circ$$

$$= 0.8902 + j0.0208 - 0.0031 + j0.4186$$

$$= 0.990\angle 26.35^\circ \text{ per unit}$$

At the sending end

$$\text{Line voltage} = 1.1102 \times 215 = 238.7 \text{ kV}$$

$$\text{Line current} = 0.990 \times 335.7 = 332.3 \text{ A}$$

Note that we multiply line-to-line voltage base by the per-unit magnitude of the voltage to find the line-to-line voltage magnitude. We could have multiplied the line-to-neutral voltage base by the per-unit voltage to find the line-to-neutral voltage magnitude. The factor $\sqrt{3}$ does not enter the calculations after we have expressed all quantities in per unit.

6.7 THE EQUIVALENT CIRCUIT OF A LONG LINE

The nominal-π circuit does not represent a transmission line exactly because it does not account for the parameters of the line being uniformly distributed. The discrepancy between the nominal π and the actual line becomes larger as the length of line increases. It is possible, however, to find the equivalent circuit of a long transmission line and to represent the line accurately, insofar as measurements at the ends of the line are concerned, by a network of lumped parameters. Let us assume that a π circuit similar to that of Fig. 6.7 is the equivalent circuit of a long line, but let us call the series arm of our equivalent-π circuit Z' and the shunt arms $Y'/2$ to distinguish them from the arms of the nominal-π circuit. Equation (6.5) gives the sending-end voltage of a symmetrical-π circuit in terms of its series and shunt arms and the voltage and current at the receiving end. By substituting Z' and $Y'/2$ for Z and $Y/2$ in Eq. (6.5), we obtain the sending-end voltage of our equivalent circuit in terms of its series and shunt arms and the voltage and current at the receiving end:

$$V_S = \left(\frac{Z'Y'}{2} + 1\right)V_R + Z'I_R \tag{6.44}$$

For our circuit to be equivalent to the long transmission line the coefficients of V_R and I_R in Eq. (6.44) must be identical, respectively, to the coefficients of V_R and I_R in Eq. (6.35). Equating the coefficients of I_R in the two equations yields

$$Z' = Z_c \sinh \gamma l \tag{6.45}$$

$$Z' = \sqrt{\frac{z}{y}} \sinh \gamma l = zl \frac{\sinh \gamma l}{\sqrt{zy}\, l}$$

$$Z' = Z \frac{\sinh \gamma l}{\gamma l} \tag{6.46}$$

where Z is equal to zl, the total series impedance of the line. The term $(\sinh \gamma l)/\gamma l$ is the factor by which the series impedance of the nominal π must be multiplied to convert the nominal π to the equivalent π. For small values of γl, *both* $\sinh \gamma l$ and γl are almost identical, and this fact shows that the nominal π represents the medium-length transmission line quite accurately, insofar as the series arm is concerned.

To investigate the shunt arms of the equivalent-π circuit, we equate the coefficients of V_R in Eqs. (6.35) and (6.44) and obtain

$$\frac{Z'Y'}{2} + 1 = \cosh \gamma l \tag{6.47}$$

Substituting $Z_c \sinh \gamma l$ for Z' gives

$$\frac{Y'Z_c \sinh \gamma l}{2} + 1 = \cosh \gamma l \tag{6.48}$$

$$\frac{Y'}{2} = \frac{1}{Z_c} \frac{\cosh \gamma l - 1}{\sinh \gamma l} \tag{6.49}$$

Another form of the expression for the shunt admittance of the equivalent circuit can be found by substituting in Eq. (6.49) the identity

$$\tanh \frac{\gamma l}{2} = \frac{\cosh \gamma l - 1}{\sinh \gamma l} \tag{6.50}$$

The identity can be verified by substituting the exponential forms of Eqs. (6.31)

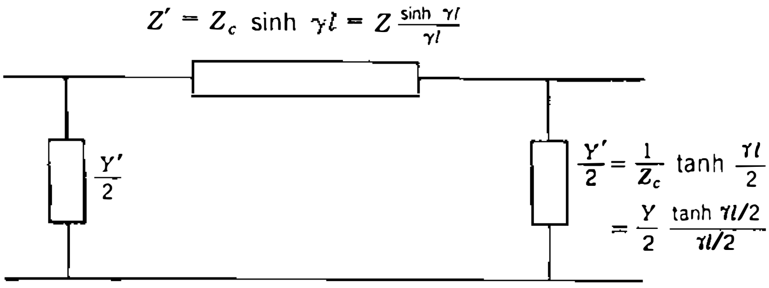


FIGURE 6.9
Equivalent-π circuit of a transmission line.

and (6.32) for the hyperbolic functions and by recalling that $\tanh \theta = \sinh \theta/\cosh \theta$. Now

$$\frac{Y'}{2} = \frac{1}{Z_c} \tanh \frac{\gamma l}{2} \tag{6.51}$$

$$\frac{Y'}{2} = \frac{Y}{2} \frac{\tanh(\gamma l/2)}{\gamma l/2} \tag{6.52}$$

where Y is equal to yl, the total shunt admittance of the line. Equation (6.52) shows the correction factor used to convert the admittance of the shunt arms of the nominal π to that of the equivalent π. Since $\tanh(\gamma l/2)$ and $\gamma l/2$ are very nearly equal for small values of γl, the nominal π represents the medium-length transmission line quite accurately, for we have seen previously that the correction factor for the series arm is negligible for medium-length lines. The equivalent-π circuit is shown in Fig. 6.9. An equivalent-T circuit can also be found for a transmission line.

Example 6.5. Find the equivalent-π circuit for the line described in Example 6.3 and compare it with the nominal-π circuit.

Solution. Since $\sinh \gamma l$ and $\cosh \gamma l$ are already known from Example 6.3, Eqs. (6.45) and (6.49) are now used.

$$Z' = 406.4\underline{/-5.48^\circ} \times 0.4597\underline{/84.93^\circ} = 186.82\underline{/79.45^\circ}\ \Omega \quad \text{in series arm}$$

$$\frac{Y'}{2} = \frac{0.8902 + j0.0208 - 1}{186.82\underline{/79.45^\circ}} = \frac{0.1118\underline{/169.27^\circ}}{186.82\underline{/79.45^\circ}}$$

$$= 0.000599\underline{/89.82^\circ}\ \text{S} \quad \text{in each shunt arm}$$

Using the values of z and y from Example 6.3, we find for the nominal-π circuit a series impedance of

$$Z = 230 \times 0.8431\underline{/79.04^\circ} = 193.9\underline{/79.04^\circ}$$