On forced convection along a vertical cylinder with uniform surface temperature and uniform surface heat flux.

Transformed model is given below:

$$(1+2\xi\eta)f''' + \left(\tfrac{1}{2}f + 2\xi\right)f'' \pm \Omega_0 \xi^{2+K}\theta = \tfrac{1}{2}\xi\left(f'\frac{\partial f'}{\partial \xi} - f''\frac{\partial f}{\partial \xi}\right) \rightarrow (1)$$

$$\frac{1}{Pr}(1+2\xi\eta)\theta'' + \left(\tfrac{1}{2}f + \frac{2}{Pr}\xi\right)\theta' + \frac{K}{2}f'\theta = \tfrac{1}{2}\xi\left(f'\frac{\partial \theta}{\partial \xi} - \theta'\frac{\partial f}{\partial \xi}\right) \rightarrow (2)$$

where $K=0$ for uniform surface temperature and $K=1$ for uniform heat flux.

Transformed Boundary Conditions:

$$\left.\begin{aligned} & f'(\xi,0) = f(\xi,0) = 0, \quad f'(\xi,\infty) = 1, \quad \theta(\xi,\infty) = 0 \\ & \theta(\xi,0) = 1 \text{ for uniform surface temperature} \end{aligned}\right] \rightarrow (3)$$

$\theta'(\xi,0) = -1$ for uniform heat flux.

With $\Omega_0 = \dfrac{Gr_0}{Re_0}$ and $Re_0 = \dfrac{U_\infty r_0}{\nu}$ and the average Grashof number is $Gr_0 = \dfrac{g_x\beta|T_w - T_\infty| r_0^3}{\nu^2}$ for uniform surface temperature and $Gr_0 = \dfrac{g\beta |q_w| r_0^4}{K\nu^2}$ for uniform heat flux.

We introduce $\xi = \left(\dfrac{x}{r_0 Re_0}\right)^{1/2}$ which yields equations in terms of a single non-similarity variable $\xi(x)$.

Solution methodology:

Now we propose the method to find the solutions of equations (1) – (2) along with boundary conditions (3). The extended series solution method (ESS) is proposed to determine the numerical solution of the flow model.

Extended Series solution: (ESS):

Near the leading edge the flow will be basically that on the flat plate, with the effects of curvature of the cylinder and the buoyancy force having only small effects. For this we have to find an ~~expassion~~ expansion for small ξ of equations (1) and (2) under the respective boundary conditions (3), which is valid near $\xi = 0$.

Thus we assume solutions of the form.

$$f(\xi,\eta) = \sum_{i=0}^{\infty} \xi^i f_i(\eta) \text{ and } \theta(\xi,\eta) = \sum_{i=0}^{\infty} \xi^i \theta_i(\eta) \rightarrow (4)$$

where $f_i(\eta)$ and $\theta_i(\eta)$ are the arbitrary functions depending on η.

We expand (4), as follows.

$$f(\xi,\eta) = f_0(\eta) + \xi f_1(\eta) + \xi^2 f_2(\eta) + \dots \rightarrow (5)$$

$$\theta(\xi,\eta) = \theta_0(\eta) + \xi \theta_1(\eta) + \xi^2 \theta_2(\eta) + \dots \rightarrow (6)$$

Differentiate w.r.t "(η)".

③

$$f'(\xi,\eta) = f_0'(\eta) + \xi f_1'(\eta) + \xi^2 f_2'(\eta) + \cdots$$

$$\theta'(\xi,\eta) = \theta_0'(\eta) + \xi \theta_1'(\eta) + \xi^2 \theta_2'(\eta) + \cdots$$

$$f''(\xi,\eta) = f_0''(\eta) + \xi f_1''(\eta) + \xi^2 f_2''(\eta) + \cdots$$

$$f'''(\xi,\eta) = f_0'''(\eta) + \xi f_1'''(\eta) + \xi^2 f_2'''(\eta) + \cdots$$

$$\theta''(\xi,\eta) = \theta_0''(\eta) + \xi \theta_1''(\eta) + \xi^2 \theta_2''(\eta) + \cdots$$

Taking the derivatives w.r.t "ξ".

$$\frac{\partial f}{\partial \xi} = f_1(\eta) + 2\xi f_2(\eta) + \cdots$$

$$\frac{\partial f'}{\partial \xi} = f_1'(\eta) + 2\xi f_2'(\eta) + \cdots$$

Putting these above expanded functions into equation (1).

we just put two terms expansions. i.e

$$f(\xi,\eta) = f_0(\eta) + \xi f_1(\eta) +$$

& $\theta(\xi,\eta) = \theta_0(\eta) + \xi \theta_1(\eta)$

So

$$(1+2\xi\eta)(f_0'''(\eta)+\xi f_1'''(\eta)) + \frac{1}{2}(f_0(\eta)+\xi f_1(\eta))(f_0''(\eta)+\xi f_1''(\eta)) + 2\xi(f_0'(\eta)+\xi f_1''(\eta))$$

$$\pm \Omega_0 \xi^{2+k}(\theta_0(\eta)+\xi\theta_1(\eta)) = \frac{1}{2}\xi\left[(f_0'(\eta)+\xi f_1'(\eta))(f_1(\eta)) - (f_0''(\eta)+\xi f_1''(\eta))(f_1)\right]$$

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$$(1+2\xi\eta) f_0''' + (1+2\xi\eta)\xi f_1''' + \frac{1}{2}\left(f_0 f_0'' + \xi f_0 f_1'' + \xi f_1 f_0'' + \xi^2 f_1 f_1''\right)$$

$$+ 2\xi f_0'' + 2\xi^2 f_1'' \pm \Omega_0 \xi^{2+K}(\theta_0 + \xi\theta_1) = \frac{1}{2}\xi\left[\left(f_0' f_1 + \xi f_1' f_1\right) - \left(f_0'' f_1 + \xi f_1'' f_1\right)\right]$$

$$f_0''' + 2\xi\eta f_0''' + \xi f_1''' + 2\xi^2\eta f_1''' + \frac{1}{2}\left\{f_0 f_0'' + \xi\left(f_0 f_1'' + f_1 f_0''\right) + \xi^2 f_1 f_1''\right\}$$

$$+ 2\xi f_0'' + 2\xi^2 f_1'' \pm \Omega_0 \xi^{2+K}(\theta_0 + \xi\theta_1) = \frac{1}{2}\xi\left[\left(f_0' f_1 - f_0'' f_1\right) + \xi\left(f_1' f_1 - f_1'' f_1\right)\right]$$

$$f_0''' + 2\xi\eta f_0''' + \xi f_1''' + 2\xi^2\eta f_1''' + \frac{1}{2}\left\{f_0 f_0'' + \xi\left(f_0 f_1'' + f_1 f_0''\right) + \xi^2 f_1 f_1''\right\}$$

$$+ 2\xi f_0'' + 2\xi^2 f_1'' \pm \Omega_0 \xi^{2+K}(\theta_0 + \xi\theta_1) = \frac{1}{2}\xi\left(f_0' f_1 - f_0'' f_1\right) + \frac{1}{2}\xi^2\left(f_1' f_1 - f_1'' f_1\right)$$

Collecting the terms of $O(\xi^0)$ and $O(\xi^1)$ on both sides.

$O(\xi^0)$:

$$\boxed{f_0''' + \frac{1}{2} f_0 f_0'' = 0,}$$

$O(\xi^1)$:

$$2\eta f_0''' + f_1''' + \frac{1}{2}\left(f_0 f_1'' + f_1 f_0''\right) + 2 f_0'' = \frac{1}{2}\left(f_0' f_1 - f_0'' f_1\right)$$

$$\Rightarrow \boxed{f_1''' + 2\eta f_0''' + 2 f_0'' + \frac{1}{2}\left(f_0 f_1'' + f_1 f_0''\right) - \frac{1}{2}\left(f_0' f_1 - f_0'' f_1\right) = 0}$$

Now we put two terms expansion into equation.

$$\frac{1}{Pr}(1+2\xi\eta)(\theta_0''+\xi\theta_1'')+\frac{1}{2}(f_0+\xi f_1)(\theta_0'+\xi\theta_1')+\frac{2}{Pr}\xi(\theta_0'+\xi\theta_1')$$
$$+\frac{K}{2}(f_0'+\xi f_1')(\theta_0+\xi\theta_1)=\frac{1}{2}\xi\left[(f_0'+\xi f_1')(\theta_1)-(\theta_0'+\xi\theta_1')(f_1)\right]$$

$$\frac{1}{Pr}(1+2\xi\eta)(\theta_0'')+\frac{\xi}{Pr}(1+2\xi\eta)\theta_1''+\frac{1}{2}\left\{f_0\theta_0'+\xi f_0\theta_1'+\xi f_1\theta_0'+\xi^2 f_1\theta_1'\right\}$$
$$+\frac{2}{Pr}\xi\theta_0'+\frac{2}{Pr}\xi^2\theta_1'+\frac{K}{2}\left\{f_0'\theta_0+\xi f_0'\theta_1+\xi f_1'\theta_0+\xi^2 f_1'\theta_1\right\}$$
$$=\frac{1}{2}\xi\left[(f_0'\theta_1+\xi f_1'\theta_1)-(\theta_0' f_1+\xi\theta_1' f_1)\right]$$

$$\frac{1}{Pr}\theta_0''+2\xi\eta\theta_0''+\frac{\xi}{Pr}\theta_1''+\frac{2\xi^2\eta}{Pr}\theta_1''+\frac{1}{2}\left\{f_0\theta_0'+\xi(f_0\theta_1'+f_1\theta_0')+\xi^2 f_1\theta_1'\right\}$$
$$+\frac{2}{Pr}\xi\theta_0'+\frac{2}{Pr}\xi^2\theta_1'+\frac{K}{2}\left\{f_0'\theta_0+\xi(f_0'\theta_1+f_1'\theta_0)+\xi^2 f_1'\theta_1\right\}$$
$$=\frac{1}{2}\xi\left[(f_0'\theta_1-\theta_0' f_1)+\xi(f_1'\theta_1-\theta_1' f_1)\right\}$$

Collecting the terms of $O(\xi^0)$ and $O(\xi^1)$ on both sides

$O(\xi^0)$:

$$\boxed{\frac{1}{Pr}\theta_0''+\frac{1}{2}f_0\theta_0'+\frac{K}{2}f_0'\theta_0=0}$$

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$o(\xi^1)$:

$$2\eta\theta_0'' + \frac{1}{Pr}\theta_1'' + \frac{1}{2}(f_0\theta_1' + f_1\theta_0') + \frac{2}{Pr}\theta_0' + \frac{K}{2}(f_0'\theta_1 + f_1'\theta_0) = \frac{1}{2}(f_0'\theta_1 - \theta_0'f_1)$$

$$\boxed{\frac{1}{Pr}\theta_1'' + 2\eta\theta_0'' + \frac{2}{Pr}\theta_0' + \frac{1}{2}(f_0\theta_1' + f_1\theta_0') - \frac{1}{2}(f_0'\theta_1 - \theta_0'f_1) + \frac{K}{2}(f_0'\theta_1 + f_1'\theta_0) = 0}$$

Now we put two terms expansion into boundary conditions,

$f_0' + \xi f_1' = 0$, $f_0 + \xi f_1 = 0$, $f_0' + \xi f_1' = 1\xi^0 + 0\xi^1$, $\theta_0 + \xi\theta_1 = 0$,

$\theta_0 + \xi\theta_1 = 1\xi^0 + 0\xi^1$ for u.s.t.

$\theta_0' + \xi\theta_1' = -1\xi^0 + 0\xi^1$ for u.h.f.

Collect terms $o(\xi^0)$ and $O(\xi^1)$.

$f_0' = 0$, $f_0 = 0$, $f_0'(\infty) = 1$, $\theta_0(0) = 0$,

$\theta_0(\infty) = 1$, $\theta_0'(\infty) = -1$

and $O(\xi^1)$:

$f_1 = f_1' = 0$, $\theta_1 = 0$,

$f_1'(\infty) = 0$, $\theta_1(\infty) = 0$, $\theta_1'(\infty) = 0$

$O(\xi^0)$:

$$f_0''' + \frac{1}{2} f_0 f_0'' = 0.$$

$$\frac{1}{Pr}\theta_0'' + \frac{1}{2} f_0 \theta_0' + \frac{K}{2} f_0' \theta_0 = 0$$

B.Cs

$f_0 = 0,\ f_0' = 0,\quad \theta_0 = 0,\qquad$ at $\eta = 0.$

$f_0' \to 1,\quad \theta_0 \to 1$ for u.s.t, $\theta_0' \to -1$ for u.h.f as $\eta \to \infty.$

$O(\xi^1)$:

$$f_1''' + 2\eta f_0''' + 2 f_0'' + \frac{1}{2}\left(f_0 f_1'' + f_1 f_0''\right) - \frac{1}{2}\left(f_0' f_1 - f_0'' f_1\right) = 0$$

$$\frac{1}{Pr}\theta_1'' + 2\xi\theta_0'' + \frac{2}{Pr}\theta_0' + \frac{1}{2}\left(f_0\theta_1' + f_1\theta_0'\right) - \frac{1}{2}\left(f_0'\theta_1 - \theta_0' f_1\right) + \frac{K}{2}\left(f_0'\theta_1 + f_1'\theta_0\right) = 0$$

B.Cs.

$f_1 = 0,\ f_1' = 0,\qquad \theta_1 = 0\qquad$ at $\eta = 0$

$f_1' \to 0,\qquad \theta_1 \to 0$ for u.s.t, $\theta_1' \to 0$, for u.h.f as $\eta \to \infty.$