To cope with this situation, let us try instead the solution

$$y(t) = Bte^{-4t}$$

with derivatives

$$y'(t) = (1 - 4t)Be^{-4t} \quad \text{and} \quad y''(t) = (-8 + 16t)Be^{-4t}$$

Substituting these into (16.48) will now yield: left side $= -5Be^{-4t}$. When this is equated to the right side, we determine the coefficient to be $B = -2/5$. Consequently, the desired particular integral of (16.48) can be written as

$$y_p = \frac{-2}{5}te^{-4t}$$

EXERCISE 16.6

1. Show that the method of undetermined coefficients is inapplicable to the differential equation $y''(t) - ay'(t) + by = t^{-1}$.
2. Find the particular integral of each of the following equations by the method of undetermined coefficients:

   (*a*) $y''(t) + 2y'(t) + y = t$

   (*b*) $y''(t) + 4y'(t) + y = 2t^2$

   (*c*) $y''(t) + y'(t) + 2y = e^t$

   (*d*) $y''(t) + y'(t) + 3y = \sin t$

16.7 Higher-Order Linear Differential Equations

The methods of solution introduced in the previous sections are readily extended to an *n*th-order linear differential equation. With constant coefficients and a constant term, such an equation can be written generally as

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + \cdots + a_{n-1}y'(t) + a_n y = b \qquad \textbf{(16.50)}$$

Finding the Solution

In this case of constant coefficients and constant term, the presence of the higher derivatives does not materially affect the method of finding the particular integral discussed earlier.

If we try the simplest possible type of solution, $y = k$, we can see that all the derivatives from $y'(t)$ to $y^{(n)}(t)$ will be zero; hence (16.50) will reduce to $a_n k = b$, and we can write

$$y_p = k = \frac{b}{a_n} \qquad (a_n \neq 0) \qquad \text{[cf. (16.3)]}$$

In case $a_n = 0$, however, we must try a solution of the form $y = kt$. Then, since $y'(t) = k$, all the higher derivatives will vanish, (16.50) can be reduced to $a_{n-1}k = b$, thereby yielding the particular integral

$$y_p = kt = \frac{b}{a_{n-1}}t \qquad (a_n = 0;\ a_{n-1} \neq 0) \qquad \text{[cf. (16.3')]}$$

If it happens that $a_n = a_{n-1} = 0$, then this last solution will fail, too; instead, a solution of the form $y = kt^2$ must be tried. Further adaptations of this procedure should be obvious.

As for the complementary function, inclusion of the higher-order derivatives in the differential equation has the effect of raising the degree of the characteristic equation. The complementary function is defined as the general solution of the reduced equation

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + \cdots + a_{n-1} y'(t) + a_n y = 0 \qquad \textbf{(16.51)}$$

Trying $y = Ae^{rt}\ (\neq 0)$ as a solution and utilizing the knowledge that this implies $y'(t) = rAe^{rt}, y''(t) = r^2Ae^{rt}, \ldots, y^{(n)}(t) = r^nAe^{rt}$, we can rewrite (16.51) as

$$Ae^{rt}(r^n + a_1 r^{n-1} + \cdots + a_{n-1}r + a_n) = 0$$

This equation is satisfied by any value of r which satisfies the following (nth-degree polynomial) characteristic equation

$$r^n + a_1 r^{n-1} + \cdots + a_{n-1}r + a_n = 0 \qquad \textbf{(16.51')}$$

There will, of course, be n roots to this polynomial, and each of these should be included in the general solution of (16.51). Thus our complementary function should in general be in the form

$$y_c = A_1 e^{r_1 t} + A_2 e^{r_2 t} + \cdots + A_n e^{r_n t} \qquad \left(= \sum_{i=1}^{n} A_i e^{r_i t}\right)$$

As before, however, some modifications must be made in case the n roots are not all real and distinct. First, suppose that there are repeated roots, say, $r_1 = r_2 = r_3$. Then, to avoid "collapsing," we must write the first three terms of the solutions as $A_1 e^{r_1 t} + A_2 t e^{r_1 t} + A_3 t^2 e^{r_1 t}$ [cf. (16.9)]. In case we have $r_4 = r_1$ as well, the fourth term must be altered to $A_4 t^3 e^{r_1 t}$, etc.

Second, suppose that two of the roots are complex, say,

$$r_5, r_6 = h \pm vi$$

then the fifth and sixth terms in the preceding solution should be combined into the following expression:

$$e^{ht}(A_5 \cos vt + A_6 \sin vt) \qquad \text{[cf. (16.24')]}$$

By the same token, if two *distinct* pairs of complex roots are found, there must be two such trigonometric expressions (with a different set of values of h, v, and two arbitrary constants for each).† As a further possibility, if there happen to be two pairs of *repeated* complex roots, then we should use e^{ht} as the multiplicative term for one but use te^{ht} for the other. Also, even though h and v have identical values in the repeated complex roots, a different pair of arbitrary constants must now be assigned to each.

Once y_p and y_c are found, the general solution of the complete equation (16.50) follows easily. As before, it is simply the sum of the complementary function and the particular integral: $y(t) = y_c + y_p$. In this general solution, we can count a total of n arbitrary constants. Thus, to definitize the solution, as many as n initial conditions will be required.

† It is of interest to note that, inasmuch as complex roots always come in conjugate pairs, we can be sure of having *at least one* real root when the differential equation is of an *odd* order, i.e., when n is an odd number.

Example 1

Find the general solution of

$$y^{(4)}(t) + 6y'''(t) + 14y''(t) + 16y'(t) + 8y = 24$$

The particular integral of this fourth-order equation is simply

$$y_p = \frac{24}{8} = 3$$

Its characteristic equation is, by (16.51′),

$$r^4 + 6r^3 + 14r^2 + 16r + 8 = 0$$

which can be factored into the form

$$(r+2)(r+2)(r^2+2r+2) = 0$$

From the first two parenthetical expressions, we can obtain the double roots $r_1 = r_2 = -2$, but the last (quadratic) expression yields the pair of complex roots $r_3, r_4 = -1 \pm i$, with $h = -1$ and $v = 1$. Consequently, the complementary function is

$$y_c = A_1 e^{-2t} + A_2 t e^{-2t} + e^{-t}(A_3 \cos t + A_4 \sin t)$$

and the general solution is

$$y(t) = A_1 e^{-2t} + A_2 t e^{-2t} + e^{-t}(A_3 \cos t + A_4 \sin t) + 3$$

The four constants A_1, A_2, A_3, and A_4 can be definitized, of course, if we are given four initial conditions.

Note that all the characteristic roots in this example either are real and negative or are complex and with a negative real part. The time path must therefore be convergent, and the intertemporal equilibrium is dynamically stable.

Convergence and the Routh Theorem

The solution of a high-degree characteristic equation is not always an easy task. For this reason, it should be of tremendous help if we can find a way of ascertaining the convergence or divergence of a time path without having to solve for the characteristic roots. Fortunately, there does exist such a method, which can provide a qualitative (though nongraphic) analysis of a differential equation.

This method is to be found in the *Routh theorem*,[†] which states that:

> The real parts of all of the roots of the *n*th-degree polynomial equation
>
> $$a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0$$
>
> are negative if and only if the first *n* of the following sequence of determinants
>
> $$|a_1|; \quad \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix}; \quad \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix}; \quad \begin{vmatrix} a_1 & a_3 & a_5 & a_7 \\ a_0 & a_2 & a_4 & a_6 \\ 0 & a_1 & a_3 & a_5 \\ 0 & a_0 & a_2 & a_4 \end{vmatrix}; \quad \ldots$$
>
> all are positive.

In applying this theorem, it should be remembered that $|a_1| \equiv a_1$. Further, it is to be understood that we should take $a_m = 0$ for all $m > n$. For example, given a third-degree

[†] For a discussion of this theorem, and a sketch of its proof, see Paul A. Samuelson, *Foundations of Economic Analysis*, Harvard University Press, 1947, pp. 429–435, and the references there cited.

polynomial equation ($n = 3$), we need to examine the signs of the first *three* determinants listed in the Routh theorem; for that purpose, we should set $a_4 = a_5 = 0$.

The relevance of this theorem to the convergence problem should become self-evident when we recall that, in order for the time path $y(t)$ to converge regardless of what the initial conditions happen to be, all the characteristic roots of the differential equation must have negative real parts. Since the characteristic equation (16.51′) is an nth-degree polynomial equation, with $a_0 = 1$, the Routh theorem can be of direct help in the testing of convergence. In fact, we note that the coefficients of the characteristic equation (16.51′) are wholly identical with those of the given differential equation (16.51), so it is perfectly acceptable to substitute the coefficients of (16.51) directly into the sequence of determinants shown in the Routh theorem for testing, provided that we always take $a_0 = 1$. Inasmuch as the condition cited in the theorem is given on the "if and only if" basis, it obviously constitutes a necessary-and-sufficient condition.

Example 2

Test by the Routh theorem whether the differential equation of Example 1 has a convergent time path. This equation is of the fourth order, so $n = 4$. The coefficient are $a_0 = 1$, $a_1 = 6$, $a_2 = 14$, $a_3 = 16$, $a_4 = 8$, and $a_5 = a_6 = a_7 = 0$. Substituting these into the first four determinants, we find their values to be 6, 68, 800, and 6,400, respectively. Because they are all positive, we can conclude that the time path is convergent.

EXERCISE 16.7

1. Find the particular integral of each of the following:
   (*a*) $y'''(t) + 2y''(t) + y'(t) + 2y = 8$
   (*b*) $y'''(t) + y''(t) + 3y'(t) = 1$
   (*c*) $3y'''(t) + 9y''(t) = 1$
   (*d*) $y^{(4)}(t) + y''(t) = 4$
2. Find the y_p and the y_c (and hence the general solution) of:
   (*a*) $y'''(t) - 2y''(t) - y'(t) + 2y = 4$
   [*Hint:* $r^3 - 2r^2 - r + 2 = (r - 1)(r + 1)(r - 2)$]
   (*b*) $y'''(t) + 7y''(t) + 15y'(t) + 9y = 0$
   [*Hint:* $r^3 + 7r^2 + 15r + 9 = (r - 1)(r^2 + 6r + 9)$]
   (*c*) $y'''(t) + 6y''(t) + 10y'(t) - 8y = 8$
   [*Hint:* $r^3 + 6r^2 + 10r + 8 = (r - 4)(r^2 + 2r + 2)$]
3. On the basis of the signs of the characteristic roots obtained in Prob. 2, analyze the dynamic stability of equilibrium. Then check your answer by the Routh theorem.
4. Without finding their characteristic roots, determine whether the following differential equations will give rise to convergent time paths:
   (*a*) $y'''(t) - 10y''(t) + 27y'(t) - 18y = 3$
   (*b*) $y'''(t) - 11y''(t) + 34y'(t) + 24y = 5$
   (*c*) $y'''(t) + 4y''(t) - 5y'(t) - 2y = -2$
5. Deduce from the Routh theorem that, for the second-order linear differential equation $y''(t) + a_1y'(t) + a_2y = b$, the solution path will be convergent regardless of initial conditions if and only if the coefficients a_1 and a_2 are both positive.