CHAPTER 5

GENERAL EQUILIBRIUM

Many scholars trace the birth of economics to the publication of Adam Smith's *The Wealth of Nations* (1776). Behind the superficial chaos of countless interdependent market actions by selfish agents, Smith saw a harmonising force serving society. This *Invisible Hand* guides the market system to an equilibrium that Smith believed possessed certain socially desirable characteristics.

One can ask many questions about competitive market systems. A fundamental one arises immediately: is Smith's vision of a smoothly functioning system composed of many self-interested individuals buying and selling on impersonal markets – with no regard for anything but their personal gain – a logically coherent vision at all? If so, is there one particular state towards which such a system will tend, or are there many such states? Are these fragile things that can be easily disrupted or are they robust?

These are questions of existence, uniqueness, and stability of general competitive equilibrium. All are deep and important, but we will only address the first.

In many ways, existence is the most fundamental question and so merits our closest attention. What is at issue is the logical coherence of the very *notion* of a competitive market system. The question is usually framed, however, as one of the existence of prices at which demand and supply are brought into balance in the market for every good and service simultaneously. The market prices of everything we buy and sell are principal determinants of what we can consume, and so, of the well-being we can achieve. Thus, market prices determine to a large extent 'who gets what' in a market economy.

In this chapter, we do not merely ask under what conditions a set of market-clearing prices exists. We also ask how well a market system solves the basic economic problem of distribution. We will begin by exploring the distribution problem in very general terms, then proceed to consider the existence of general competitive equilibrium itself. Along the way, we will focus particular scrutiny on Smith's claim that a competitive market system promotes society's welfare through no conscious collective intention of its members.

5.1 EQUILIBRIUM IN EXCHANGE

Here we explore the basic economic problem of distribution in a very simple society *without* organised markets. Our objective is to describe what outcomes might arise through a process of voluntary exchange. By examining the outcomes of this process, we can establish a benchmark against which the equilibria achieved under competitive market systems can be compared.

The society we consider is very stark. First, there is no production. Commodities exist, but for now we do not ask how they came to be. Instead, we merely assume each consumer is 'endowed' by nature with a certain amount of a finite number of consumable goods. Each consumer has preferences over the available commodity bundles, and each cares only about his or her individual well-being. Agents may consume their endowment of commodities or may engage in barter exchange with others. We admit the institution of private ownership into this society and assume that the principle of voluntary, non-coercive trade is respected. In the absence of coercion, and because consumers are self-interested, voluntary exchange is the only means by which commodities may be redistributed from the initial distribution. In such a setting, what outcomes might we expect to arise? Or, rephrasing the question, where might this system come to rest through the process of voluntary exchange? We shall refer to such rest points as barter equilibria.

To simplify matters, suppose there are only two consumers in this society, consumer 1 and consumer 2, and only two goods, x_1 and x_2. Let $\mathbf{e}^1 \equiv (e_1^1, e_2^1)$ denote the non-negative endowment of the two goods owned by consumer 1, and $\mathbf{e}^2 \equiv (e_1^2, e_2^2)$ the endowment of consumer 2. The total amount of each good available in this society then can be summarised by the vector $\mathbf{e}^1 + \mathbf{e}^2 = (e_1^1 + e_1^2, e_2^1 + e_2^2)$. (From now on, superscripts will be used to denote consumers and subscripts to denote goods.)

The essential aspects of this economy can be analysed with the ingenious **Edgeworth box**, familiar from intermediate theory courses. In Fig. 5.1, units of x_1 are measured along each horizontal side and units of x_2 along each vertical side. The south-west corner is consumer 1's origin and the north-east corner consumer 2's origin.

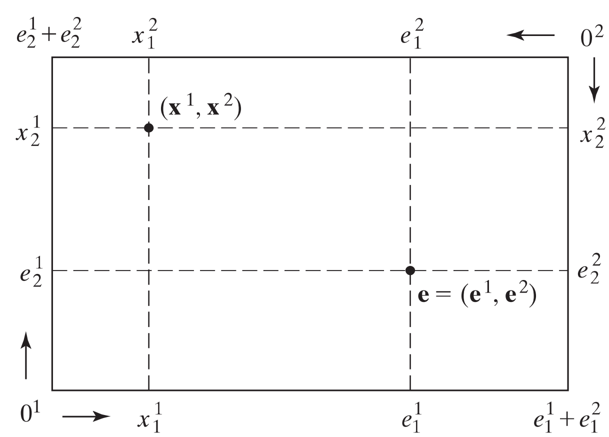

Figure 5.1. The Edgeworth box.

Increasing amounts of x_1 for consumer 1 are measured rightwards from 0^1 along the bottom side, and increasing amounts of x_1 for consumer 2 are measured leftwards from 0^2 along the top side. Similarly, x_2 for consumer 1 is measured vertically up from 0^1 on the left, and for consumer 2, vertically down on the right. The box is constructed so that its width measures the total endowment of x_1 and its height the total endowment of x_2.

Notice carefully that each point in the box has *four* coordinates – two indicating some amount of each good for consumer 1 and two indicating some amount of each good for consumer 2. Because the dimensions of the box are fixed by the total endowments, each set of four coordinates represents some division of the total amount of each good between the two consumers. For example, the point labelled $\mathbf{e}$ denotes the pair of initial endowments $\mathbf{e}^1$ and $\mathbf{e}^2$. Every other point in the box represents some other way the totals can be allocated between the consumers, and every possible allocation of the totals between the consumers is represented by some point in the box. The box therefore provides a complete picture of every feasible distribution of existing commodities between consumers.

To complete the description of the two-person exchange economy, suppose each consumer has preferences represented by a usual, convex indifference map. In Fig. 5.2, consumer 1's indifference map increases north-easterly, and consumer 2's increases south-westerly. One indifference curve for each consumer passes through *every* point in the box. The line labelled *CC* is the subset of allocations where the consumers' indifference curves through the point are tangent to each other, and it is called the **contract curve**. At any point off the contract curve, the consumers' indifference curves through that point must cut each other.

Given initial endowments at $\mathbf{e}$, which allocations will be barter equilibria in this exchange economy? Obviously, the first requirement is that the allocations be somewhere,

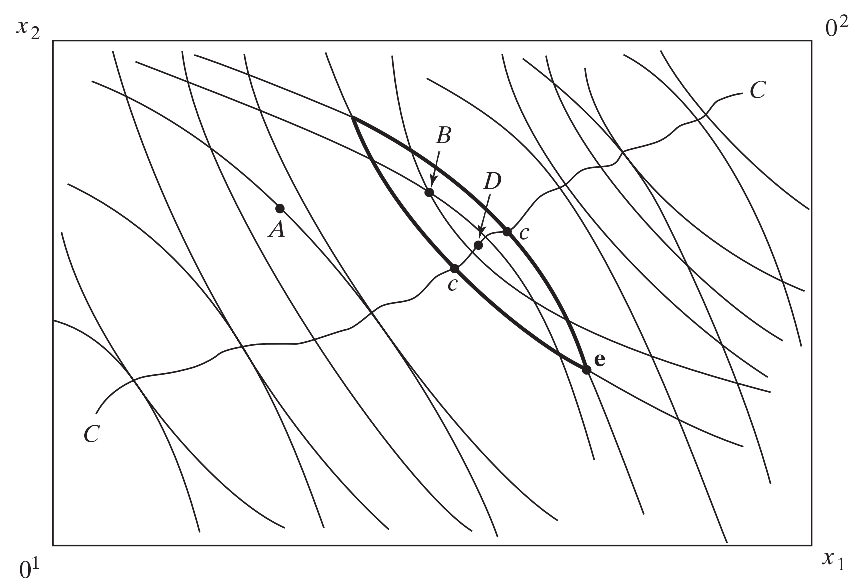

Figure 5.2. Equilibrium in two-person exchange.

'in the box', because only those are feasible. But not every feasible allocation can be a barter equilibrium. For example, suppose a redistribution from **e** to point A were proposed. Consumer 2 would be better off, but consumer 1 would clearly be worse off. Because this economy relies on voluntary exchange, and because consumers are self-interested, the redistribution to A would be refused, or 'blocked', by consumer 1, and so could not arise as an equilibrium given the initial endowment. By the same argument, all allocations to the left of consumer 1's indifference curve through **e** would be blocked by consumer 1, and all allocations to the right of consumer 2's indifference curve through **e** would be blocked by consumer 2.

This leaves only allocations inside and on the boundary of the lens-shaped area delineated by the two consumers' indifference curves through **e** as potential barter equilibria. At every point along the boundary, one consumer will be better off and the other no worse off than they are at **e**. At every allocation inside the lens, however, both consumers will be strictly better off than they are at **e**. To achieve these gains, the consumers must arrange a trade. Consumer 1 must give up some x_1 in exchange for some of consumer 2's x_2, and consumer 2 must give up some x_2 in exchange for some of consumer 1's x_1.

But are all allocations inside the lens barter equilibria? Suppose a redistribution to B within that region were to occur. Because B is off the contract curve, the two indifference curves passing through it must cut each other, forming another lens-shaped region contained entirely within the original one. Consequently, both consumers once again can be made strictly better off by arranging an appropriate trade away from B and inside the lens it determines. Thus, B and every such point inside the lens through **e** but off the contract curve can be ruled out as barter equilibria.

Now consider a point like D on segment cc of the contract curve. A move from **e** to any such point will definitely make both parties better off. Moreover, once the consumers trade to D, there are no feasible trades that result in further *mutual* gain. Thus, once D is achieved, no further trades will take place: D is a barter equilibrium. Indeed, any point along cc is a barter equilibrium. Should the consumers agree to trade and so find themselves at *any* allocation on cc, and should a redistribution to *any* other allocation in the box then be proposed, that redistribution would be blocked by one or both of them. (This includes, of course, any movement from one point on cc to another on cc.) Pick any point on cc, consider several possible reallocations, and convince yourself of this. Once on cc, we can be sure there will be no subsequent movement away.

Clearly, there are many barter equilibria toward which the system might evolve. We are content with having identified all of the possibilities. Note that these equilibria all share the property that once there, it is not possible to move elsewhere in the box without making at least one of the consumers worse off. Thus, each point of equilibrium in exchange is Pareto efficient in the sense described in Chapter 4.

Consider now the case of many consumers and many goods. Let

$$\mathcal{I} = \{1, \ldots, I\}$$

index the set of consumers, and suppose there are n goods. Each consumer $i \in \mathcal{I}$ has a preference relation, $\succsim^i$, and is endowed with a non-negative vector of the n goods, $\mathbf{e}^i = (e^i_1, \ldots, e^i_n)$. Altogether, the collection $\mathcal{E} = (\succsim^i, \mathbf{e}^i)_{i \in \mathcal{I}}$ defines an **exchange economy**.

What conditions characterise barter equilibria in this exchange economy? As before, the first requirement is that the assignment of goods to individuals not exceed the amounts available. Let

$$\mathbf{e} \equiv (\mathbf{e}^1, \ldots, \mathbf{e}^I)$$

denote the economy's endowment vector, and define an **allocation** as a vector

$$\mathbf{x} \equiv (\mathbf{x}^1, \ldots, \mathbf{x}^I),$$

where $\mathbf{x}^i \equiv (x_1^i, \ldots, x_n^i)$ denotes consumer i's bundle according to the allocation. The set of **feasible allocations** in this economy is given by

$$F(\mathbf{e}) \equiv \left\{ \mathbf{x} \,\middle|\, \sum_{i \in \mathcal{I}} \mathbf{x}^i = \sum_{i \in \mathcal{I}} \mathbf{e}^i \right\}, \tag{5.1}$$

and it contains all allocations of goods across individuals that, in total, exhaust the available amount of every good. The first requirement on $\mathbf{x}$ as a barter equilibrium is therefore that $\mathbf{x} \in F(\mathbf{e})$.

Now in the two-consumer case, we noted that if both consumers could be made better off by trading with one another, then we could not yet be at a barter equilibrium. Thus, at a barter equilibrium, no Pareto improvements were possible. This also carries over to the more general case. To formalise this, let us begin with the following.

DEFINITION 5.1 ***Pareto-Efficient Allocations***

A feasible allocation, $\mathbf{x} \in F(\mathbf{e})$, is Pareto efficient if there is no other feasible allocation, $\mathbf{y} \in F(\mathbf{e})$, such that $\mathbf{y}^i \succsim^i \mathbf{x}^i$ for all consumers, i, with at least one preference strict.

So, an allocation is Pareto efficient if it is not possible to make someone strictly better off without making someone else strictly worse off.

Now if $\mathbf{x} \in F(\mathbf{e})$ is not Pareto efficient, then there is another feasible allocation $\mathbf{y}$ making someone strictly better off and no one worse off. Consequently, the consumer who can be made strictly better off can arrange a trade with the others by announcing: 'I'll give each consumer i the bundle $\mathbf{y}^i$ in exchange for the bundle $\mathbf{x}^i$'. Because both allocations $\mathbf{x}$ and $\mathbf{y}$ are feasible, this trade is feasible. No consumer will object to it because it makes everyone at least as well off as they were before. Moreover it makes (at least) the one consumer strictly better off. Consequently, $\mathbf{x}$ would not be an equilibrium. Thus, to be a barter equilibrium, $\mathbf{x}$ must be feasible and Pareto efficient.

Suppose now that $\mathbf{x}$ is Pareto efficient. Can we move away from $\mathbf{x}$? No, we cannot. Because $\mathbf{x}$ is Pareto efficient, every other feasible allocation that makes someone better off must make at least one other consumer worse off. Hence, the latter consumer will not agree to the trade that is involved in the move.

So, we now know that *only* Pareto-efficient allocations are candidates for barter equilibrium, and whenever a Pareto-efficient allocation is reached, it will indeed be an equilibrium of our process of voluntary exchange. Thus, it remains to describe the set of Pareto-efficient allocations that can be reached through voluntary exchange.

Recall from the two-consumer case that not all Pareto-efficient allocations were equilibria there. That is, only those allocations on the contract curve and within the lens created by the indifference curves through the endowment point were equilibria. The reason for this was that the other Pareto-efficient allocations – those on the contract curve but outside the lens – made at least one of the consumers worse off than they would be by simply consuming their endowment. Thus, each such Pareto-efficient allocation was 'blocked' by one of the consumers.

Similarly, when there are more than two consumers, no equilibrium allocation can make any consumer worse off than he would be consuming his endowment. That consumer would simply refuse to make the necessary trade. But in fact there are now additional reasons you might refuse to trade to some Pareto-efficient allocation. Indeed, although you might prefer the bundle assigned to you in the proposed allocation over your own endowment, you might be able to find another consumer to strike a trade with such that you do even better as a result of that trade and he does no worse than he would have done had you both gone along with the proposed allocation. Consequently, although you *alone* are unable to block the proposal, you are able to block it *together with someone else*. Of course, the potential for blocking is not limited to coalitions of size 2. Three or more of you might be able to get together to block an allocation. With all of this in mind, consider the following.

DEFINITION 5.2 ***Blocking Coalitions***

Let $S \subset \mathcal{I}$ denote a coalition of consumers. We say that S blocks $\mathbf{x} \in F(\mathbf{e})$ if there is an allocation $\mathbf{y}$ such that:[1]

1. $\sum_{i\in S} \mathbf{y}^i = \sum_{i\in S} \mathbf{e}^i$.
2. $\mathbf{y}^i \succsim^i \mathbf{x}^i$ *for all $i \in S$, with at least one preference strict.*

Together, the first and second items in the definition say that the consumers in S must be able to take what they themselves have and divide it up differently among themselves so that none is worse off and at least one is better off than with their assignment under $\mathbf{x}$. Thus, an allocation $\mathbf{x}$ is blocked whenever some group, no matter how large or small, can do better than they do under $\mathbf{x}$ by simply 'going it alone'. By contrast, we say that an allocation is 'unblocked' if *no* coalition can block it. Our final requirement for equilibrium, then, is that the allocation be unblocked.

Note that this takes care of the two-consumer case because all allocations outside the lens are blocked by a coalition consisting of a single consumer (sometimes consumer 1, sometimes consumer 2). In addition, note that in general, if $\mathbf{x} \in F(\mathbf{e})$ is unblocked, then it must be Pareto efficient, because otherwise it would be blocked by the grand

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[1]Note that there is no need to insist that $\mathbf{y} \in F(\mathbf{e})$, because one can always make it so by replacing the bundles in it going to consumers $j \notin S$ by $\mathbf{e}^j$.

coalition $S = \mathcal{I}$. This lets us summarise the requirements for equilibrium in exchange very compactly.

Specifically, *an allocation* $\mathbf{x} \in F(\mathbf{e})$ *is an equilibrium in the exchange economy with endowments* $\mathbf{e}$ if $\mathbf{x}$ *is not blocked by any coalition of consumers*. Take a moment to convince yourself that this definition reduces to the one we developed earlier when there were only two goods and two consumers.

The set of allocations we have identified as equilibria of the process of voluntary exchange is known as the 'core', and we define this term for future reference.

DEFINITION 5.3 ***The Core of an Exchange Economy***

The core of an exchange economy with endowment $\mathbf{e}$, *denoted* $C(\mathbf{e})$, *is the set of all unblocked feasible allocations.*

Can we be assured that every exchange economy possesses at least one allocation in the core? That is, must there exist at least one feasible and unblocked allocation? As we shall later show, the answer is yes under a number of familiar conditions.

We have argued that under ideal circumstances, including the costless nature of both the formation of coalitions and the acquisition of the information needed to arrange mutually beneficial trades, consumers are led, through the process of voluntary exchange, to pursue the attainment of allocations in the core. From this point of view, points in the core seem very far indeed from becoming a reality in a real-world economy. After all, most of us have little or no direct contact with the vast majority of other consumers. Consequently, one would be quite surprised were there not substantial gains from trade left unrealised, regardless of how the economy were organised – centrally planned, market-based, or otherwise. In the next section, we investigate economies organised by competitive markets. Prepare for a surprise.

5.2 EQUILIBRIUM IN COMPETITIVE MARKET SYSTEMS

In the preceding section, we examined a very primitive economic system based wholly on voluntary barter exchange. Here we take a first look at questions of equilibrium and distribution in a more sophisticated economic system. In a **perfectly competitive market system**, all transactions between individuals are mediated by impersonal markets. Consumers' market behaviour is guided solely by their personal self-interest, and each consumer, whether acting as buyer or seller, is individually insignificant on every market, with no power to affect prevailing prices. Equilibrium on each market separately is achieved when the totality of buyers' decisions are compatible with the totality of sellers' decisions at the prevailing market price. Equilibrium in the market *system* is achieved when the demands of buyers match the supplies of sellers at prevailing prices in every market simultaneously.

A noteworthy feature of the competitive model we shall develop here is its *decentralised* nature. Each consumer, fully aware of the prices of goods prevailing in all markets, demands a bundle that is best for him, *without the need to consider what other consumers might demand, being fully confident that sufficient production has taken place*. Similarly, producers, also fully aware of the prevailing prices of all goods (both inputs and outputs), choose amounts of production that maximise their profits, *without the need to consider how much other producers are producing, being fully confident that their output will be purchased*.

The naivete expressed in the decentralised aspect of the competitive model (i.e., that every agent acts in his own self-interest while ignoring the actions of others) should be viewed as a strength. Because in equilibrium consumers' demands *will* be satisfied, and because producers' outputs *will* be purchased, the actions of the other agents *can* be ignored and the *only* information required by consumers and producers is the *prevailing prices*. Consequently, the informational requirements of this model are minimal. This is in stark contrast to the barter model of trade developed in the previous section in which each consumer requires very detailed information about all other consumers' preferences and bundles.

Clearly, the optimality of ignoring others' actions requires that at prevailing prices consumer demands are met and producer supplies are sold. So, it is essential that prices are able to clear all markets simultaneously. But is it not rather bold to presume that a suitable vector of prices will ensure that the diverse tastes of consumers and the resulting totality of their demands will be exactly matched by the supplies coming from the production side of the market, with its many distinct firms, each being more or less adept at producing one good or another? The existence of such a vector of prices is not obvious at all, but the coherence of our competitive model requires such a price vector to exist.

To give you a feeling for the potential for trouble on this front, suppose that there are just three goods and that at current prices the demand for good 1 is equal to its supply, so this market is in equilibrium. However, suppose that there is excess demand for good 2 and excess supply of good 3, so that neither of these markets clears at current prices. It would be natural to suppose that one can achieve equilibrium in these markets by increasing the price of good 2 and decreasing the price of good 3. Now, while this might help to reduce the difference between demand and supply in these markets, these price changes may very well affect the demand for good 1! After all if goods 1 and 2 are substitutes, then increases in the price of good 2 can lead to increases in the demand for good 1. So, changing the prices of goods 2 and 3 in an attempt to equilibrate those markets can upset the equilibrium in the market for good 1.

The *interdependence* of markets renders the existence of an equilibrium price vector a subtle issue indeed. But again, the existence of a vector of prices that *simultaneously* clears all markets is *essential* for employing the model of the consumer and producer developed in Chapters 1 and 3, where we assumed that demands were always met and supplies always sold. Fortunately, even though it is not at all obvious, we can show (with a good deal of effort) that under some economically meaningful conditions, there *does* exist at least one vector of prices that simultaneously clears all markets. We now turn to this critical question.

5.2.1 EXISTENCE OF EQUILIBRIUM

For simplicity, let us first consider an economy without the complications of production in the model. Again let $\mathcal{I} = \{1, \ldots, I\}$ index the set of consumers and assume that each is endowed with a non-negative vector $\mathbf{e}^i$ of n goods. Further, suppose each consumer's preferences on the consumption set $\mathbb{R}^n_+$ can be represented by a utility function u^i satisfying the following.[2]

ASSUMPTION 5.1 ***Consumer Utility***

Utility u^i is continuous, strongly increasing, and strictly quasiconcave on $\mathbb{R}^n_+$.

On competitive markets, every consumer takes prices as given, whether acting as a buyer or a seller. If $\mathbf{p} \equiv (p_1, \ldots, p_n) \gg \mathbf{0}$ is the vector of market prices, then each consumer solves

$$\max_{\mathbf{x}^i \in \mathbb{R}^n_+} u^i(\mathbf{x}^i) \qquad \text{s.t.} \qquad \mathbf{p} \cdot \mathbf{x}^i \leq \mathbf{p} \cdot \mathbf{e}^i. \tag{5.2}$$

The constraint in (5.2) simply expresses the consumer's usual budget constraint but explicitly identifies the source of a consumer's income. Intuitively, one can imagine a consumer selling his entire endowment at prevailing market prices, receiving income, $\mathbf{p} \cdot \mathbf{e}^i$, and then facing the ordinary constraint that expenditures, $\mathbf{p} \cdot \mathbf{x}^i$, not exceed income. The solution $\mathbf{x}^i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i)$ to (5.2) is the consumer's demanded bundle, which depends on market prices and the consumer's endowment income. We record here a familiar result that we will need later.

THEOREM 5.1 ***Basic Properties of Demand***

If u^i satisfies Assumption 5.1 then for each $\mathbf{p} \gg \mathbf{0}$, the consumer's problem (5.2) has a unique solution, $\mathbf{x}^i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i)$. In addition, $\mathbf{x}^i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i)$ is continuous in $\mathbf{p}$ on $\mathbb{R}^n_{++}$.

Recall that existence of a solution follows because $\mathbf{p} \gg \mathbf{0}$ implies that the budget set is bounded, and uniqueness follows from the strict quasiconcavity of u^i. Continuity at $\mathbf{p}$ follows from Theorem A2.21 (the theorem of the maximum), and this requires $\mathbf{p} \gg \mathbf{0}$. We emphasise here that $\mathbf{x}^i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i)$ *is not continuous in* $\mathbf{p}$ *on all of* $\mathbb{R}^n_+$ because demand may well be infinite if one of the prices is zero. We will have to do a little work later to deal with this unpleasant, yet unavoidable, difficulty.

We can interpret the consumer's endowment $\mathbf{e}^i$ as giving the quantity of each of the n goods that he inelastically supplies on the various markets.

[2]Recall that a function is strongly increasing if strictly raising one component in the domain vector and lowering none strictly increases the value of the function. Note also that Cobb-Douglas utilities are neither strongly increasing nor strictly quasiconcave on all of $\mathbb{R}^n_+$ and so are ruled out by Assumption 5.1.

We now can begin to build a description of the system of markets we intend to analyse. The market demand for some good will simply be the sum of every individual consumer's demand for it. Market supply will be the sum of every consumer's supply. With n goods, the market system will consist of n markets, each with its market demand and market supply. Because consumers' demand for any one good depends on the prices of every good, the system of markets so constructed will be a completely *interdependent* system, with conditions in any one market affecting and being affected by conditions in every other market.

The earliest analysis of market systems, undertaken by Léon Walras (1874), proceeded along these lines, with each market described by separate demand and supply functions. Today, largely as a matter of convenience and notational simplicity, it is more common to describe each separate market by a single *excess demand function*. Then, the market system may be described compactly by a single n-dimensional *excess demand vector*, each of whose elements is the excess demand function for one of the n markets.

DEFINITION 5.4 ***Excess Demand***

The aggregate excess demand function for good k is the real-valued function,

$$z_k(\mathbf{p}) \equiv \sum_{i \in \mathcal{I}} x_k^i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i) - \sum_{i \in \mathcal{I}} e_k^i.$$

The aggregate excess demand function is the vector-valued function,

$$\mathbf{z}(\mathbf{p}) \equiv (z_1(\mathbf{p}), \ldots, z_n(\mathbf{p})).$$

When $z_k(\mathbf{p}) > 0$, the aggregate demand for good k exceeds the aggregate endowment of good k and so there is excess demand for good k. When $z_k(\mathbf{p}) < 0$, there is excess supply of good k.

Aggregate excess demand functions possess certain properties. We detail these here.

THEOREM 5.2 ***Properties of Aggregate Excess Demand Functions***

If for each consumer i, u^i satisfies Assumption 5.1, then for all $\mathbf{p} \gg \mathbf{0}$,

1. *Continuity:* $\mathbf{z}(\cdot)$ *is continuous at* $\mathbf{p}$.
2. *Homogeneity:* $\mathbf{z}(\lambda\mathbf{p}) = \mathbf{z}(\mathbf{p})$ *for all* $\lambda > 0$.
3. *Walras' law:* $\mathbf{p} \cdot \mathbf{z}(\mathbf{p}) = 0$.

Proof: Continuity follows from Theorem 5.1.

Homogeneity: A glance at the constraint in (5.2) should convince you that individual demands, and excess demands, are homogeneous of degree zero in prices. It follows immediately that aggregate excess demand is also homogeneous of degree zero in prices.

Walras' law: The third property, Walras' law, is important. It says that the *value* of aggregate excess demand will always be zero at *any* set of positive prices. Walras' law follows because when u^i is strongly increasing, each consumer's budget constraint holds with *equality*.

When the budget constraint in (5.2) holds with equality,

$$\sum_{k=1}^{n} p_k\big(x_k^i(\mathbf{p}, \mathbf{p}\cdot\mathbf{e}^i) - e_k^i\big) = 0.$$

Summing over individuals gives

$$\sum_{i\in\mathcal{I}}\sum_{k=1}^{n} p_k\big(x_k^i(\mathbf{p}, \mathbf{p}\cdot\mathbf{e}^i) - e_k^i\big) = 0.$$

Because the order of summation is immaterial, we can reverse it and write this as

$$\sum_{k=1}^{n}\sum_{i\in\mathcal{I}} p_k\big(x_k^i(\mathbf{p}, \mathbf{p}\cdot\mathbf{e}^i) - e_k^i\big) = 0.$$

This, in turn, is equivalent to the expression

$$\sum_{k=1}^{n} p_k\left(\sum_{i\in\mathcal{I}} x_k^i(\mathbf{p}, \mathbf{p}\cdot\mathbf{e}^i) - \sum_{i\in\mathcal{I}} e_k^i\right) = 0.$$

From Definition 5.4, the term in parentheses is the aggregate excess demand for good k, so we have

$$\sum_{k=1}^{n} p_k z_k(\mathbf{p}) = 0,$$

and the claim is proved. ∎

Walras' law has some interesting implications. For example, consider a two-good economy and suppose that prices are strictly positive. By Walras' law, we know that

$$p_1 z_1(\mathbf{p}) = -p_2 z_2(\mathbf{p}).$$

If there is excess demand in market 1, say, so that $z_1(\mathbf{p}) > 0$, we know immediately that we must have $z_2(\mathbf{p}) < 0$, or excess supply in market 2. Similarly, if market 1 is in equilibrium at $\mathbf{p}$, so that $z_1(\mathbf{p}) = 0$, Walras' law ensures that market 2 is also in equilibrium with $z_2(\mathbf{p}) = 0$. Both of these ideas generalise to the case of n markets. Any excess demand in the system of markets must be exactly matched by excess supply of equal value at the given prices somewhere else in the system. Moreover, if at some set of prices $n - 1$ markets are in equilibrium, Walras' law ensures the nth market is also in equilibrium. This is often quite useful to remember.

Now consider a market system described by some excess demand function, $\mathbf{z}(\mathbf{p})$. We know that excess demand in any particular market, $z_k(\mathbf{p})$, may depend on the prices prevailing in *every* market, so that the system of markets is completely interdependent. There is a *partial equilibrium* in the single market k when the quantity of commodity k demanded is equal to the quantity of k supplied at prevailing prices, or when $z_k(\mathbf{p}) = 0$. If, at some prices $\mathbf{p}$, we had $\mathbf{z}(\mathbf{p}) = \mathbf{0}$, or demand equal to supply in *every* market, then we would say that the *system* of markets is in *general equilibrium*. Prices that equate demand and supply in every market are called **Walrasian**.[3]

DEFINITION 5.5 ***Walrasian Equilibrium***

A vector $\mathbf{p}^* \in \mathbb{R}^n_{++}$ *is called a Walrasian equilibrium if* $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$.

We now turn to the question of existence of Walrasian equilibrium. This is indeed an important question because it speaks directly to the logical coherence of Smith's vision of a market economy. One certainly cannot explore sensibly the social and economic properties of equilibria in market economies without full confidence that they exist, and without full knowledge of the circumstances under which they can be expected to exist. This central question in economic theory has attracted the attention of a great many theorists over time. We have mentioned that Walras was the first to attempt an answer to the question of existence by reducing it to a question of whether a system of market demand and market supply equations possessed a solution. However, Walras cannot be credited with providing a satisfactory answer to the question because his conclusion rested on the fallacious assumption that any system of equations with as many unknowns as equations always possesses a solution. Abraham Wald (1936) was the first to point to Walras' error by offering a simple counterexample: the two equations in two unknowns, $x^2 + y^2 = 0$ and $x^2 - y^2 = 1$, have no solution, as you can easily verify. Wald is credited with providing the first mathematically correct proof of existence, but his includes what many would regard as unnecessarily restrictive assumptions on consumers' preferences. In effect, he required that preferences be strongly separable and that every good exhibit 'diminishing marginal utility'. McKenzie (1954) and Arrow and Debreu (1954) were the first to offer significantly more general proofs of existence. Each framed their search for market-clearing prices as the search for a fixed point to a carefully chosen mapping and employed powerful fixed-point theorems to reach their conclusion. In what follows, we too shall employ the fixed-point method to demonstrate existence. However, we encourage the reader to consult both McKenzie (1954) and Arrow and Debreu (1954) for a more general treatment.

We begin by presenting a set of conditions on aggregate excess demand that guarantee a Walrasian equilibrium price vector exists.

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[3]Note that we restrict attention to positive prices. Strictly speaking, there is no reason to do so. However, under our assumption that consumers' utility functions are strongly increasing, aggregate excess demand can be zero only if all prices are positive. See Exercise 5.3.

THEOREM 5.3 ***Aggregate Excess Demand and Walrasian Equilibrium***

Suppose $\mathbf{z}: \mathbb{R}^n_{++} \rightarrow \mathbb{R}^n$ *satisfies the following three conditions:*

1. $\mathbf{z}(\cdot)$ *is continuous on* $\mathbb{R}^n_{++}$*;*
2. $\mathbf{p} \cdot \mathbf{z}(\mathbf{p}) = 0$ *for all* $\mathbf{p} \gg \mathbf{0}$*;*
3. *If* $\{\mathbf{p}^m\}$ *is a sequence of price vectors in* $\mathbb{R}^n_{++}$ *converging to* $\bar{\mathbf{p}} \neq 0$, *and* $\bar{p}_k = 0$ *for some good k, then for some good* k' *with* $\bar{p}_{k'} = 0$, *the associated sequence of excess demands in the market for good* k', $\{z_{k'}(\mathbf{p}^m)\}$, *is unbounded above.*

Then there is a price vector $\mathbf{p}^* \gg \mathbf{0}$ *such that* $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$.

Before giving the proof, let us consider the three conditions in the theorem. The first two are familiar and are guaranteed to hold under the hypotheses of Theorem 5.2. Only the third, rather ominous-looking condition, is new. What it says is actually very easy to understand, however. It says roughly that if the prices of some but not all goods are arbitrarily close to zero, then the (excess) demand for at least one of those goods is arbitrarily high. Put this way, the condition sounds rather plausible. Later, we will show that under Assumption 5.1, condition 3 is satisfied.

Before getting into the proof of the theorem, we remark that it is here where the lack of continuity of consumer demand, and hence aggregate excess demand, on the boundary of the non-negative orthant of prices requires us to do some hard work. In particular, you will note that in a number of places, we take extra care to stay away from that boundary.

Proof: For each good, k, let $\bar{z}_k(\mathbf{p}) = \min(z_k(\mathbf{p}), 1)$ for all $\mathbf{p} \gg \mathbf{0}$, and let $\bar{\mathbf{z}}(\mathbf{p}) = (\bar{z}_1(\mathbf{p}), \ldots, \bar{z}_n(\mathbf{p}))$. Thus, we are assured that $\bar{z}_k(\mathbf{p})$ is bounded above by 1.

Now, fix $\varepsilon \in (0, 1)$, and let

$$S_\varepsilon = \left\{ \mathbf{p} \;\middle|\; \sum_{k=1}^{n} p_k = 1 \text{ and } p_k \geq \frac{\varepsilon}{1+2n} \quad \forall\, k \right\}.$$

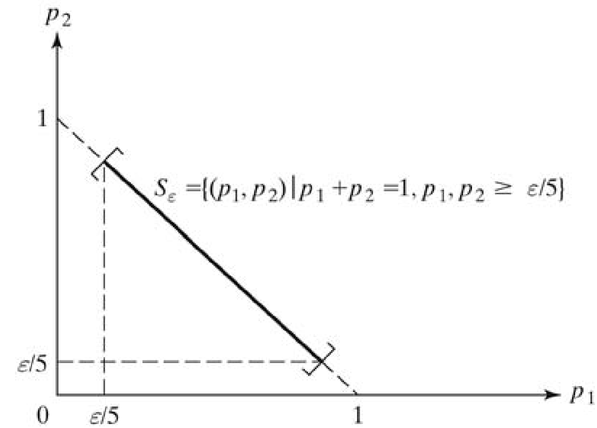
In searching for $\mathbf{p}^*$ satisfying $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$, we shall begin by restricting our search to the set S_ε. It is depicted in Fig. 5.3 for the two-good case. Note how prices on and near the boundary of the non-negative orthant are excluded from S_ε. Note also that as ε is allowed to approach zero, S_ε includes more and more prices. Thus, we can expand the scope of our search by letting ε tend to zero. We shall do so a little later. For now, however, ε remains fixed.

Note the following properties of the set S_ε: it is compact, convex, and non-empty. Compactness follows because it is both closed and bounded (check this), and convexity can be easily checked. To see that it is non-empty, note that the price vector with each component equal to $(2 + 1/n)/(1 + 2n)$ is always a member because $\varepsilon < 1$.

For each good k and every $\mathbf{p} \in S_\varepsilon$, define $f_k(\mathbf{p})$ as follows:

$$f_k(\mathbf{p}) = \frac{\varepsilon + p_k + \max(0, \bar{z}_k(\mathbf{p}))}{n\varepsilon + 1 + \sum_{m=1}^{n} \max(0, \bar{z}_m(\mathbf{p}))},$$

Figure 5.3. The set S_ε in $\mathbb{R}^2_+$.



and let $f(\mathbf{p}) = (f_1(\mathbf{p}), \ldots, f_n(\mathbf{p}))$. Consequently, $\sum_{k=1}^n f_k(\mathbf{p}) = 1$ and $f_k(\mathbf{p}) \geq \varepsilon/(n\varepsilon + 1 + n \cdot 1)$, because $\bar{z}_m(\mathbf{p}) \leq 1$ for each m. Hence, $f_k(\mathbf{p}) \geq \varepsilon/(1 + 2n)$ because $\varepsilon < 1$. Therefore $f\colon S_\varepsilon \to S_\varepsilon$.

Note now that each f_k is continuous on S_ε because, by condition 1 of the statement of the theorem, $z_k(\cdot)$, and therefore $\bar{z}_k(\cdot)$, is continuous on S_ε, so that both the numerator and denominator defining f_k are continuous on S_ε. Moreover, the denominator is bounded away from zero because it always takes on a value of at least 1.

Therefore, f is a continuous function mapping the non-empty, compact, convex set S_ε into itself. We may then appeal to Brouwer's fixed-point theorem (Theorem A1.11) to conclude that there exists $\mathbf{p}^\varepsilon \in S_\varepsilon$ such that $f(\mathbf{p}^\varepsilon) = \mathbf{p}^\varepsilon$, or, equivalently, that $f_k(\mathbf{p}^\varepsilon) = p_k^\varepsilon$ for every $k = 1, 2, \ldots, n$. But this means, using the definition of $f_k(\mathbf{p}^\varepsilon)$ and rearranging, that for every k

$$p_k^\varepsilon\left[n\varepsilon + \sum_{m=1}^n \max(0, \bar{z}_m(\mathbf{p}^\varepsilon))\right] = \varepsilon + \max(0, \bar{z}_k(\mathbf{p}^\varepsilon)). \tag{P.1}$$

So, up to this point, we have shown that for every $\varepsilon \in (0, 1)$ there is a price vector in S_ε satisfying (P.1).

Now allow ε to approach zero and consider the associated sequence of price vectors $\{\mathbf{p}^\varepsilon\}$ satisfying (P.1). Note that the price sequence is bounded, because $\mathbf{p}^\varepsilon \in S_\varepsilon$ implies that the price in every market always lies between zero and one. Consequently, by Theorem A1.8, some subsequence of $\{\mathbf{p}^\varepsilon\}$ must converge. To keep the notation simple, let us suppose that we were clever enough to choose this convergent subsequence right from the start so that $\{\mathbf{p}^\varepsilon\}$ itself converges to $\mathbf{p}^*$, say. Of course, $\mathbf{p}^* \geq \mathbf{0}$ and $\mathbf{p}^* \neq \mathbf{0}$ because its components sum to 1. We argue that in fact, $\mathbf{p}^* \gg \mathbf{0}$. This is where condition 3 enters the picture.

Let us argue by way of contradiction. So, suppose it is not the case that $\mathbf{p}^* \gg \mathbf{0}$. Then for some $\bar{k}$, we must have $p_{\bar{k}}^* = 0$. But condition 3 of the statement of the theorem then implies that there must be some good k' with $p_{k'}^* = 0$ such that $z_{k'}(\mathbf{p}^\varepsilon)$ is unbounded above as ε tends to zero.

But note that because $\mathbf{p}^{\varepsilon} \to \mathbf{p}^*$, $p^*_{k'} = 0$ implies that $p^{\varepsilon}_{k'} \to 0$. Consequently, the left-hand side of (P.1) for $k = k'$ must tend to zero, because the term in square brackets is bounded above by the definition of $\bar{\mathbf{z}}$. However, the right-hand side apparently does not tend to zero, because the unboundedness above of $z_{k'}(\mathbf{p}^{\varepsilon})$ implies that $\bar{z}_{k'}(\mathbf{p}^{\varepsilon})$ assumes its maximum value of 1 infinitely often. Of course, this is a contradiction because the two sides are equal for all values of ε. We conclude, therefore, that $\mathbf{p}^* \gg \mathbf{0}$.

Thus, $\mathbf{p}^{\varepsilon} \to \mathbf{p}^* \gg \mathbf{0}$ as $\varepsilon \to 0$. Because $\bar{\mathbf{z}}(\cdot)$ inherits continuity on $\mathbb{R}^n_{++}$ from $\mathbf{z}(\cdot)$, we may take the limit as $\varepsilon \to 0$ in (P.1) to obtain

$$p^*_k \sum_{m=1}^{n} \max(0, \bar{z}_m(\mathbf{p}^*)) = \max(0, \bar{z}_k(\mathbf{p}^*)) \tag{P.2}$$

for all $k = 1, 2, \ldots, n$. Multiplying both sides by $z_k(\mathbf{p}^*)$ and summing over k yields

$$\mathbf{p}^* \cdot \mathbf{z}(\mathbf{p}^*)\left(\sum_{m=1}^{n} \max(0, \bar{z}_m(\mathbf{p}^*))\right) = \sum_{k=1}^{n} z_k(\mathbf{p}^*) \max(0, \bar{z}_k(\mathbf{p}^*)).$$

Now, condition 2 in the statement of the theorem (Walras' law) says that $\mathbf{p}^*\mathbf{z}(\mathbf{p}^*) = 0$, so we may conclude that the left-hand side and therefore also the right-hand side of the preceding equation is zero. But because the sign of $\bar{z}_k(\mathbf{p}^*)$ is the same as that of $z_k(\mathbf{p}^*)$, the sum on the right-hand side can be zero only if $z_k(\mathbf{p}^*) \leq 0$ for all k. This, together with $\mathbf{p}^* \gg \mathbf{0}$ and Walras' law implies that each $z_k(\mathbf{p}^*) = 0$, as desired. ∎

Thus, as long as on $\mathbb{R}^n_{++}$ aggregate excess demand is continuous, satisfies Walras' law, and is unbounded above as some, but not all, prices approach zero, a Walrasian equilibrium (with the price of every good strictly positive) is guaranteed to exist.

One might be tempted to try to obtain the same result without condition 3 on the unboundedness of excess demand. However, you are asked to show in Exercise 5.7 that the result simply does not hold without it.

We already know that when each consumer's utility function satisfies Assumption 5.1, conditions 1 and 2 of Theorem 5.3 will hold. (This is the content of Theorem 5.2.) It remains to show when condition 3 holds. We do so now.

THEOREM 5.4 ***Utility and Aggregate Excess Demand***

If each consumer's utility function satisfies Assumption 5.1, and if the aggregate endowment of each good is strictly positive (i.e., $\sum_{i=1}^{I} \mathbf{e}^i \gg \mathbf{0}$), then aggregate excess demand satisfies conditions 1 through 3 of Theorem 5.3.

Proof: Conditions 1 and 2 follow from Theorem 5.2. Thus, it remains only to verify condition 3. Consider a sequence of strictly positive price vectors, $\{\mathbf{p}^m\}$, converging to $\bar{\mathbf{p}} \neq \mathbf{0}$, such that $\bar{p}_k = 0$ for some good k. Because $\sum_{i=1}^{I} \mathbf{e}^i \gg \mathbf{0}$, we must have $\bar{\mathbf{p}} \cdot \sum_{i=1}^{I} \mathbf{e}^i > 0$. Consequently, $\bar{\mathbf{p}} \cdot \sum_{i=1}^{I} \mathbf{e}^i = \sum_{i=1}^{I} \bar{\mathbf{p}} \cdot \mathbf{e}^i > 0$, so that there must be at least one consumer i for whom $\bar{\mathbf{p}} \cdot \mathbf{e}^i > 0$.

Consider this consumer i's demand, $\mathbf{x}^i(\mathbf{p}^m, \mathbf{p}^m \cdot \mathbf{e}^i)$, along the sequence of prices. Now, let us suppose, by way of contradiction, that this sequence of demand vectors is bounded. Then, by Theorem A1.8, there must be a convergent subsequence. So we may assume without any loss (by reindexing the subsequence, for example) that the original sequence of demands converges to $\mathbf{x}^*$, say. That is, $\mathbf{x}^i(\mathbf{p}^m, \mathbf{p}^m \cdot \mathbf{e}^i) \to \mathbf{x}^*$.

To simplify the notation, let $\mathbf{x}^m \equiv \mathbf{x}^i(\mathbf{p}^m, \mathbf{p}^m \cdot \mathbf{e}^i)$ for every m. Now, because $\mathbf{x}^m$ maximises u^i subject to i's budget constraint given the prices $\mathbf{p}^m$, and because u^i is strongly (and, therefore, strictly) increasing, the budget constraint must be satisfied with equality. That is,

$$\mathbf{p}^m \cdot \mathbf{x}^m = \mathbf{p}^m \cdot \mathbf{e}^i$$

for every m.

Taking the limit as $m \to \infty$ yields

$$\bar{\mathbf{p}} \cdot \mathbf{x}^* = \bar{\mathbf{p}} \cdot \mathbf{e}^i > 0, \tag{P.1}$$

where the strict inequality follows from our choice of consumer i.

Now let $\hat{\mathbf{x}} = \mathbf{x}^* + (0, \ldots, 0, 1, 0, \ldots, 0)$, where the 1 occurs in the kth position. Then because u^i is strongly increasing on $\mathbb{R}^n_+$,

$$u^i(\hat{\mathbf{x}}) > u^i(\mathbf{x}^*). \tag{P.2}$$

In addition, because $\bar{p}_k = 0$, (P.1) implies that

$$\bar{\mathbf{p}} \cdot \hat{\mathbf{x}} = \bar{\mathbf{p}} \cdot \mathbf{e}^i > 0. \tag{P.3}$$

So, because u^i is continuous, (P.2) and (P.3) imply that there is a $t \in (0, 1)$ such that

$$u^i(t\hat{\mathbf{x}}) > u^i(\mathbf{x}^*),$$
$$\bar{\mathbf{p}} \cdot (t\hat{\mathbf{x}}) < \bar{\mathbf{p}} \cdot \mathbf{e}^i.$$

But because $\mathbf{p}^m \to \bar{\mathbf{p}}$, $\mathbf{x}^m \to \mathbf{x}^*$ and u^i is continuous, this implies that for m large enough,

$$u^i(t\hat{\mathbf{x}}) > u^i(\mathbf{x}^m)$$
$$\mathbf{p}^m \cdot (t\hat{\mathbf{x}}) < \mathbf{p}^m \cdot \mathbf{e}^i,$$

contradicting the fact that $\mathbf{x}^m$ solves the consumer's problem at prices $\mathbf{p}^m$. We conclude therefore that consumer i's sequence of demand vectors must be unbounded.

Now because i's sequence of demand vectors, $\{\mathbf{x}^m\}$, is unbounded yet non-negative, there must be some good k' such that $\{x^m_{k'}\}$ is unbounded above. But because i's income converges to $\bar{\mathbf{p}} \cdot \mathbf{e}^i$, the sequence of i's income $\{\mathbf{p}^m \cdot \mathbf{e}^i\}$ is bounded. (See Exercise 5.8.) Consequently, we must have $p^m_{k'} \to 0$, because this is the only way that the demand for good k' can be unbounded above and affordable. Consequently, $\bar{p}_{k'} = \lim_m p^m_{k'} = 0$.

Finally, note that because the aggregate supply of good k' is fixed and equal to the total endowment of it, and all consumers demand a non-negative amount of good k', the fact that i's demand for good k' is unbounded above implies that the aggregate excess demand for good k' is unbounded above. Consequently, beginning with the assumption that $\mathbf{p}^m \to \bar{\mathbf{p}} \neq \mathbf{0}$ and $\bar{p}_k = 0$ for some k, we have shown that there exists some good k', with $\bar{p}'_k = 0$, such that the aggregate excess demand for good k' is unbounded above along the sequence of prices $\{\mathbf{p}^m\}$, as desired. ∎

We now can state an existence result in terms of the more primitive elements of the model. The next theorem follows directly from Theorems 5.4 and 5.3.

THEOREM 5.5 ***Existence of Walrasian Equilibrium***

If each consumer's utility function satisfies Assumption 5.1, and $\sum_{i=1}^{I} \mathbf{e}^i \gg \mathbf{0}$, *then there exists at least one price vector,* $\mathbf{p}^* \gg \mathbf{0}$, *such that* $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$.

The assumption that utilities be strongly increasing is somewhat restrictive, although it has allowed us to keep the analysis relatively simple. As mentioned earlier, the otherwise very well-behaved Cobb-Douglas functional form of utility is not strongly increasing on $\mathbb{R}^n_+$. You are asked to show in Exercise 5.14 that existence of a Walrasian equilibrium with Cobb-Douglas preferences is nonetheless guaranteed.

When utilities satisfy Assumption 5.1, we know from Theorem 5.2 that the excess demand vector will be homogeneous of degree zero. The behavioural significance of homogeneity is that only relative prices matter in consumers' choices. Thus, if $\mathbf{p}^*$ is a Walrasian equilibrium in such an economy, we will have $\mathbf{z}(\mathbf{p}^*) = \mathbf{z}(\lambda\mathbf{p}^*) = \mathbf{0}$ for all $\lambda > 0$. So, should there exist some set of prices at which all markets clear, those markets will also clear at any other prices obtained by multiplying all prices by any positive constant. This fact often can be exploited to help simplify calculations when solving for Walrasian equilibria.

EXAMPLE 5.1 Let us take a simple two-person economy and solve for a Walrasian equilibrium. Let consumers 1 and 2 have identical CES utility functions,

$$u^i(x_1, x_2) = x_1^\rho + x_2^\rho, \qquad i = 1, 2,$$

where $0 < \rho < 1$. Let there be 1 unit of each good and suppose each consumer owns all of one good, so initial endowments are $\mathbf{e}^1 = (1, 0)$ and $\mathbf{e}^2 = (0, 1)$. Because the aggregate endowment of each good is strictly positive and the CES form of utility is strongly increasing and strictly quasiconcave on $\mathbb{R}^n_+$ when $0 < \rho < 1$, the requirements of Theorem 5.5 are satisfied, so we know a Walrasian equilibrium exists in this economy.

From (E.10) and (E.11) in Example 1.1, consumer i's demand for good j at prices $\mathbf{p}$ will be $x_j^i(\mathbf{p}, y^i) = p_j^{r-1} y^i/(p_1^r + p_2^r)$, where $r \equiv \rho/(\rho - 1)$, and y^i is the consumer's income. Here, income is equal to the market value of the endowment, so $y^1 = \mathbf{p} \cdot \mathbf{e}^1 = p_1$ and $y^2 = \mathbf{p} \cdot \mathbf{e}^2 = p_2$.

Because only *relative* prices matter, and because we know from Theorem 5.5 that there is an equilibrium in which all prices are strictly positive, we can choose a convenient normalisation to simplify calculations. Let $\bar{\mathbf{p}} \equiv (1/p_2)\mathbf{p}$. Here, $\bar{p}_1 \equiv p_1/p_2$ and $\bar{p}_2 \equiv 1$, so $\bar{p}_1$ is just the relative price of the good x_1. Because each consumer's demand at $\mathbf{p}$ is the same as the demand at $\bar{\mathbf{p}}$, we can frame our problem as one of finding an equilibrium set of relative prices, $\bar{\mathbf{p}}$.

Now consider the market for good 1. Assuming an interior solution, equilibrium requires $\bar{\mathbf{p}}^*$ where total quantity demanded equals total quantity supplied, or where

$$x_1^1(\bar{\mathbf{p}}^*, \bar{\mathbf{p}}^* \cdot \mathbf{e}^1) + x_1^2(\bar{\mathbf{p}}^*, \bar{\mathbf{p}}^* \cdot \mathbf{e}^2) = e_1^1 + e_1^2.$$

Substituting from before, this requires

$$\frac{\bar{p}_1^{*r-1}\bar{p}_1^*}{\bar{p}_1^{*r}+1} + \frac{\bar{p}_1^{*r-1}}{\bar{p}_1^{*r}+1} = 1.$$

Solving, we obtain $\bar{p}_1^* = 1$. We conclude that any vector $\mathbf{p}^*$ where $p_1^* = p_2^*$, equates demand and supply in market 1. By Walras' law, those same prices must equate demand and supply in market 2, so we are done. □

5.2.2 EFFICIENCY

We can adapt the Edgeworth box description of a two-person economy to gain useful perspective on the nature of Walrasian equilibrium. Fig. 5.4 represents an economy where preferences satisfy the requirements of Theorem 5.5. Initial endowments are (e_1^1, e_2^1) and (e_1^2, e_2^2), and the box is constructed so these two points coincide at $\mathbf{e}$, as before. At relative prices p_1^*/p_2^*, consumer 1's budget constraint is the straight line through $\mathbf{e}$ when viewed from 1's origin. Facing the same prices, consumer 2's budget constraint will coincide with that same straight line when viewed (upside down) from 2's origin. Consumer 1's most preferred bundle within her budget set is (x_1^1, x_2^1), giving the quantities of each good consumer 1 demands facing prices p_1^*/p_2^* and having income equal to the market value of her endowment, $p_1^*e_1^1 + p_2^*e_2^1$. Similarly, consumer 2's demanded bundle at these same prices with income equal to the value of his endowment is (x_1^2, x_2^2). Equilibrium in the market for good 1 requires $x_1^1 + x_1^2 = e_1^1 + e_1^2$, or that total quantity demanded equal total quantity supplied. This, of course, is equivalent to the requirement $x_1^2 - e_1^2 = e_1^1 - x_1^1$, or that consumer 2's *net* demand be equal to consumer 1's *net* supply of good 1. A similar description of equilibrium in the market for good 2 also can be given.

A little experimentation with different relative prices, and so different budget sets for the two consumers, should convince you that these conditions for market equilibrium will obtain only when the demanded bundles – viewed from the consumers' respective origins – coincide with the same point in the box, as in Fig. 5.4. Because by construction one indifference curve for each consumer passes through every point in the box, and because equilibrium requires the demanded bundles coincide, it is clear that equilibrium

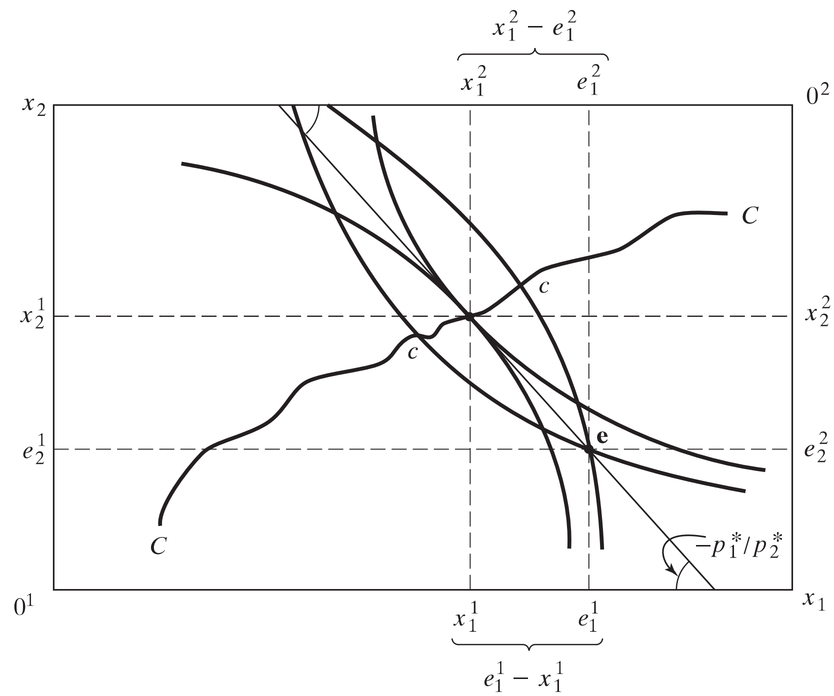


Figure 5.4. Walrasian equilibrium in the Edgeworth box.

will involve *tangency* between the two consumers' indifference curves through their demanded bundles, as illustrated in the figure.

There are several interesting features of Walrasian equilibrium that become immediately apparent with the perspective of the box. First, as we have noted, consumers' supplies and demands depend only on relative prices. Doubling or tripling all prices will not change the consumers' budget sets, so will not change their utility-maximising market behaviour. Second, Fig. 5.4 reinforces our understanding that market equilibrium amounts to the simultaneous compatibility of the actions of independent, decentralised, utility-maximising consumers.

Finally, Fig. 5.4 gives insight into the distributional implications of competitive market equilibrium. We have noted that equilibrium there is characterised by a tangency of the consumers' indifference curves through their respective demanded bundles. These bundles, in turn, give the final amount of each good owned and consumed by the consumer in the market system equilibrium. Thus, having begun with some initial distribution of the goods given by **e**, the maximising actions of self-interested consumers on impersonal markets has led to a redistribution of goods that is both 'inside the lens' formed by the indifference curves of each consumer through their respective endowments and 'on the contract curve'. In the preceding section, we identified allocations such as these as in the 'core' of the economy with endowments **e**. Thus, despite the fact that in the competitive market we have considered here, consumers do not require knowledge of other consumers' preferences or endowments, the allocation resulting from Walrasian equilibrium prices is in the core, at least for the Edgeworth box economy. As we now proceed to show, this remarkable property holds in general. We begin by defining some notation.

DEFINITION 5.6 ***Walrasian Equilibrium Allocations (WEAs)***

Let $\mathbf{p}^*$ be a Walrasian equilibrium for some economy with initial endowments $\mathbf{e}$, and let

$$\mathbf{x}(\mathbf{p}^*) \equiv (\mathbf{x}^1(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^1), \ldots, \mathbf{x}^I(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^I)),$$

where component i gives the n-vector of goods demanded and received by consumer i at prices $\mathbf{p}^*$. Then $\mathbf{x}(\mathbf{p}^*)$ is called a Walrasian equilibrium allocation, or WEA.

Now consider an economy with initial endowments $\mathbf{e}$ and feasible allocations $F(\mathbf{e})$ defined in (5.1). We should note some basic properties of the WEA in such economies. First, it should be obvious that any WEA will be feasible for this economy. Second, Fig. 5.4 makes clear that the bundle received by every consumer in a WEA is the most preferred bundle in that consumer's budget set at the Walrasian equilibrium prices. It therefore follows that any other allocation that is both feasible and preferred by some consumer to their bundle in the WEA must be too expensive for that consumer. Indeed, this would follow even if the price vector were not a Walrasian equilibrium. We record both of these facts as lemmas and leave the proof of the first and part of the proof of the second as exercises.

LEMMA 5.1 *Let $\mathbf{p}^*$ be a Walrasian equilibrium for some economy with initial endowments $\mathbf{e}$. Let $\mathbf{x}(\mathbf{p}^*)$ be the associated WEA. Then $\mathbf{x}(\mathbf{p}^*) \in F(\mathbf{e})$.*

LEMMA 5.2 *Suppose that u^i is strictly increasing on $\mathbb{R}^n_+$, that consumer i's demand is well-defined at $\mathbf{p} \geq \mathbf{0}$ and equal to $\hat{\mathbf{x}}^i$, and that $\mathbf{x}^i \in \mathbb{R}^n_+$.*

i. If $u^i(\mathbf{x}^i) > u^i(\hat{\mathbf{x}}^i)$, then $\mathbf{p} \cdot \mathbf{x}^i > \mathbf{p} \cdot \hat{\mathbf{x}}^i$.

ii. If $u^i(\mathbf{x}^i) \geq u^i(\hat{\mathbf{x}}^i)$, then $\mathbf{p} \cdot \mathbf{x}^i \geq \mathbf{p} \cdot \hat{\mathbf{x}}^i$.

Proof: We leave the first for you to prove as an exercise. So let us suppose that (i) holds. We therefore can employ it to prove (ii).

Suppose, by way of contradiction, that (ii) does not hold. Then $u^i(\mathbf{x}^i) \geq u^i(\hat{\mathbf{x}}^i)$ and $\mathbf{p} \cdot \mathbf{x}^i < \mathbf{p} \cdot \hat{\mathbf{x}}^i$. Consequently, beginning with $\mathbf{x}^i$, we may increase the amount of every good consumed by a small enough amount so that the resulting bundle, $\bar{\mathbf{x}}^i$, remains strictly less expensive than $\hat{\mathbf{x}}^i$. But because u^i is strictly increasing, we then have $u^i(\bar{\mathbf{x}}^i) > u^i(\mathbf{x}^i) \geq u^i(\hat{\mathbf{x}}^i)$, and $\mathbf{p} \cdot \bar{\mathbf{x}}^i < \mathbf{p} \cdot \hat{\mathbf{x}}^i$. But this contradicts (i) with $\mathbf{x}^i$ replaced by $\bar{\mathbf{x}}^i$. ∎

It bears noting, in general, that we have no reason to expect that when WEAs exist, they will be unique. Even in the two-person Edgeworth box economy, it is easy to construct examples where preferences satisfy very ordinary properties yet multiple Walrasian equilibrium allocations exist. Fig. 5.5 illustrates such a case. It seems prudent, therefore, to keep such possibilities in mind and avoid slipping into the belief that Walrasian equilibria

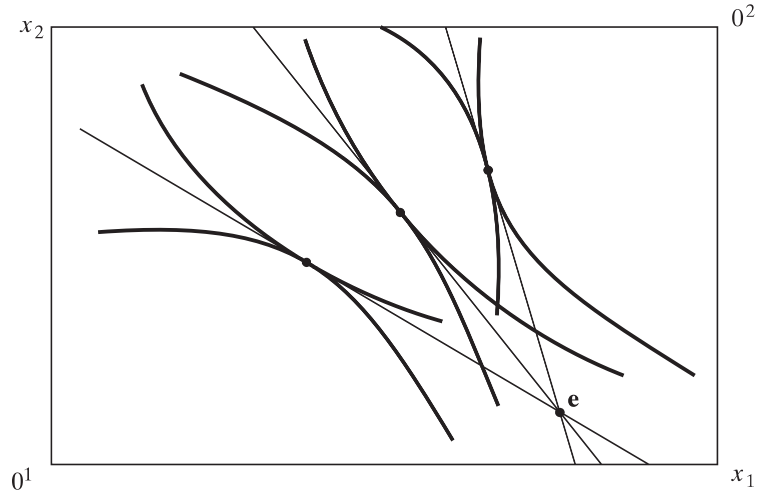


Figure 5.5. Multiple equilibria in a two-person market economy.

are 'usually' unique. As a matter of notation, then, let us give a name to the *set* of WEAs in an economy.

DEFINITION 5.7 ***The Set of WEAs***

For any economy with endowments $\mathbf{e}$, *let* $W(\mathbf{e})$ *denote the set of Walrasian equilibrium allocations.*

We now arrive at the crux of the matter. It is clear in both Figs. 5.4 and 5.5 that the WEAs involve allocations of goods to consumers that lie on the segment *cc* of the contract curve representing the *core* of those economies. It remains to show that WEAs have this property in arbitrary economies. Recall that $C(\mathbf{e})$ denotes the set of allocations in the core.

THEOREM 5.6 ***Core and Equilibria in Competitive Economies***

Consider an exchange economy $(u^i, \mathbf{e}^i)_{i\in\mathcal{I}}$. *If each consumer's utility function,* u^i, *is strictly increasing on* $\mathbb{R}^n_+$, *then every Walrasian equilibrium allocation is in the core. That is,*

$$W(\mathbf{e}) \subset C(\mathbf{e}).$$

Proof: The theorem claims that if $\mathbf{x}(\mathbf{p}^*)$ is a WEA for equilibrium prices $\mathbf{p}^*$, then $\mathbf{x}(\mathbf{p}^*) \in C(\mathbf{e})$. To prove it, suppose $\mathbf{x}(\mathbf{p}^*)$ is a WEA, and assume $\mathbf{x}(\mathbf{p}^*) \notin C(\mathbf{e})$.

Because $\mathbf{x}(\mathbf{p}^*)$ is a WEA, we know from Lemma 5.1 that $\mathbf{x}(\mathbf{p}^*) \in F(\mathbf{e})$, so $\mathbf{x}(\mathbf{p}^*)$ is feasible. However, because $\mathbf{x}(\mathbf{p}^*) \notin C(\mathbf{e})$, we can find a coalition S and another allocation $\mathbf{y}$ such that

$$\sum_{i\in S} \mathbf{y}^i = \sum_{i\in S} \mathbf{e}^i \tag{P.1}$$

and

$$u^i(\mathbf{y}^i) \geq u^i(\mathbf{x}^i(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i)) \qquad \text{for all } i \in S, \tag{P.2}$$

with at least one inequality strict. (P.1) implies

$$\mathbf{p}^* \cdot \sum_{i \in S} \mathbf{y}^i = \mathbf{p}^* \cdot \sum_{i \in S} \mathbf{e}^i. \tag{P.3}$$

Now from (P.2) and Lemma 5.2, we know that for each $i \in S$, we must have

$$\mathbf{p}^* \cdot \mathbf{y}^i \geq \mathbf{p}^* \cdot \mathbf{x}^i(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i) = \mathbf{p}^* \cdot \mathbf{e}^i, \tag{P.4}$$

with at least one inequality strict. Summing over all consumers in S, we obtain

$$\mathbf{p}^* \cdot \sum_{i \in S} \mathbf{y}^i > \mathbf{p}^* \cdot \sum_{i \in S} \mathbf{e}^i,$$

contradicting (P.3). Thus, $\mathbf{x}(\mathbf{p}^*) \in C(\mathbf{e})$ and the theorem is proved. ∎

Note that as a corollary to Theorem 5.5, we immediately have a result on the non-emptiness of the core. That is, under the conditions of Theorem 5.5, a Walrasian equilibrium allocation exists, and by Theorem 5.6, this allocation is in the core. Hence, the conditions of Theorem 5.5 guarantee that the core is non-empty.

Before moving on, we pause to consider what we have shown here. In a Walrasian equilibrium, each consumer acts completely independently of all other consumers in the sense that he simply chooses to demand a bundle that maximises his utility given the prevailing prices and given his income determined by the value of his endowment. In particular, he does not consider the amount demanded by others or the total amount supplied of any good. He knows only his own preferences and the prices at which he can carry out transactions.

Contrast this with the story of pure barter exchange with which we began the chapter. There, it was crucial that consumers actually could get together, take stock of the total resources available to them, and then exploit all potential gains from trade. In particular, each consumer would have to be keenly aware of when a mutually beneficial trade could be made with some other consumer – *any* other consumer! As we remarked earlier, it would be astonishing if such complete coordination could be even approximated, let alone achieved in practice. And even if it could be approximated, it would appear to require the aid of some central authority charged with coordinating the appropriate coalitions and trades.

But we have now shown in Theorem 5.6 that it is possible to achieve outcomes in the core *without* the aid of a central planner. Indeed, no one in our competitive economy requires direction or advice from anyone else. Each consumer simply observes the prices and places his utility-maximising demands and supplies on the market. In this sense, the competitive market mechanism is said to be **decentralised**.

Note, in particular, that because all core allocations are Pareto efficient, so, too, must be all Walrasian equilibrium allocations. Although we have proven more, this alone is quite remarkable. Imagine being charged with allocating all the economy's resources, so that in the end, the allocation is Pareto efficient. To keep you from giving all the resources to one person, let us also insist that in the end, every consumer must be at least as well off as they would have been just consuming their endowment. Think about how you might accomplish this. You might start by trying to gather information about the preferences of all consumers in the economy. (What a task that would be!) Only then could you attempt to redistribute goods in a manner that left no further gains from trade. As incredibly difficult as this task is, the competitive market mechanism achieves it, and more. To emphasise the fact that competitive outcomes are Pareto efficient, we state it as a theorem, called the *First Welfare Theorem*.

THEOREM 5.7 ***First Welfare Theorem***

Under the hypotheses of Theorem 5.6, every Walrasian equilibrium allocation is Pareto efficient.

Proof: The proof follows immediately from Theorem 5.6 and the observation that all core allocations are Pareto efficient. ∎

Theorem 5.7 provides some specific support for Adam Smith's contention that society's interests are served by an economic system where self-interested actions of individuals are mediated by impersonal markets. If conditions are sufficient to ensure that Walrasian equilibria exist, then regardless of the initial allocation of resources, the allocation realised in market equilibrium will be Pareto efficient.

It is extremely important to appreciate the scope of this aspect of competitive market systems. It is equally important to realise its limitations and to resist the temptation to read more into what we have shown than is justified. Nothing we have argued so far should lead us to believe that WEAs are necessarily 'socially optimal' if we include in our notion of social optimality any consideration for matters of 'equity' or 'justice' in distribution. Most would agree that an allocation that is not Pareto efficient is not even a candidate for the socially best, because it would always be possible to redistribute goods and make someone better off and no one worse off. At the same time, few could argue persuasively that every Pareto-efficient distribution has an equal claim to being considered the best or 'most just' from a social point of view.

In a later chapter, we give fuller consideration to normative issues such as these. For now, a simple example will serve to illustrate the distinction. Consider an economy with total endowments given by the dimensions of the Edgeworth box in Fig. 5.6. Suppose by some unknown means society has identified the distribution $\bar{\mathbf{x}}$ as the socially best. Suppose, in addition, that initial endowments are given by the allocation $\mathbf{e}$. Theorem 5.6 tells us that an equilibrium allocation under a competitive market system will be some allocation in $C(\mathbf{e})$, such as $\mathbf{x}'$, which in this case is quite distinct from $\bar{\mathbf{x}}$. Thus, while competitive market systems can improve on an initial distribution that is not itself Pareto efficient, there is no

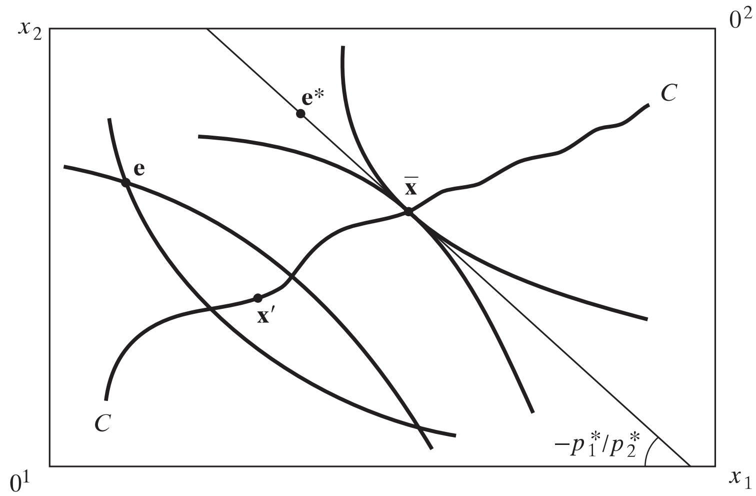


Figure 5.6. Efficiency and social optimality in a two-person economy.

assurance a competitive system, by itself, will lead to a final distribution that society as a whole views as best.

Before we become unduly pessimistic, let us consider a slightly different question. If by some means, we can determine the allocation we would like to see, can the power of a decentralised market system be used to achieve it? From Fig. 5.6, it seems this should be so. If initial endowments could be redistributed to $\mathbf{e}^*$, it is clear that $\bar{\mathbf{x}}$ is the allocation that would be achieved in competitive equilibrium with those endowments and prices $\mathbf{p}^*$.

In fact, this is an example of a rather general principle. It can be shown that under certain conditions, *any* Pareto-efficient allocation can be achieved by competitive markets and *some* initial endowments. This result is called the *Second Welfare Theorem*.

THEOREM 5.8 ***Second Welfare Theorem***

Consider an exchange economy $(u^i, \mathbf{e}^i)_{i\in\mathcal{I}}$ *with aggregate endowment* $\sum_{i=1}^{I} \mathbf{e}^i \gg \mathbf{0}$, *and with each utility function* u^i *satisfying Assumption 5.1. Suppose that* $\bar{\mathbf{x}}$ *is a Pareto-efficient allocation for* $(u^i, \mathbf{e}^i)_{i\in\mathcal{I}}$, *and that endowments are redistributed so that the new endowment vector is* $\bar{\mathbf{x}}$. *Then* $\bar{\mathbf{x}}$ *is a Walrasian equilibrium allocation of the resulting exchange economy* $(u^i, \bar{\mathbf{x}}^i)_{i\in\mathcal{I}}$.

Proof: Because $\bar{\mathbf{x}}$ is Pareto efficient, it is feasible. Hence, $\sum_{i=1}^{I} \bar{\mathbf{x}}^i = \sum_{i=1}^{I} \mathbf{e}^i \gg \mathbf{0}$. Consequently, we may apply Theorem 5.5 to conclude that the exchange economy $(u^i, \bar{\mathbf{x}}^i)_{i\in\mathcal{I}}$ possesses a Walrasian equilibrium allocation $\hat{\mathbf{x}}$. It only remains to show that $\hat{\mathbf{x}} = \bar{\mathbf{x}}$.

Now in the Walrasian equilibrium, each consumer's demand is utility maximising subject to her budget constraint. Consequently, because i demands $\hat{\mathbf{x}}^i$, and has endowment

$\bar{\mathbf{x}}^i$, we must have

$$u^i(\hat{\mathbf{x}}^i) \geq u^i(\bar{\mathbf{x}}^i) \qquad \text{for all } i \in \mathcal{I}. \tag{P.1}$$

But because $\hat{\mathbf{x}}$ is an equilibrium allocation, it must be feasible for the economy $(u^i, \bar{\mathbf{x}}^i)_{i\in\mathcal{I}}$. Consequently, $\sum_{i=1}^{I} \hat{\mathbf{x}}^i = \sum_{i=1}^{I} \bar{\mathbf{x}}^i = \sum_{i=1}^{I} \mathbf{e}^i$, so that $\hat{\mathbf{x}}$ is feasible for the original economy as well.

Thus, by (P.1), $\hat{\mathbf{x}}$ is feasible for the original economy and makes no consumer worse off than the Pareto-efficient (for the original economy) allocation $\bar{\mathbf{x}}$. Therefore, $\hat{\mathbf{x}}$ cannot make anyone strictly better off; otherwise, $\bar{\mathbf{x}}$ would not be Pareto efficient. Hence, every inequality in (P.1) must be an equality.

To see now that $\hat{\mathbf{x}}^i = \bar{\mathbf{x}}^i$ for every i, note that if for some consumer this were not the case, then in the Walrasian equilibrium of the new economy, that consumer could afford the average of the bundles $\hat{\mathbf{x}}^i$ and $\bar{\mathbf{x}}^i$ and strictly increase his utility (by strict quasiconcavity), contradicting the fact that $\hat{\mathbf{x}}^i$ is utility-maximising in the Walrasian equilibrium. ∎

One can view the Second Welfare Theorem as an affirmative answer to the following question: is a system that depends on decentralised, self-interested decision making by a large number of consumers capable of sustaining the socially 'best' allocation of resources, if we could just agree on what that was? Under the conditions stated before, the Second Welfare Theorem says yes, as long as socially 'best' requires, at least, Pareto efficiency.

Although we did not explicitly mention prices in the statement of the Second Welfare Theorem, or in its proof, they are there in the background. Specifically, the theorem says that there are Walrasian equilibrium prices, $\bar{\mathbf{p}}$, such that when the endowment allocation is $\bar{\mathbf{x}}$, each consumer i will maximise $u^i(\mathbf{x}^i)$ subject to $\bar{\mathbf{p}} \cdot \mathbf{x}^i \leq \bar{\mathbf{p}} \cdot \bar{\mathbf{x}}^i$ by choosing $\mathbf{x}^i = \bar{\mathbf{x}}^i$. Because of this, the prices $\bar{\mathbf{p}}$ are sometimes said to *support* the allocation $\bar{\mathbf{x}}$.

We began discussing the Second Welfare Theorem by asking whether redistribution to a point like $\mathbf{e}^*$ in Fig. 5.6 could yield the allocation $\bar{\mathbf{x}}$ as a WEA. In the theorem, we showed that the answer is yes if endowments were redistributed to $\bar{\mathbf{x}}$ itself. It should be clear from Fig. 5.6, however, that $\bar{\mathbf{x}}$ in fact will be a WEA for market prices $\bar{\mathbf{p}}$ under a redistribution of initial endowments to *any* point along the price line through $\bar{\mathbf{x}}$, including, of course, to $\mathbf{e}^*$. This same principle applies generally, so we have an immediate corollary to Theorem 5.8. The proof is left as an exercise.

COROLLARY 5.1 ***Another Look at the Second Welfare Theorem***

Under the assumptions of the preceding theorem, if $\bar{\mathbf{x}}$ is Pareto efficient, then $\bar{\mathbf{x}}$ is a WEA for some Walrasian equilibrium $\bar{\mathbf{p}}$ after redistribution of initial endowments to any allocation $\mathbf{e}^* \in F(\mathbf{e})$, such that $\bar{\mathbf{p}} \cdot \mathbf{e}^{*i} = \bar{\mathbf{p}} \cdot \bar{\mathbf{x}}^i$ for all $i \in \mathcal{I}$.

5.3 EQUILIBRIUM IN PRODUCTION

Now we expand our description of the economy to include production as well as consumption. We will find that most of the important properties of competitive market systems uncovered earlier continue to hold. However, production brings with it several new issues that must be addressed.

For example, the profits earned by firms must be distributed back to the consumers who own them. Also, in a single firm, the distinction between what constitutes an input and what constitutes an output is usually quite clear. This distinction becomes blurred when we look *across* firms and view the production side of the economy as a whole. An input for one firm may well be the output of another. To avoid hopelessly entangling ourselves in notation, it seems best to resist making any a priori distinctions between inputs and outputs and instead let the distinction depend on the context. Thus, we will view every type of good or service in a neutral way as just a different kind of *commodity*. We will suppose throughout that there is a fixed and finite number n of such commodities. In the case of producers, we will then adopt simple sign conventions to distinguish inputs from outputs in any particular context.

Again, we formalise the competitive structure of the economy by supposing consumers act to maximise utility subject to their budget constraints and that firms seek to maximise profit. Both consumers and firms are price takers.

5.3.1 PRODUCERS

To describe the production sector, we suppose there is a fixed number J of firms that we index by the set

$$\mathcal{J} = \{1, \ldots, J\}.$$

We now let $\mathbf{y}^j \in \mathbb{R}^n$ be a production plan for some firm, and observe the convention of writing $y_k^j < 0$ if commodity k is an input used in the production plan and $y_k^j > 0$ if it is an output produced from the production plan. If, for example, there are two commodities and $\mathbf{y}^j = (-7, 3)$, then the production plan requires 7 units of commodity one as an input, to produce 3 units of commodity two as an output.

To summarise the technological possibilities in production, we return to the most general description of the firm's technology, first encountered in Section 3.2, and suppose each firm possesses a production possibility set, $Y^j, j \in \mathcal{J}$. We make the following assumptions on production possibility sets.

ASSUMPTION 5.2 ***The Individual Firm***

1. $\mathbf{0} \in Y^j \subseteq \mathbb{R}^n$.
2. Y^j *is closed and bounded.*
3. Y^j *is strongly convex. That is, for all distinct* $\mathbf{y}^1, \mathbf{y}^2 \in Y^j$ *and all* $t \in (0, 1)$*, there exists* $\bar{\mathbf{y}} \in Y^j$ *such that* $\bar{\mathbf{y}} \geq t\mathbf{y}^1 + (1-t)\mathbf{y}^2$ *and equality does not hold.*

The first of these guarantees firm profits are bounded from below by zero, and the second that production of output always requires some inputs. The closedness part of the second condition imposes continuity. It says that the limits of possible production plans are themselves possible production plans. The boundedness part of this condition is very restrictive and is made only to keep the analysis simple to follow. Do not be tempted into thinking that it merely expresses the idea that resources are limited. For the time being, regard it as a simplifying yet dispensable assumption. We shall discuss the importance of removing this assumption a little later. The third assumption, strong convexity, is new. Unlike all the others, which are fairly weak restrictions on the technology, strong convexity is a more demanding requirement. In effect, strong convexity rules out constant and increasing returns to scale in production and ensures that the firm's profit-maximising production plan is unique. Although Assumption 5.2 does not impose it, all of our results to follow are consistent with the assumption of 'no free production' (i.e., $Y^j \cap \mathbb{R}^n_+ = \{\mathbf{0}\}$).

Each firm faces fixed commodity prices $\mathbf{p} \geq \mathbf{0}$ and chooses a production plan to maximise profit. Thus, each firm solves the problem

$$\max_{\mathbf{y}^j \in Y^j} \mathbf{p} \cdot \mathbf{y}^j \tag{5.3}$$

Note how our sign convention ensures that inputs are accounted for in profits as costs and outputs as revenues. Because the objective function is continuous and the constraint set closed and bounded, a maximum of firm profit will exist. So, for all $\mathbf{p} \geq \mathbf{0}$ let

$$\Pi^j(\mathbf{p}) \equiv \max_{\mathbf{y}^j \in Y^j} \mathbf{p} \cdot \mathbf{y}^j$$

denote firm j's profit function. By Theorem A2.21 (the theorem of the maximum), $\Pi^j(\mathbf{p})$ is continuous on $\mathbb{R}^n_+$. As you are asked to show in Exercise 5.23, strong convexity ensures that the profit-maximising production plan, $\mathbf{y}^j(\mathbf{p})$, will be unique whenever $\mathbf{p} \gg \mathbf{0}$. Finally, from Theorem A2.21 (the theorem of the maximum), $\mathbf{y}^j(\mathbf{p})$ will be continuous on $\mathbb{R}^n_{++}$. Note that for $\mathbf{p} \gg \mathbf{0}$, $\mathbf{y}^j(\mathbf{p})$ is a vector-valued function whose components are the firm's output supply and input demand functions. However, we often simply refer to $\mathbf{y}^j(\mathbf{p})$ as firm j's supply function. We record these properties for future reference.

THEOREM 5.9 ***Basic Properties of Supply and Profits***

If Y^j satisfies conditions 1 through 3 of Assumption 5.2, then for every price $\mathbf{p} \gg \mathbf{0}$, the solution to the firm's problem (5.3) is unique and denoted by $\mathbf{y}^j(\mathbf{p})$. Moreover, $\mathbf{y}^j(\mathbf{p})$ is continuous on $\mathbb{R}^n_{++}$. In addition, $\Pi^j(\mathbf{p})$ is well-defined and continuous on $\mathbb{R}^n_+$.

Finally, note that maximum firm profits are homogeneous of degree 1 in the vector of commodity prices. Each output supply and input demand function will be homogeneous of degree zero in prices. (See Theorems 3.7 and 3.8.)

Next we consider *aggregate* production possibilities economy-wide. We suppose there are no externalities in production between firms, and define the aggregate production possibilities set,

$$Y \equiv \left\{ \mathbf{y} \mid \mathbf{y} = \sum_{j \in \mathcal{J}} \mathbf{y}^j, \quad \text{where } \mathbf{y}^j \in Y^j \right\}.$$

The set Y will inherit all the properties of the individual production sets, and we take note of that formally.

THEOREM 5.10 ***Properties of*** Y

If each Y^j satisfies Assumption 5.2, then the aggregate production possibility set, Y, also satisfies Assumption 5.2.

We shall leave the proof of this as an exercise. Conditions 1, 3, and the boundedness of Y follow directly from those properties of the Y^j. The closedness of Y does not follow simply from the closedness of the individual Y^j's. However, under our additional assumption that the Y^j's are bounded, Y can be shown to be closed.

Now consider the problem of maximising *aggregate* profits. Under Theorem 5.10, a maximum of $\mathbf{p} \cdot \mathbf{y}$ over the aggregate production set Y will exist and be unique when $\mathbf{p} \gg \mathbf{0}$. In addition, the aggregate profit-maximising production plan $\mathbf{y}(\mathbf{p})$ will be a continuous function of $\mathbf{p}$. Moreover, we note the close connection between aggregate profit-maximising production plans and individual firm profit-maximising production plans.

THEOREM 5.11 ***Aggregate Profit Maximisation***

For any prices $\mathbf{p} \geq \mathbf{0}$, we have

$$\mathbf{p} \cdot \bar{\mathbf{y}} \geq \mathbf{p} \cdot \mathbf{y} \qquad \textit{for all} \quad \mathbf{y} \in Y$$

if and only if for some $\bar{\mathbf{y}}^j \in Y^j, j \in \mathcal{J}$, we may write $\bar{\mathbf{y}} = \sum_{j \in \mathcal{J}} \bar{\mathbf{y}}^j$, and

$$\mathbf{p} \cdot \bar{\mathbf{y}}^j \geq \mathbf{p} \cdot \mathbf{y}^j \qquad \textit{for all} \quad \mathbf{y}^j \in Y^j, \ j \in \mathcal{J}.$$

In words, the theorem says that $\bar{\mathbf{y}} \in Y$ maximises aggregate profit if and only if it can be decomposed into individual firm profit-maximising production plans. The proof is straightforward.

Proof: Let $\bar{\mathbf{y}} \in Y$ maximise aggregate profits at price $\mathbf{p}$. Suppose that $\bar{\mathbf{y}} \equiv \sum_{j \in \mathcal{J}} \bar{\mathbf{y}}^j$ for $\bar{\mathbf{y}}^j \in Y^j$. If $\bar{\mathbf{y}}^k$ does not maximise profits for firm k, then there exists some other $\tilde{\mathbf{y}}^k \in Y^k$ that gives firm k higher profits. But then the aggregate production vector $\tilde{\mathbf{y}} \in Y$ composed of $\tilde{\mathbf{y}}^k$ and the sum of the $\bar{\mathbf{y}}^j$ for $j \neq k$ must give higher aggregate profits than the aggregate vector $\bar{\mathbf{y}}$, contradicting the assumption that $\bar{\mathbf{y}}$ maximises aggregate profits at price $\mathbf{p}$.

Next, suppose feasible production plans $\bar{\mathbf{y}}^1, \ldots, \bar{\mathbf{y}}^j$ maximise profits at price $\mathbf{p}$ for the individual firms in $\mathcal{J}$. Then

$$\mathbf{p} \cdot \bar{\mathbf{y}}^j \geq \mathbf{p} \cdot \mathbf{y}^j \qquad \text{for } \mathbf{y}^j \in Y^j \text{ and } j \in \mathcal{J}.$$

Summing over all firms yields

$$\sum_{j \in \mathcal{J}} \mathbf{p} \cdot \bar{\mathbf{y}}^j \geq \sum_{j \in \mathcal{J}} \mathbf{p} \cdot \mathbf{y}^j \qquad \text{for } \mathbf{y}^j \in Y^j \text{ and } j \in \mathcal{J}.$$

Rearranging, we can write this as

$$\mathbf{p} \cdot \sum_{j \in \mathcal{J}} \bar{\mathbf{y}}^j \geq \mathbf{p} \cdot \sum_{j \in \mathcal{J}} \mathbf{y}^j \qquad \text{for } \mathbf{y}^j \in Y^j \text{ and } j \in \mathcal{J}.$$

But from the definitions of $\bar{\mathbf{y}}$ and Y, this just says

$$\mathbf{p} \cdot \bar{\mathbf{y}} \geq \mathbf{p} \cdot \mathbf{y} \qquad \text{for } \mathbf{y} \in Y,$$

so $\bar{\mathbf{y}}$ maximises aggregate profits at price $\mathbf{p}$, completing the proof. ∎

5.3.2 CONSUMERS

Formally, the description of consumers is just as it has always been. However, we need to modify some of the details to account for the distribution of firm *profits* because firms are owned by consumers. As before, we let

$$\mathcal{I} \equiv \{1, \ldots, I\}$$

index the set of consumers and let u^i denote i's utility function over the consumption set $\mathbb{R}^n_+$.

Before continuing, note that our assumption that consumer bundles are non-negative does not preclude the possibility that consumers supply goods and services to the market. Indeed, labour services are easily included by endowing the consumer with a fixed number of hours that are available for consumption. Those that are not consumed as 'leisure' are then supplied as labour services. If the consumer's only source of income is his endowment, then just as before, whether a consumer is a net demander or supplier of a good depends upon whether his (total) demand falls short of or exceeds his endowment of that good.

Of course, we must here also take account of the fact that consumers receive income from the profit earned by firms they own. In a *private ownership economy*, which we shall consider here, consumers own shares in firms and firm profits are distributed to shareholders. Consumer i's shares in firm j entitle him to some proportion $0 \leq \theta^{ij} \leq 1$ of the profits of firm j. Of course, these shares, summed over all consumers in the economy, must sum to 1. Thus,

$$0 \leq \theta^{ij} \leq 1 \qquad \text{for all } i \in \mathcal{I} \text{ and } j \in J,$$

where

$$\sum_{i \in \mathcal{I}} \theta^{ij} = 1 \qquad \text{for all } j \in J.$$

In our economy with production and private ownership of firms, a consumer's income can arise from two sources – from selling an endowment of commodities already owned, and from shares in the profits of any number of firms. If $\mathbf{p} \geq \mathbf{0}$ is the vector of market prices, one for each commodity, the consumer's budget constraint is

$$\mathbf{p} \cdot \mathbf{x}^i \leq \mathbf{p} \cdot \mathbf{e}^i + \sum_{j \in J} \theta^{ij} \Pi^j(\mathbf{p}). \tag{5.4}$$

By letting $m^i(\mathbf{p})$ denote the right-hand side of (5.4), the consumer's problem is

$$\max_{\mathbf{x}^i \in \mathbb{R}^n_+} u^i(\mathbf{x}^i) \qquad \text{s.t.} \qquad \mathbf{p} \cdot \mathbf{x}^i \leq m^i(\mathbf{p}). \tag{5.5}$$

Now, under Assumption 5.2, each firm will earn non-negative profits because each can always choose the zero production vector. Consequently, $m^i(\mathbf{p}) \geq 0$ because $\mathbf{p} \geq \mathbf{0}$ and $\mathbf{e}^i \geq \mathbf{0}$. Therefore, under Assumptions 5.1 and 5.2, a solution to (5.5) will exist and be unique whenever $\mathbf{p} \gg \mathbf{0}$. Again, we denote it by $\mathbf{x}^i(\mathbf{p}, m^i(\mathbf{p}))$, where $m^i(\mathbf{p})$ is just the consumer's income.

Recall from Chapter 1 that under the assumptions we made there (and also here), $\mathbf{x}^i(\mathbf{p}, y)$ is continuous in $(\mathbf{p}, y) \in \mathbb{R}^n_{++} \times \mathbb{R}^n_+$. Consequently, as long as $m^i(\mathbf{p})$ is continuous in $\mathbf{p}$, $\mathbf{x}^i(\mathbf{p}, m^i(\mathbf{p}))$ will be continuous in $\mathbf{p}$. By appealing to Theorem 5.9, we see that $m_i(\mathbf{p})$ is continuous on $\mathbb{R}^n_+$ under Assumption 5.2. Putting this all together we have the following theorem.

THEOREM 5.12 ***Basic Property of Demand with Profit Shares***

If each Y^j satisfies Assumption 5.2 and if u^i satisfies Assumption 5.1, then a solution to the consumer's problem (5.5) exists and is unique for all $\mathbf{p} \gg \mathbf{0}$. Denoting it by $\mathbf{x}^i(\mathbf{p}, m^i(\mathbf{p}))$, we have furthermore that $\mathbf{x}^i(\mathbf{p}, m^i(\mathbf{p}))$ is continuous in $\mathbf{p}$ on $\mathbb{R}^n_{++}$. In addition, $m_i(\mathbf{p})$ is continuous on $\mathbb{R}^n_+$.

This completes the description of the economy. Altogether, we can represent it as the collection $(u^i, \mathbf{e}^i, \theta^{ij}, Y^j)_{i \in \mathcal{I}, j \in \mathcal{J}}$.

5.3.3 EQUILIBRIUM

As in the case with no production, we can again define a real-valued aggregate excess demand function for each commodity market and a vector-valued aggregate excess demand function for the economy as a whole. Aggregate excess demand for commodity k is

$$z_k(\mathbf{p}) \equiv \sum_{i\in\mathcal{I}} x_k^i(\mathbf{p}, m^i(\mathbf{p})) - \sum_{j\in J} y_k^j(\mathbf{p}) - \sum_{i\in\mathcal{I}} e_k^i,$$

and the aggregate excess demand vector is

$$\mathbf{z}(\mathbf{p}) \equiv (z_1(\mathbf{p}), \ldots, z_n(\mathbf{p})).$$

As before (see Definition 5.5), a Walrasian equilibrium price vector $\mathbf{p}^* \gg \mathbf{0}$ clears all markets. That is, $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$.

THEOREM 5.13 ***Existence of Walrasian Equilibrium with Production***

Consider the economy $(u^i, \mathbf{e}^i, \theta^{ij}, Y^j)_{i\in\mathcal{I}, j\in\mathcal{J}}$. If each u^i satisfies Assumption 5.1, each Y^j satisfies Assumption 5.2, and $\mathbf{y} + \sum_{i\in\mathcal{I}} \mathbf{e}^i \gg \mathbf{0}$ for some aggregate production vector $\mathbf{y} \in \sum_{j\in\mathcal{J}} Y^j$, then there exists at least one price vector $\mathbf{p}^* \gg \mathbf{0}$, such that $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$.

Recall that when there was no production, we required the aggregate endowment vector to be strictly positive to guarantee existence. With production, that condition can be weakened to requiring that there is a feasible production vector for this economy whose net result is a strictly positive amount of every good (i.e., $\mathbf{y} + \sum_{i\in\mathcal{I}} \mathbf{e}^i \gg \mathbf{0}$ for some aggregate production vector $\mathbf{y}$).

Proof: We shall get the proof started, and leave the rest for you to complete as an exercise. The idea is to show that under the assumptions above, the aggregate excess demand function satisfies the conditions of Theorem 5.3. Because production sets are bounded and consumption is non-negative, this reduces to showing that some consumer's demand for some good is unbounded as some, but not all, prices approach zero. (However, you should check even this logic as you complete the proof for yourself.) Therefore, we really need only mimic the proof of Theorem 5.4.

So, consider a sequence of strictly positive price vectors, $\{\mathbf{p}^m\}$, converging to $\bar{\mathbf{p}} \neq \mathbf{0}$, such that $\bar{p}_k = 0$ for some good k. We would like to show that for some, possibly other, good k' with $\bar{p}_{k'} = 0$, the sequence $\{z_{k'}(\mathbf{p}^m)\}$, of excess demands for good k' is unbounded.

Recall that our first step in the proof of Theorem 5.4 was to identify a consumer whose income was strictly positive at the limit price vector $\bar{\mathbf{p}}$. This is where we shall use the new condition on net aggregate production.

Because $\mathbf{y} + \sum_{i=1}^{I} \mathbf{e}^i \gg \mathbf{0}$ for some aggregate production vector $\mathbf{y}$, and because the non-zero price vector $\bar{\mathbf{p}}$ has no negative components, we must have $\bar{\mathbf{p}} \cdot (\mathbf{y} + \sum_{i=1}^{I} \mathbf{e}^i) > 0$.

Consequently, recalling that both $m^i(\mathbf{p})$ and $\Pi^j(\mathbf{p})$ are well-defined for all $\mathbf{p} \geq \mathbf{0}$,

$$\begin{aligned}\sum_{i\in\mathcal{I}} m^i(\bar{\mathbf{p}}) &= \sum_{i\in\mathcal{I}}\left(\bar{\mathbf{p}}\cdot\mathbf{e}^i + \sum_{j\in\mathcal{J}}\theta^{ij}\Pi^j(\bar{\mathbf{p}})\right)\\ &= \sum_{i\in\mathcal{I}}\bar{\mathbf{p}}\cdot\mathbf{e}^i + \sum_{j\in\mathcal{J}}\Pi^j(\bar{\mathbf{p}})\\ &\geq \sum_{i\in\mathcal{I}}\bar{\mathbf{p}}\cdot\mathbf{e}^i + \bar{\mathbf{p}}\cdot\mathbf{y}\\ &= \bar{\mathbf{p}}\cdot\left(\mathbf{y}+\sum_{i=1}^{I}\mathbf{e}^i\right)\\ &> 0,\end{aligned}$$

where the first equality follows by the definition of $m^i(\bar{\mathbf{p}})$, the second follows because total non-endowment income is simply aggregate profits, and the weak inequality follows from Theorem 5.11, which ensures that the sum of individual firm maximised profits must be at least as large as maximised aggregate profits and hence aggregate profits from $\mathbf{y}$. Therefore, there must exist at least one consumer whose income at prices $\bar{\mathbf{p}}$, $m^i(\bar{\mathbf{p}})$, is strictly positive. The rest of the proof proceeds now as in the proof of Theorem 5.4, and we leave it for you to complete as an exercise. (You will need to use the result noted in Theorem 5.12 that $m^i(\mathbf{p})$ is continuous on $\mathbb{R}^n_+$.) ∎

As before, because excess demand is homogeneous of degree zero, when Walrasian equilibrium prices exist, they will not be unique. Also, note that once again the assumption that each u^i is strongly increasing (and strictly quasiconcave) on all of $\mathbb{R}^n_+$ rules out Cobb-Douglas utility functions. However, you are asked to show in Exercise 5.14 that, under Assumption 5.2 on the production sector, the aggregate excess demand function nonetheless satisfies all the conditions of Theorem 5.3 even when utilities are of the Cobb-Douglas form.

EXAMPLE 5.2 In the classic Robinson Crusoe economy, all production and all consumption is carried out by a single consumer. Robinson the consumer sells his labour time h (in hours) to Robinson the producer, who in turn uses the consumer's labour services for that amount of time to produce coconuts, y, which he then sells to Robinson the consumer. All profits from the production and sale of coconuts are distributed to Robinson the consumer.

With only one firm, the production possibility set for the firm and the economy coincide. Let that set be

$$Y = \{(-h, y) \mid 0 \leq h \leq b, \text{ and } 0 \leq y \leq h^\alpha\},$$

where $b > 0$, and $\alpha \in (0, 1)$.

So, for example, the production vector $(-2, 2^{\alpha})$ is in the production set, which means that it is possible to produce 2^{α} coconuts by using 2 hours of Robinson's time.

The set Y is illustrated in Fig. 5.7(a), and it is easy to verify that it satisfies all the requirements of Assumption 5.2. Note that parameter b serves to bound the production set. Because this bound is present for purely technical purposes, do not give it much thought. In a moment, we will choose it to be large enough so that it is irrelevant.

As usual, the consumption set for Robinson the consumer is just the non-negative orthant, which in this two-good case is $\mathbb{R}^2_+$. Robinson's utility function is

$$u(h, y) = h^{1-\beta} y^{\beta},$$

where $\beta \in (0, 1)$. Here, h denotes the number of hours consumed by Robinson (leisure, if you will), and y denotes the number of coconuts consumed. We will suppose that Robinson is endowed with $T > 0$ units of h (i.e., T hours), and no coconuts. That is, $\mathbf{e} = (T, 0)$.

We will now choose b large enough so that $b > T$. Consequently, in any Walrasian equilibrium, the constraint for the firm that $h \leq b$ will not be binding because in equilibrium the number of hours demanded by the firm cannot exceed the total available number of hours, T.

This economy satisfies all the hypotheses of Theorem 5.13 except that Robinson's utility function, being of the Cobb-Douglas form, is neither strongly increasing nor strictly quasiconcave on all of $\mathbb{R}^n_+$. However, as you are asked to show in Exercise 5.14, the resulting aggregate excess demand function nonetheless satisfies the conditions of Theorem 5.3. Consequently, a Walrasian equilibrium in strictly positive prices is guaranteed to exist. We now calculate one.

Let $p > 0$ denote the price of coconuts, y, and $w > 0$ denote the price per hour of Robinson's time, h. (Thus, it makes sense to think of w as a wage rate.) Consumer Robinson's budget set, before including income from profits, is depicted in Fig. 5.7(b), and Fig. 5.7(c) shows Robinson's budget set when he receives his (100 per cent) share of the firm's profits, equal to $\bar{\pi}$ in the figure.

To determine Walrasian equilibrium prices (w^*, p^*), we shall first determine the firm's supply function (which, in our terminology also includes the firm's demand for hours of labour), then determine the consumer's demand function, and finally put them together to find market-clearing prices. We begin with Robinson the firm. From this point, we use the terms firm and consumer and trust that you will keep in mind that both are in fact Robinson.

Because it never pays the firm to waste hours purchased, it will always choose $(-h, y) \in Y$, so that $y = h^{\alpha}$. Consequently, because we have chosen b large enough so that it will not be a binding constraint, the firm will choose $h \geq 0$ to maximise

$$ph^{\alpha} - wh.$$

When $\alpha < 1$, $h = 0$ will not be profit-maximising (as we shall see); hence, the first-order conditions require setting the derivative with respect to h equal to zero, i.e., $\alpha ph^{\alpha-1} - w = 0$. Rewriting this, and recalling that $y = h^{\alpha}$, gives the firm's demand for

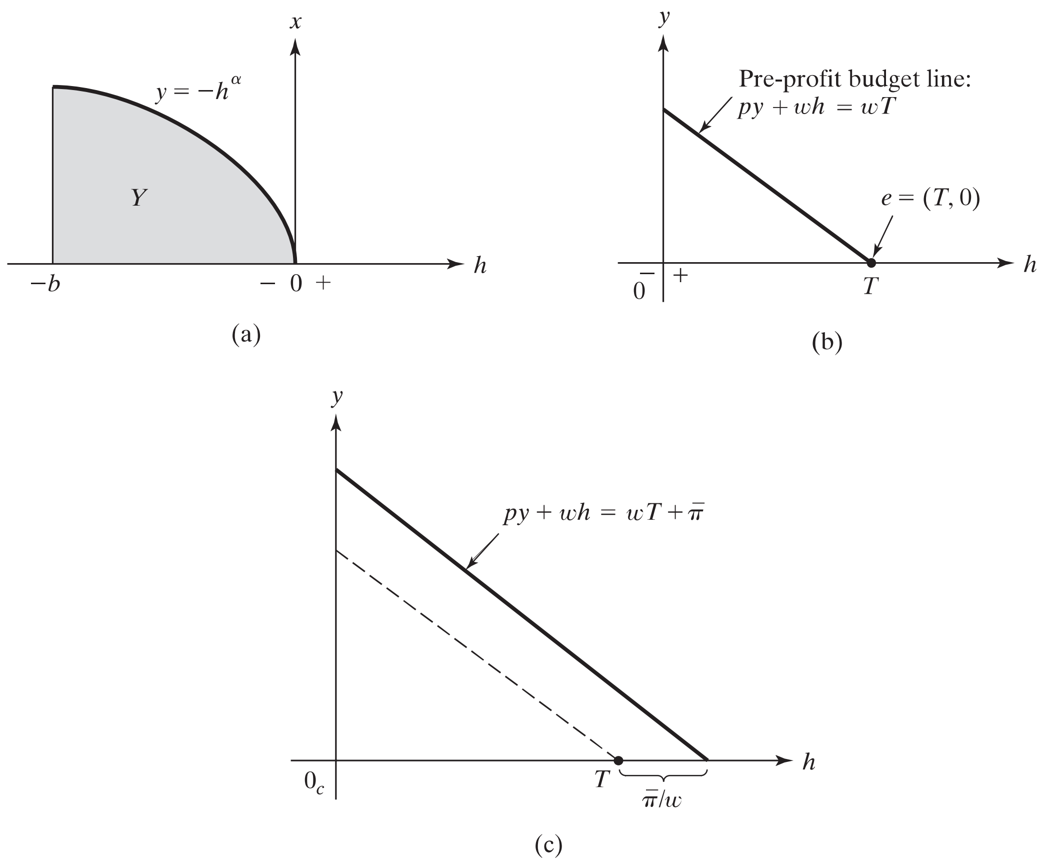


Figure 5.7. Production possibility set, Y, pre-profit budget line, and post-profit budget line in the Robinson Crusoe economy.

hours, denoted h^f, and its supply of output, denoted y^f, as functions of the prices w, p:[4]

$$h^f = \left(\frac{\alpha p}{w}\right)^{1/(1-\alpha)},$$

$$y^f = \left(\frac{\alpha p}{w}\right)^{\alpha/(1-\alpha)}.$$

Consequently, the firm's profits are

$$\pi(w, p) = \frac{1-\alpha}{\alpha} w \left(\frac{\alpha p}{w}\right)^{1/1-\alpha}.$$

Note that profits are positive as long as prices are. (This shows that choosing $h = 0$ is not profit-maximising just as we claimed earlier.)

[4]In case you are keeping track of sign conventions, this means that $(-h^f, y^f) \in Y$.

We now turn to the consumer's problem. Robinson's income is the sum of his endowment income, $(w, p) \cdot (T, 0) = wT$, and his income from his 100 per cent ownership in the firm, $\pi(w, p)$, the firm's profits. So the consumer's budget constraint, which will be satisfied with equality because his utility function is strictly increasing, is

$$py + wh = wT + \pi(w, p).$$

He chooses $(h, y) \geq (0, 0)$ to maximise utility subject to this constraint. By now, you are familiar with the demand functions of a consumer with Cobb-Douglas utility. He will spend the fraction $1 - \beta$ of his total income on h and fraction β of it on y. So, letting h^c and y^c denote the consumer's demands, we have

$$h^c = \frac{(1-\beta)(wT + \pi(w, p))}{w},$$
$$y^c = \frac{\beta(wT + \pi(w, p))}{p}.$$

We can now put all of this together to search for a price vector (w, p) that will clear both markets. There are two simplifications we can make, however. The first is that because aggregate excess demand is homogeneous of degree zero, and we are guaranteed a Walrasian equilibrium in strictly positive prices, we may set the Walrasian equilibrium price of y, p^*, equal to one without any loss. The second is that we need now only find a price w^* so that the market for h clears, because by Walras' law, the market for y will then clear as well.

It thus remains to find w^* such that $h^c + h^f = T$, or using the equations above and setting $p^* = 1$,

$$\frac{(1-\beta)(w^*T + \pi(w^*, 1))}{w^*} + \left(\frac{\alpha}{w^*}\right)^{1/(1-\alpha)} = T,$$

or

$$\frac{(1-\beta)(1-\alpha)}{\alpha}\left(\frac{\alpha}{w^*}\right)^{1/(1-\alpha)} + \left(\frac{\alpha}{w^*}\right)^{1/(1-\alpha)} = \beta T,$$

where we have substituted for the firm's profits to arrive at the second equality. It is straightforward now to solve for w^* to obtain the equilibrium wage

$$w^* = \alpha\left(\frac{1-\beta(1-\alpha)}{\alpha\beta T}\right)^{1-\alpha} > 0.$$

We invite you to check that for this value of w^*, and with $p^* = 1$, both markets do indeed clear.

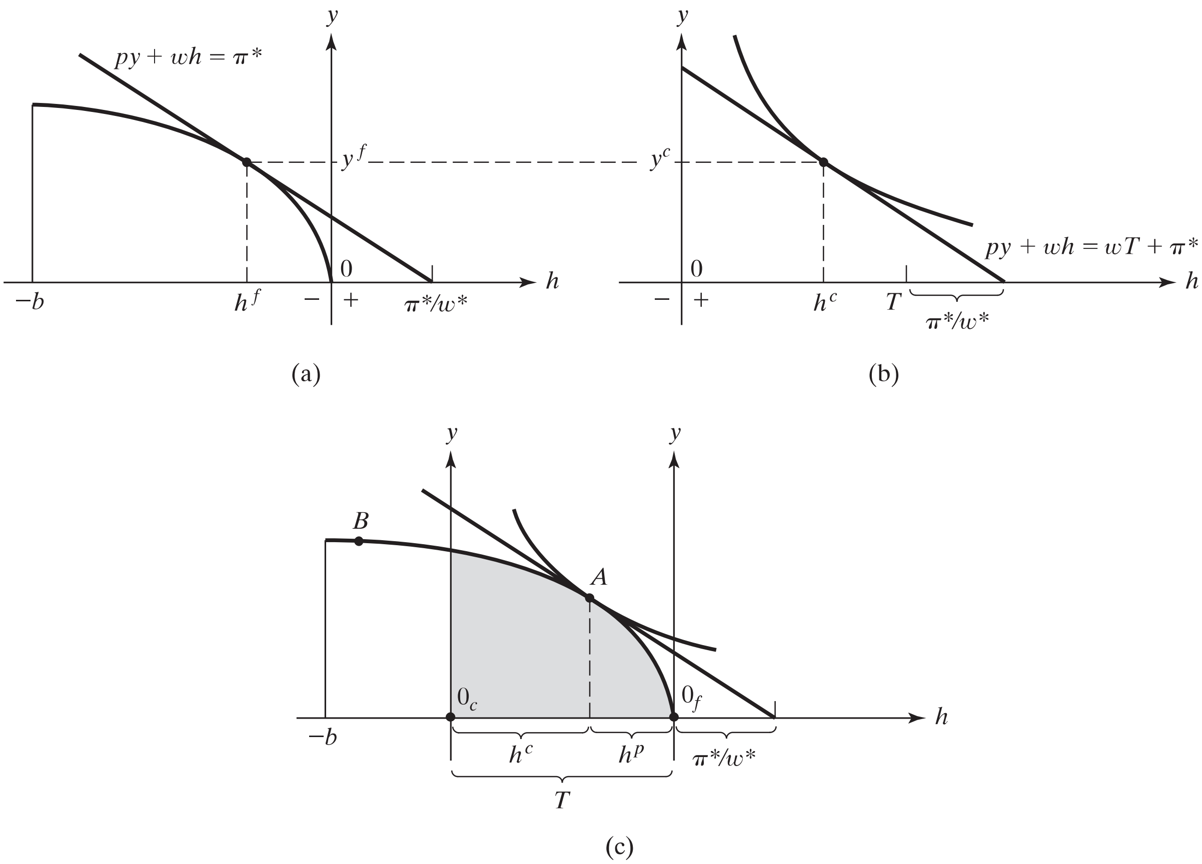


Figure 5.8. Equilibrium in a Robinson Crusoe economy.

We can illustrate the equilibrium diagrammatically. Fig. 5.8(a) shows the firm's profit-maximising solution. The line given by $\pi^* = py + wh$ is an **iso-profit line** for the firm, because profits are constant and equal to π^* for every (h, y) on it. Note that when $(h, y) \in Y$, $h \leq 0$, so that $py + wh$ is indeed the correct formula for profits in the figure. Also note that this iso-profit line (and all others) has slope $-w/p$. Moreover, the iso-profit line depicted yields the highest possible profits for the firm because higher profits would require a production plan above the π^* iso-profit line, and none of those is in the production set. Therefore, $\pi^* = \pi(w^*, 1)$.

Fig. 5.8(b) shows the consumer's utility-maximising solution given the budget constraint $py + wh = wT + \pi^*$. Note the slope of the consumer's budget constraint is $-w/p$, which is the same as the slope of the firm's iso-profit line.

Fig. 5.8(c) puts Figs. 5.8(a) and 5.8(b) together by superimposing the consumer's figure over the firm's, placing the point marked T in the consumer's figure onto the origin in the firm's figure. The origin for the consumer is marked as 0_c and the origin for the firm is 0_f. Point A shows the Walrasian equilibrium allocation.

Fig. 5.8(c) allows us to conclude that this competitive equilibrium with production is Pareto efficient. Consider the shaded region in the figure. With the origin at 0_f, the shaded region denotes the set of feasible production plans – those that can be actually implemented in this economy, taking into account the available resources. Any production plan in the shaded region can be carried out because it calls for no more than T hours, and this is the total number of hours with which the economy is endowed. On the other hand, a production plan like point B is technologically possible because it is in the production set, but it is infeasible because it requires more than T hours.

Switching our point of view, considering 0_c as the origin, the shaded region indicates the set of feasible consumption bundles for this economy. With this in mind, it is clear that the Walrasian allocation at A is Pareto efficient. It maximises Robinson's utility among all feasible consumption bundles.

Soon, we shall show that, just as in the case of a pure exchange economy, this is a rather general result even with production. □

We now return to the assumption of boundedness of the firms' production sets. As mentioned earlier, this assumption can be dispensed with. Moreover, there is very good reason to do so.

The production possibilities set is meant to describe the firm's technology, nothing more. It describes how much of various outputs can be produced with different amounts of various inputs. Thus, if the amount of inputs applied to the process increases without bound, so too might the amount of output produced. So, the first point is that there is simply no place in the description of the *technology* itself for bounds on the amounts of inputs that are available.

However, this might not impress a practical person. After all, who cares if it is *possible* to fill the universe with fountain pens if most of the universe were filled with ink! Is it not sufficient to describe the technology for only those production plans that are actually feasible? On the one hand, the answer is yes, because in equilibrium the production plans in fact be must feasible. But there is a more subtle and important difficulty. When we impose constraints on production possibilities based on aggregate supply, then we are implicitly assuming that *the firm takes these aggregate input constraints into account when making its profit-maximising decisions*. For example, if we bound the production set of a producer of pens because the supply of ink is finite, then at very low ink prices, the producer's demand for ink will be up against this constraint. But were it not for this constraint, the producer would demand even more ink at the current low price. Thus, by imposing this seemingly innocent feasibility constraint on production possibilities, we have severed the all-important connection between price and (excess) demand. And indeed, this is the essence of the competitive model. Producers (and consumers) make demand and supply decisions based on the prevailing prices, not on whether there is enough of the good to supply their demands (or vice versa). Thus, imposing boundedness on the production set runs entirely against the decentralised aspect of the competitive market that we are trying to capture. (A similar argument can be made against placing upper bounds on the consumption set.)

Fortunately, the boundedness assumption is not needed. However, do not despair that all of the hard work we have done has been wasted. It turns out that a standard method of proving existence without bounded production sets is to first prove it by placing artificial bounds on them (which is essentially what we have done) and then letting the artificial bounds become arbitrarily large (which we will not do). Under suitable conditions, this will yield a competitive equilibrium of the economy with unbounded production sets.

For the record, strict convexity of preferences and strong convexity of firm production possibility sets assumed in Theorem 5.13 are more stringent than needed to prove existence of equilibrium. If, instead, merely convexity of preferences and production possibility sets is assumed, existence can still be proved, though the mathematical techniques required are outside the scope of this book. If production possibility sets are convex, we allow the possibility of constant returns to scale for firms. Constant returns introduces the possibility that firm output supply and input demand functions will be set-valued relationships and that they will not be continuous in the usual way. Similarly, mere convexity of preferences raises the possibility of set-valued demand functions together with similar continuity problems. All of these can be handled by adopting generalised functions (called 'correspondences'), an appropriately generalised notion of continuity, and then applying a generalised version of Brouwer's fixed-point theorem due to Kakutani (1941). In fact, we can even do without convexity of individual *firm* production possibility sets altogether, as long as the *aggregate* production possibility set is convex. The reader interested in exploring all of these matters should consult Debreu (1959). But see also Exercise 5.22.

5.3.4 WELFARE

Here we show how Theorems 5.7 and 5.8 can be extended to an economy with production. As before, we focus on properties of the allocations consumers receive in a Walrasian equilibrium. In a production economy, we expand our earlier definition of Walrasian equilibrium allocations as follows.

DEFINITION 5.8 ***WEAs in a Production Economy***

Let $\mathbf{p}^* \gg \mathbf{0}$ be a Walrasian equilibrium for the economy $(u^i, \mathbf{e}^i, \theta^{ij}, Y^j)_{i\in\mathcal{I}, j\in\mathcal{J}}$. Then the pair $(\mathbf{x}(\mathbf{p}^*), \mathbf{y}(\mathbf{p}^*))$ is a Walrasian equilibrium allocation (WEA) where $\mathbf{x}(\mathbf{p}^*)$ denotes the vector, $(\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^I)$, whose ith entry is the utility-maximising bundle demanded by consumer i facing prices $\mathbf{p}^*$ and income $m^i(\mathbf{p}^*)$; and where $\mathbf{y}(\mathbf{p}^*)$ denotes the vector, $(\mathbf{y}^1, \mathbf{y}^2, \ldots, \mathbf{y}^J)$, of profit-maximising production vectors at prices $\mathbf{p}^*$. (Note then that because $\mathbf{p}^*$ is a Walrasian equilibrium, $\sum_{i\in\mathcal{I}} \mathbf{x}^i = \sum_{i\in\mathcal{I}} \mathbf{e}^i + \sum_{j\in\mathcal{J}} \mathbf{y}^j$).

In other words, a consumption and production allocation is a WEA at prices $\mathbf{p}^*$ if (1) each consumer's commodity bundle is the most preferred in his budget set at prices $\mathbf{p}^*$, (2) each firm's production plan is profit-maximising in its production possibility set at prices $\mathbf{p}^*$, and (3) demand equals supply in every market.

We are now ready to extend the First Welfare Theorem to economies with production. Recall from our Robinson Crusoe example that the Walrasian equilibrium allocation there was such that no other feasible allocation could make Robinson better off. We now define Pareto efficiency when there are many consumers and firms based on the same idea.

Throughout the remainder of this section, we shall be concerned with the fixed economy $(u^i, \mathbf{e}^i, \theta^{ij}, Y^j)_{i\in\mathcal{I},j\in\mathcal{J}}$. Thus, all definitions and theorems are stated with this economy in mind.

An allocation, $(\mathbf{x}, \mathbf{y}) = ((\mathbf{x}^1, \ldots, \mathbf{x}^I), (\mathbf{y}^1, \ldots, \mathbf{y}^J))$, of bundles to consumers and production plans to firms is **feasible** if $\mathbf{x}^i \in \mathbb{R}^n_+$ for all i, $\mathbf{y}^j \in Y^j$ for all j, and $\sum_{i\in\mathcal{I}} \mathbf{x}^i = \sum_{i\in\mathcal{I}} \mathbf{e}^i + \sum_{j\in\mathcal{J}} \mathbf{y}^j$.

DEFINITION 5.9 ***Pareto-Efficient Allocation with Production***

The feasible allocation $(\mathbf{x}, \mathbf{y})$ is Pareto efficient if there is no other feasible allocation $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ such that $u^i(\bar{\mathbf{x}}^i) \geq u^i(\mathbf{x}^i)$ for all $i \in \mathcal{I}$ with at least one strict inequality.

Thus, a feasible allocation of bundles to consumers and production plans to firms is Pareto efficient if there is no other feasible allocation that makes at least one consumer strictly better off and no consumer worse off.

It would be quite a task indeed to attempt to allocate resources in a manner that was Pareto efficient. Not only would you need information on consumer preferences, you would also require detailed knowledge of the technologies of all firms and the productivity of all inputs. In particular, you would have to assign individuals with particular skills to the firms that require those skills. It would be a massive undertaking. And yet, with apparently no central direction, the allocations obtained as Walrasian equilibria are Pareto efficient as we now demonstrate.

THEOREM 5.14 ***First Welfare Theorem with Production***

If each u^i is strictly increasing on $\mathbb{R}^n_+$, then every Walrasian equilibrium allocation is Pareto efficient.

Proof: We suppose $(\mathbf{x}, \mathbf{y})$ is a WEA at prices $\mathbf{p}^*$, but is not Pareto efficient, and derive a contradiction.

Because $(\mathbf{x}, \mathbf{y})$ is a WEA, it is feasible, so

$$\sum_{i\in\mathcal{I}} \mathbf{x}^i = \sum_{j\in\mathcal{J}} \mathbf{y}^j + \sum_{i\in\mathcal{I}} \mathbf{e}^i. \tag{P.1}$$

Because $(\mathbf{x}, \mathbf{y})$ is not Pareto efficient, there exists some feasible allocation $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ such that

$$u^i(\hat{\mathbf{x}}^i) \geq u^i(\mathbf{x}^i), \quad i \in \mathcal{I}, \tag{P.2}$$

with at least one strict inequality. By Lemma 5.2, this implies that

$$\mathbf{p}^* \cdot \hat{\mathbf{x}}^i \geq \mathbf{p}^* \cdot \mathbf{x}^i, \qquad i \in \mathcal{I}, \tag{P.3}$$

with at least one strict inequality. Summing over consumers in (P.3) and rearranging gives

$$\mathbf{p}^* \cdot \sum_{i \in \mathcal{I}} \hat{\mathbf{x}}^i > \mathbf{p}^* \cdot \sum_{i \in \mathcal{I}} \mathbf{x}^i. \tag{P.4}$$

Now (P.4) together with (P.1) and the feasibility of $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ tell us

$$\mathbf{p}^* \cdot \left(\sum_{j \in \mathcal{J}} \hat{\mathbf{y}}^j + \sum_{i \in \mathcal{I}} \mathbf{e}^i \right) > \mathbf{p}^* \cdot \left(\sum_{j \in \mathcal{J}} \mathbf{y}^j + \sum_{i \in \mathcal{I}} \mathbf{e}^i \right),$$

so

$$\mathbf{p}^* \cdot \sum_{j \in \mathcal{J}} \hat{\mathbf{y}}^j > \mathbf{p}^* \cdot \sum_{j \in \mathcal{J}} \mathbf{y}^j.$$

However, this means that $\mathbf{p}^* \cdot \hat{\mathbf{y}}^j > \mathbf{p}^* \cdot \mathbf{y}^j$ for some firm j, where $\hat{\mathbf{y}}^j \in Y^j$. This contradicts the fact that in the Walrasian equilibrium, $\mathbf{y}^j$ maximises firm j's profit at prices $\mathbf{p}^*$. ∎

Next we show that competitive markets can support Pareto-efficient allocations after appropriate income transfers.

THEOREM 5.15 ***Second Welfare Theorem with Production***

Suppose that (i) each u^i satisfies Assumption 5.1, (ii) each Y^j satisfies Assumption 5.2, (iii) $\mathbf{y} + \sum_{i \in \mathcal{I}} \mathbf{e}^i \gg \mathbf{0}$ for some aggregate production vector $\mathbf{y}$, and (iv) the allocation $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is Pareto efficient.

Then there are income transfers, $T_1, \ldots, T_I$, satisfying $\sum_{i \in \mathcal{I}} T_i = 0$, and a price vector, $\bar{\mathbf{p}}$, such that

1. *$\hat{\mathbf{x}}^i$ maximises $u^i(\mathbf{x}^i)$ s.t. $\bar{\mathbf{p}} \cdot \mathbf{x}^i \leq m^i(\bar{\mathbf{p}}) + T_i$, $i \in \mathcal{I}$.*
2. *$\hat{\mathbf{y}}^j$ maximises $\bar{\mathbf{p}} \cdot \mathbf{y}^j$ s.t. $\mathbf{y}^j \in Y^j$, $j \in \mathcal{J}$.*

Proof: For each $j \in \mathcal{J}$, let $\bar{Y}^j \equiv Y^j - \{\hat{\mathbf{y}}^j\}$, and note that so defined, each $\bar{Y}^j$ satisfies Assumption 5.2. Consider now the economy $\bar{\mathcal{E}} = (u^i, \hat{\mathbf{x}}^i, \theta^{ij}, \bar{Y}^j)_{i \in \mathcal{I}, j \in \mathcal{J}}$ obtained from the original economy by replacing consumer i's endowment, $\mathbf{e}^i$, with the endowment $\hat{\mathbf{x}}^i$, and replacing each production set, Y^j, with the production set $\bar{Y}^j$. It is straightforward to show using hypotheses (i) to (iii) that $\bar{\mathcal{E}}$ satisfies all the assumptions of Theorem 5.13. Consequently, $\bar{\mathcal{E}}$ possesses a Walrasian equilibrium, $\bar{\mathbf{p}} \gg \mathbf{0}$, and an associated WEA, $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$.

Now because $\mathbf{0} \in \bar{Y}^j$ for every firm j, profits of every firm are non-negative in equilibrium, so that each consumer can afford his endowment vector. Consequently,

$$u^i(\bar{\mathbf{x}}^i) \geq u^i(\hat{\mathbf{x}}^i), \quad i \in \mathcal{I}. \tag{P.1}$$

Next we shall argue that for some aggregate production vector $\tilde{\mathbf{y}}$, $(\bar{\mathbf{x}}, \tilde{\mathbf{y}})$ is feasible for the original economy. To see this, note that each $\bar{\mathbf{y}}^j \in \bar{Y}^j$ is of the form $\bar{\mathbf{y}}^j = \tilde{\mathbf{y}}^j - \hat{\mathbf{y}}^j$ for some $\tilde{\mathbf{y}}^j \in Y^j$, by the definition of $\bar{Y}^j$. Now, because $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is a WEA for $\bar{\mathcal{E}}$, it must be feasible in that economy. Therefore,

$$\begin{aligned}\sum_{i\in\mathcal{I}} \bar{\mathbf{x}}^i &= \sum_{i\in\mathcal{I}} \hat{\mathbf{x}}^i + \sum_{j\in\mathcal{J}} \bar{\mathbf{y}}^j \\ &= \sum_{i\in\mathcal{I}} \hat{\mathbf{x}}^i + \sum_{j\in\mathcal{J}} (\tilde{\mathbf{y}}^j - \hat{\mathbf{y}}^j) \\ &= \sum_{i\in\mathcal{I}} \hat{\mathbf{x}}^i - \sum_{j\in\mathcal{J}} \hat{\mathbf{y}}^j + \sum_{j\in\mathcal{J}} \tilde{\mathbf{y}}^j \\ &= \sum_{i\in\mathcal{I}} \mathbf{e}^i + \sum_{j\in\mathcal{J}} \tilde{\mathbf{y}}^j,\end{aligned}$$

where the last equality follows from the feasibility of $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ in the original economy. Consequently, $(\bar{\mathbf{x}}, \tilde{\mathbf{y}})$ is feasible for the original economy, where $\tilde{\mathbf{y}} = \sum_{j\in\mathcal{J}} \tilde{\mathbf{y}}^j$.

We may conclude that every inequality in (P.1) must be an equality, otherwise $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ would not be Pareto efficient. But the strict quasiconcavity of u^i then implies that

$$\bar{\mathbf{x}}^i = \hat{\mathbf{x}}^i, \quad i \in \mathcal{I},$$

because otherwise some consumer would strictly prefer the average of the two bundles to $\bar{\mathbf{x}}^i$, and the average is affordable at prices $\bar{\mathbf{p}}$ because both bundles themselves are affordable. This would contradict the fact that $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is a WEA for $\bar{\mathcal{E}}$ at prices $\bar{\mathbf{p}}$. Thus, we may conclude that

$$\hat{\mathbf{x}}^i \text{ maximises } u^i(\mathbf{x}^i) \quad \text{s.t.} \quad \bar{\mathbf{p}} \cdot \mathbf{x}^i \leq \bar{\mathbf{p}} \cdot \hat{\mathbf{x}}^i + \sum_{j\in\mathcal{J}} \theta^{ij} \bar{\mathbf{p}} \cdot \bar{\mathbf{y}}^j, \quad i \in \mathcal{I}.$$

But because utility is strongly increasing, the budget constraint holds with equality at $\mathbf{x}^i = \hat{\mathbf{x}}^i$, which implies that each consumer i's income from profits is zero. This means that every firm must be earning zero profits, which in turn means that $\bar{\mathbf{y}}^j = \mathbf{0}$ for every firm j.

We leave it as an exercise to show that because $\bar{\mathbf{y}}^j = \mathbf{0}$ maximises firm j's profits at prices $\bar{\mathbf{p}}$ when its production set is $\bar{Y}^j$, then (by the definition of $\bar{Y}^j$) $\hat{\mathbf{y}}^j$ maximises firm j's profits at prices $\bar{\mathbf{p}}$ when its production set is Y^j (i.e., in the original economy).

So altogether, we have shown the following:

$$\hat{\mathbf{x}}^i \text{ maximises } u^i(\mathbf{x}^i) \quad \text{s.t.} \quad \bar{\mathbf{p}} \cdot \mathbf{x}^i \leq \bar{\mathbf{p}} \cdot \hat{\mathbf{x}}^i, \quad i \in \mathcal{I}, \tag{P.2}$$

$$\hat{\mathbf{y}}^j \text{ maximises } \bar{\mathbf{p}} \cdot \mathbf{y}^j \quad \text{s.t.} \quad \mathbf{y}^j \in Y^j, \quad j \in \mathcal{J}. \tag{P.3}$$

Note then that setting $T_i \equiv \bar{\mathbf{p}} \cdot \hat{\mathbf{x}}^i - m^i(\bar{\mathbf{p}})$ provides the appropriate transfers, where $m^i(\bar{\mathbf{p}}) = \bar{\mathbf{p}} \cdot \mathbf{e}^i + \sum_{j \in \mathcal{J}} \theta^{ij} \bar{\mathbf{p}} \cdot \hat{\mathbf{y}}^j$ is consumer i's income in the original economy at prices $\bar{\mathbf{p}}$. These transfers sum to zero by the feasibility of $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$, and when employed (in the original economy), they reduce each consumer's problem to that in (P.2). Consequently, both (1) and (2) are satisfied. ∎

5.4 CONTINGENT PLANS

Up to now we have considered the problem of how a market economy allocates resources through a competitive price system in what appears to be a static environment. There has been no mention of *time* in the model. So, for example, discussions of interest rates, inflation, borrowing, and lending seem to be out of reach. But in fact this is not so. The model we have developed is actually quite capable of including not only time, interest rates, borrowing, and lending, but also uncertainty about many things, including the future state of the economy, the value of stocks and bonds, and more. The key idea is to refine the notion of a good to include all of the characteristics of interest to us.

5.4.1 TIME

If we wish to include time in our model, then we simply index goods not only by what they are, e.g. apples, oranges, etc., but also by the date at which they are consumed (or produced). So instead of keeping track only of x_k, the amount of good k consumed by a consumer, we also keep track of the date t at which it is consumed. Thus, we let x_{kt} denote the amount of good k consumed at date t. If there are two goods, $k = 1, 2$, and two dates $t = 1, 2$, then a consumption bundle is a vector of four numbers $(x_{11}, x_{12}, x_{21}, x_{22})$, where, for example, x_{12} is the amount of good $k = 1$ consumed at date $t = 2$.

But if a consumption bundle is $(x_{11}, x_{12}, x_{21}, x_{22})$, then in keeping with our convention up to now, we should really think of each of the four coordinates of the consumption bundle as representing the quantities of *distinct goods*. That is, with two ‘ basic’ goods, apples and oranges, and two dates, today and tomorrow, we actually have four goods – apples today, apples tomorrow, oranges today, and oranges tomorrow.

5.4.2 UNCERTAINTY

Uncertainty, too, can be captured using the same technique. For example, suppose there is uncertainty about today's weather and that this is important because the weather might affect the desirability of certain products (e.g., umbrellas, sunscreen, vacations,. . .) and/or the production possibilities for certain products (e.g., agriculture). To keep things simple, let us suppose that there are just two possibilities for the state of the weather. In state $s = 1$ it rains, and in state $s = 2$ it is sunny. Then, analogous to what we did with time, we can index each good k with the state in which it is consumed (or produced) by letting x_{ks} denote the amount of good k consumed in state s, and letting y_{ks} denote the amount of good k produced in state s. This permits consumers to have quite distinct preferences over umbrellas when it is sunny and umbrellas when it rains, and it also permits production possibilities, for agricultural products for example, to be distinct in the two states. We can also model the demand for insurance by allowing a consumer's endowment vector to depend upon the state, with low endowments being associated with one state (fire or flood, for example) and high endowments with another.

5.4.3 WALRASIAN EQUILIBRIUM WITH CONTINGENT COMMODITIES

Let us put all of this together by incorporating both time and uncertainty. We will then step back and interpret the meaning of a Walrasian equilibrium of the resulting model.

There are N basic goods, $k = 1, 2, \ldots, N$, T dates, $t = 1, 2, \ldots, T$, and for each date t there are S_t mutually exclusive and exhaustive events $s_t = 1, 2, \ldots, S_t$ that can occur. Consequently, the *state of the world* at date t is described by the vector $(s_1, \ldots, s_t)$ of the t events that occurred at the start of dates 1 through t inclusive. A consumption bundle is a non-negative vector $\mathbf{x} = (x_{kts})$, where k runs from 1 to N, t runs from 1 to T, and given t, $s = (s_1, \ldots, s_t)$ is one of the $S_1 S_2 \ldots S_t$ states of the world describing the events that have occurred up to date t. Thus, $\mathbf{x} \in \mathbb{R}^{NM}_+$, where $M = S_1 + S_1 S_2 + \ldots + S_1 S_2 \ldots S_T$ is the total number of date–state pairs (t, s).

There are J firms and each firm $j \in \mathcal{J}$ has a production possibilities set, Y^j, contained in $\mathbb{R}^{NM}$.

There are I consumers. Each consumer $i \in \mathcal{I}$ has preferences over the set of consumption bundles in $\mathbb{R}^{NM}_+$ and i's preferences are represented by a utility function $u^i(\cdot)$. Consumer i has an endowment vector $\mathbf{e}^i \in \mathbb{R}^{NM}_+$ and ownership share θ^{ij} of each firm $j \in \mathcal{J}$.[5] Note that the endowment vector $\mathbf{e}^i$ specifies that at date t and in state s, consumer i's endowment of the N goods is $(e^i_{1ts}, \ldots, e^i_{Nts})$.

In terms of our previous definitions, this is simply a private ownership economy with $n = NM$ goods. For example $x_{kts} = 2$ denotes two units of good kts or equivalently it denotes two units of the basic good k at date t in state s. Thus, we are treating the same basic good as distinct when consumed at distinct dates or in distinct states. After all, the amount one is willing to pay for an automobile delivered today might well be higher than the amount one is willing to pay for delivery of an otherwise identical automobile six months from today. From this perspective, treating the same basic good at distinct dates (or in distinct states) as distinct goods is entirely natural.

Under the hypotheses of Theorem 5.13, there is a price vector $\mathbf{p}^* \in \mathbb{R}^{NM}_{++}$ constituting a Walrasian equilibrium for this private ownership economy. In particular, demand must equal supply for each of the NM goods, that is for every basic good at every date and in every state of the world. Let us now understand what this means starting with firms.

For each firm $j \in \mathcal{J}$, let $\hat{\mathbf{y}}^j = (\hat{y}^j_{kts}) \in Y^j \subseteq \mathbb{R}^{NM}$ denote its (unique) profit-maximising production plan given the price vector $\mathbf{p}^*$. Consequently, at date t in state s, firm j will produce $\hat{y}^j_{kts}$ units of the basic good (output) k if $\hat{y}^j_{kts} \geq 0$ and will demand

[5]One could allow ownership shares to depend upon the date and the state, but we shall not do so.

$\left|\hat{y}^j_{kts}\right|$ units of the basic good (input) k if $\hat{y}^j_{kts} < 0$. Thus, $\hat{\mathbf{y}}^j$ is a profit-maximising *contingent production plan*, describing output supply and input demand for the N basic goods contingent upon each date and state. Let us now turn to consumers.

For each $i \in \mathcal{I}$, let $\hat{\mathbf{x}}^i = (\hat{x}^i_{kts}) \in \mathbb{R}^{NM}_+$ denote consumer i's (unique) utility-maximising affordable consumption bundle given prices $\mathbf{p}^*$ and income $m^i(\mathbf{p}^*)$. Consequently, at date t in state s consumer i will consume $\hat{x}^i_{kts}$ units of the basic good k. Thus $\hat{\mathbf{x}}^i$ is a utility-maximising affordable *contingent consumption plan* for consumer i, specifying his consumption of each of the basic goods contingent on each date and state.

Now, on the one hand, because demand equals supply for every good, we have

$$\sum_{i \in \mathcal{I}} \hat{x}^i_{kts} = \sum_{j \in \mathcal{J}} \hat{y}^j_{kts} + \sum_{i \in \mathcal{I}} e^i_{kts}, \text{ for every } k, t, s. \tag{5.6}$$

Consequently, at every date and in every state, demand equals supply for each of the basic goods. On the other hand, each consumer i has only a single budget constraint linking his expenditures on all goods as follows:

$$\sum_{k,t,s} p^*_{kts}\hat{x}^i_{kts} = \sum_{k,t,s} p^*_{kts}e^i_{kts} + \sum_{j \in \mathcal{J}} \theta^{ij} \sum_{k,t,s} p^*_{kts}\hat{y}^j_{kts}, \text{ for every } i \in \mathcal{I}. \tag{5.7}$$

In particular, when state s' occurs at date t', it may turn out that for some consumer(s) i,

$$\sum_k p^*_{kt's'}\hat{x}^i_{kt's'} > \sum_k p^*_{kt's'}e^i_{kt's'} + \sum_{j \in \mathcal{J}} \theta^{ij} \sum_k p^*_{kt's'}\hat{y}^j_{kt's'}.$$

That is, consumer i's expenditures on basic goods at date t' in state s' might exceed his income at that date and in that state. Does this make sense? The answer is 'yes, it absolutely makes sense'. Indeed, this budget shortfall is an expression of two important economic phenomena, namely borrowing and insurance. When one borrows at date t, one is effectively spending more than one's endowment and profit-share income at date t, and when one receives an insurance payment due to loss in state s (e.g., fire or flood) then again one is able to spend in state s more than one's endowment and profit-share income. On the other side of the coin, there can very well be some states and dates associated with budget surpluses (e.g., when one lends or when one provides insurance on states that did not occur).

But if each consumer's budget need balance only overall, as given in (5.7), then how is this Walrasian equilibrium allocation actually implemented? The answer is as follows. Think of a prior date zero at which firms and consumers participate in a market for binding contracts. A contract is a piece of paper on which is written a non-negative real number, a basic good k, a date t, and a state, s. For example, the contract $(107.6, k = 3, t = 2, s = 7)$ entitles the bearer to 107.6 units of basic good $k = 3$ at date $t = 2$ in state $s = 7$. Notice that each consumer's equilibrium net consumption bundle $\hat{\mathbf{x}}^i - \mathbf{e}^i = (\hat{x}^i_{kts} - e^i_{kts})$ can be reinterpreted as a vector of contracts. That is, for each k, t, and s, if $\hat{x}^i_{kts} - e^i_{kts} \geq 0$ then consumer i is entitled to receive from the market $\hat{x}^i_{kts} - e^i_{kts}$ units of basic good k at date t in state s. If $\hat{x}^i_{kts} - e^i_{kts} < 0$, consumer i is required to supply to the market $|\hat{x}^i_{kts} - e^i_{kts}|$ units of basic good k at date t in state s.

Similarly, each firm's production plan $\hat{\mathbf{y}}^j = (\hat{y}^j_{kts})$ can be reinterpreted as the vector of contracts requiring firm j to supply to the market $\hat{y}^j_{kts}$ units of basic good k at date t in state s if $\hat{y}^j_{kts} \geq 0$ and entitling firm j to receive from the market $\left|\hat{y}^j_{kts}\right|$ units of basic good k at date t in state s if $\hat{y}^j_{kts} < 0$.

Finally, note that if for each k, t, and s, the price of a contract per unit of basic good k at date t in state s is p^*_{kts}, then at date zero the market for contracts will clear with consumers maximising utility and firms maximising profits. When each date t arrives and any state s occurs, the contracts that are relevant for that date and state are executed. The market-clearing condition (5.6) ensures that this is feasible. After the initial trading of contracts in period zero, no further trade takes place. The only activity taking place as time passes and states occur is the execution of contracts that were purchased and sold at date zero.

Let us now provide several important remarks on this interpretation of our model. First, we have implicitly assumed that there is perfect monitoring in the sense that it is not possible for a firm or consumer to claim that he can supply more units of a basic good in state s at date t than he actually can supply. Thus, bankruptcy is assumed away. Second, it is assumed that there is perfect information in the sense that all firms and consumers are informed of the state when it occurs at each date. Otherwise, if only some agents were informed of the state, they might have an incentive to lie about which state actually did occur. Third, it is assumed that all contracts are perfectly enforced. Clearly, each of these assumptions is strong and rules out important economic settings. Nonetheless, it is quite remarkable how much additional mileage we are able to get from a model that appears entirely static and deterministic simply by reinterpreting its variables! The exercises explore this model further, examining how it provides theories of insurance, borrowing and lending, interest rates, and asset pricing.

5.5 CORE AND EQUILIBRIA

In this final section, we return to the world of pure exchange economies and pursue further the relation between the core of an economy and the set of Walrasian equilibrium allocations. As we have seen, every Walrasian equilibrium allocation is also a core allocation. On the other hand, simple Edgeworth box examples can be constructed that yield core allocations that are not Walrasian. Thus, it would seem that the connection between the two ideas is limited.

Edgeworth (1881), however, conjectured a more intimate relationship between Walrasian allocations and the core. He suggested that when the economy is 'large', and so when the Walrasian assumption of price-taking behaviour by consumers makes most sense, the distinction between core allocations and Walrasian equilibrium ones disappears. In considering that possibility anew, Debreu and Scarf (1963) extended Edgeworth's framework and proved him to be correct. Loosely speaking, they showed that as an economy becomes 'larger', its core 'shrinks' to include *only* those allocations that are Walrasian!

All in all, their result is heartening to those who believe in the special qualities of a market system, where the only information a consumer requires is the set of market prices he faces. It suggests a tantalising comparison between the polar paradigms of central planning and laissez-faire in very large economies. If the objective of the planning process is to identify and then implement some distribution of goods that is in the core, and if there are no other allocations in the core but those that would be picked out by a competitive market system, why go to the bother (and expense) of planning at all? To find the core, a central planner needs information on consumers' preferences, and consumers have selfish incentives to be less than completely honest in revealing that information to the planner. The *market* does not need to know anything about consumers' preferences at all, and in fact depends on consumers' selfishness. What is a vice in one case is a virtue of sorts in the other.

There is, of course, a great deal of loose language in this discussion. On a broad plane, the choice between planning and market systems would never hinge on efficiency alone. In addition, we know that core allocations from arbitrary initial endowments need not be equitable in any sense of the word. Planning may still be justified as a means of achieving a desired redistribution of endowments. On a narrower plane, there are technical issues unaddressed. What does it mean for an economy to be 'large', or to be 'larger', than another? Moreover, because an 'allocation' involves a vector of goods for each consumer, and because presumably a larger economy has a greater number of consumers, is not the 'dimensionality' of the core in large economies different from that in small economies? If so, how can we speak of the core 'shrinking'? We will answer each of these questions before we finish.

5.5.1 REPLICA ECONOMIES

To keep the analysis manageable, we follow Debreu and Scarf by formalising the notion of a large economy in a very particular way. We start with the idea of a basic exchange economy consisting of a finite but arbitrary finite number I of consumers, each with his or her own preferences and endowments. Now think of each consumer's preferences and/or endowments as making that consumer a different 'type' of consumer from all the rest. Two consumers with different preferences but the same endowments are considered different types. So, too, are two consumers with the same preferences but different endowments.[6] Thus, we now think of there being an arbitrary finite number of different *types* of consumers, and the basic exchange economy consists of one consumer of each type.

[6] In fact, we would also call two consumers with the same preferences and endowments different types even though the distinction would just be a formal one. For now, however, it is best to think of no two consumers as having both the same preferences and the same endowments.

Now imagine that each consumer suddenly acquires a twin. The twins are completely identical, having the same preferences and the same endowments. The new economy, consisting of all the original consumers and their twins, now has two consumers of each type rather than one. This new economy is clearly larger than the original one because it contains exactly twice as many consumers. We call this new economy the twofold *replica* of the original one. If each original consumer was tripled, or quadrupled, we could similarly construct threefold or fourfold replicas of the original economy, each in turn being larger than the preceding one in a well-defined way. Now you get the idea of a **replica economy**. It is one with a finite number of 'types' of consumers, an equal number of consumers of each type, and all individuals of the same type are *identical* in that they have identical preferences and identical endowments. Formally, we have the following definition and assumptions.

DEFINITION 5.10 ***An r-Fold Replica Economy***

Let there be I types of consumers in the basic exchange economy and index these types by the set $\mathcal{I} = \{1, \ldots, I\}$. By the r-fold replica economy, denoted $\mathcal{E}_r$, we mean the economy with r consumers of each type for a total of rI consumers. For any type $i \in \mathcal{I}$, all r consumers of that type share the common preferences $\succsim^i$ on $\mathbb{R}^n_+$ and have identical endowments $\mathbf{e}^i \gg \mathbf{0}$. We further assume for $i \in \mathcal{I}$ that preferences $\succsim^i$ can be represented by a utility function u^i satisfying Assumption 5.1.

Thus, when comparing two replica economies, we can unambiguously say which of them is larger. It will be the one having more of every type of consumer.

Let us now think about the core of the r-fold replica economy $\mathcal{E}_r$. Under the assumptions we have made, all of the hypotheses of Theorem 5.5 will be satisfied. Consequently, a WEA will exist, and by Theorem 5.5, it will be in the core. So we have made enough assumptions to ensure that the core of $\mathcal{E}_r$ is non-empty.

To keep track of all of the consumers in each replica economy, we shall index each of them by *two* superscripts, i and q, where $i = 1, \ldots, I$ runs through all the types, and $q = 1, \ldots, r$ runs through all consumers of a particular type. For example, the index $iq = 23$ refers to the type 2 consumer labelled by the number 3, or simply the third consumer of type 2. So, an allocation in $\mathcal{E}_r$ takes the form

$$\mathbf{x} = (\mathbf{x}^{11}, \mathbf{x}^{12}, \ldots, \mathbf{x}^{1r}, \ldots, \mathbf{x}^{I1}, \ldots, \mathbf{x}^{Ir}), \tag{5.8}$$

where $\mathbf{x}^{iq}$ denotes the bundle of the qth consumer of type i. The allocation is then feasible if

$$\sum_{i\in\mathcal{I}} \sum_{q=1}^{r} \mathbf{x}^{iq} = r \sum_{i\in\mathcal{I}} \mathbf{e}^i, \tag{5.9}$$

because each of the r consumers of type i has endowment vector $\mathbf{e}^i$.

The theorem below exploits this fact and the strict convexity of preferences.

THEOREM 5.16 ***Equal Treatment in the Core***

If $\mathbf{x}$ *is an allocation in the core of* $\mathcal{E}_r$*, then every consumer of type i must have the same bundle according to* $\mathbf{x}$*. That is, for every* $i = 1, \ldots, I$, $\mathbf{x}^{iq} = \mathbf{x}^{iq'}$ *for every* $q, q' = 1, \ldots, r$.

This theorem with the delightfully democratic name identifies a crucial property of core allocations in replica economies. It is therefore important that we not only believe equal treatment of like types occurs in the core but that we also have a good feel for why it is true. For that reason, we will give a leisurely 'proof' for the simplest, two-type, four-person economy. Once you understand this case, you should be able to derive the formal proof of the more general case for yourself, and that will be left as an exercise.

Proof: Let $I = 2$, and consider $\mathcal{E}_2$, the replica economy with two types of consumers and two consumers of each type, for a total of four consumers in the economy. Suppose that

$$\mathbf{x} \equiv (\mathbf{x}^{11}, \mathbf{x}^{12}, \mathbf{x}^{21}, \mathbf{x}^{22})$$

is an allocation in the core of $\mathcal{E}_2$. First, we note that because $\mathbf{x}$ is in the core, it must be feasible, so

$$\mathbf{x}^{11} + \mathbf{x}^{12} + \mathbf{x}^{21} + \mathbf{x}^{22} = 2\mathbf{e}^1 + 2\mathbf{e}^2 \tag{P.1}$$

because both consumers of each type have identical endowments.

Now suppose that $\mathbf{x}$ does not assign identical bundles to some pair of identical types. Let these be consumers 11 and 12, so $\mathbf{x}^{11}$ and $\mathbf{x}^{12}$ are distinct. Remember that they each have the same preferences, $\succsim^1$.

Because $\succsim^1$ is complete, it must rank one of the two bundles as being at least as good as the other. Let us assume that

$$\mathbf{x}^{11} \succsim^1 \mathbf{x}^{12}. \tag{P.2}$$

Of course, the preference may be strict, or the two bundles may be ranked equally. Figs. 5.9(a) and 5.9(b) illustrate both possibilities. Either way, we would like to show that because $\mathbf{x}^{11}$ and $\mathbf{x}^{12}$ are distinct, $\mathbf{x}$ cannot be in the core of $\mathcal{E}_2$. To do this, we will show that $\mathbf{x}$ can be blocked.

Now, consider the two consumers of type 2. Their bundles according to $\mathbf{x}$ are $\mathbf{x}^{21}$ and $\mathbf{x}^{22}$, and they each have preferences $\succsim^2$. Let us assume (again without loss of generality) that

$$\mathbf{x}^{21} \succsim^2 \mathbf{x}^{22}. \tag{P.3}$$

So, consumer 2 of type 1 is the worst off type 1 consumer, and consumer 2 of type 2 is the worst off type 2 consumer. Let us see if these *worst off* consumers of each type can get together and block the allocation $\mathbf{x}$.

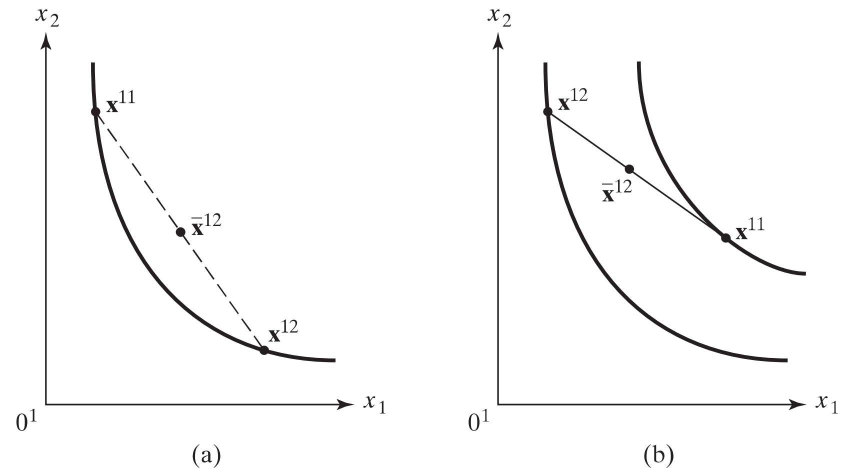


Figure 5.9. Demonstration of equal treatment in the core.

Let the bundles $\bar{\mathbf{x}}^{12}$ and $\bar{\mathbf{x}}^{22}$ be defined as follows:

$$\bar{\mathbf{x}}^{12} = \frac{\mathbf{x}^{11} + \mathbf{x}^{12}}{2},$$
$$\bar{\mathbf{x}}^{22} = \frac{\mathbf{x}^{21} + \mathbf{x}^{22}}{2}.$$

The first bundle is the average of the bundles going to the type 1 consumers and the second is the average of the bundles going to the type 2 consumers. See Fig. 5.9 for the placement of $\bar{\mathbf{x}}^{12}$.

Now, suppose it were possible to give consumer 12 the bundle $\bar{\mathbf{x}}^{12}$. How would this compare to giving him the bundle he's getting under $\bar{\mathbf{x}}$, namely, $\mathbf{x}^{12}$? Well, remember that according to (P.2), consumer 12 was the worst off consumer of type 1. Consequently, because bundles $\mathbf{x}^{11}$ and $\mathbf{x}^{12}$ are distinct, consumer 12 would strictly prefer $\bar{\mathbf{x}}^{12}$ to $\mathbf{x}^{12}$ because his preferences, $\succsim^1$, are strictly convex. That is,

$$\bar{\mathbf{x}}^{12} \succ^1 \mathbf{x}^{12}.$$

This is shown in Figs. 5.9(a) and 5.9(b).

Similarly, the strict convexity of consumer 22's preferences, $\succsim^2$, together with (P.3) imply

$$\bar{\mathbf{x}}^{22} \succsim^2 \mathbf{x}^{22},$$

where the preference need not be strict because we may have $\mathbf{x}^{21} = \mathbf{x}^{22}$.

The pair of bundles $(\bar{\mathbf{x}}^{12}, \bar{\mathbf{x}}^{22})$ therefore makes consumer 12 strictly better off and consumer 22 no worse off than the allocation $\mathbf{x}$. If this pair of bundles can be achieved by consumers 12 and 22 alone, then they can block the allocation $\mathbf{x}$, and the proof will be complete.

To see that together they can achieve $(\bar{\mathbf{x}}^{12}, \bar{\mathbf{x}}^{22})$, note the following:

$$\begin{aligned}\bar{\mathbf{x}}^{12} + \bar{\mathbf{x}}^{22} &= \frac{\mathbf{x}^{11} + \mathbf{x}^{12}}{2} + \frac{\mathbf{x}^{21} + \mathbf{x}^{22}}{2} \\ &= \tfrac{1}{2}(\mathbf{x}^{11} + \mathbf{x}^{12} + \mathbf{x}^{21} + \mathbf{x}^{22}) \\ &= \tfrac{1}{2}(2\mathbf{e}^{1} + 2\mathbf{e}^{2}) \\ &= \mathbf{e}^{1} + \mathbf{e}^{2},\end{aligned}$$

where the third equality follows from (P.1). Consequently, the two worst off consumers of each type can together achieve a pair of bundles that makes one of them strictly better off and the other no worse off. The coalition $S = \{12, 22\}$ therefore can block $\mathbf{x}$. But this contradicts the fact that $\mathbf{x}$ is in the core.

We conclude then that $\mathbf{x}$ must give consumers of the same type the same bundle. ∎

Now that we have made clear what it means for one economy to be larger than another, and have demonstrated the equal treatment property in the core of a replica economy, we can clarify what we mean when we say the core 'shrinks' as the economy gets larger by replication. First, recognise that when we replicate some basic economy, we increase the number of consumers in the economy and so increase the number of bundles in an allocation. There should be no confusion about that. However, when we restrict our attention to *core* allocations in these economies, the equal-treatment property allows us to completely describe any allocation in the core of $\mathcal{E}_r$ by reference to a similar allocation in the basic economy, $\mathcal{E}_1$.

To see this, suppose that $\mathbf{x}$ is in the core of $\mathcal{E}_r$. Then by the equal treatment property, $\mathbf{x}$ must be of the form

$$\mathbf{x} = \Big(\underbrace{\mathbf{x}^1, \ldots, \mathbf{x}^1}_{r \text{ times}}, \underbrace{\mathbf{x}^2, \ldots, \mathbf{x}^2}_{r \text{ times}}, \ldots, \underbrace{\mathbf{x}^I, \ldots, \mathbf{x}^I}_{r \text{ times}}\Big),$$

because all consumers of the same type must receive the same bundle. Consequently, *core* allocations in $\mathcal{E}_r$ are just r-fold copies of allocations in $\mathcal{E}_1$ – i.e., the above core allocation is just the r-fold copy of the $\mathcal{E}_1$ allocation

$$(\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^I). \tag{5.10}$$

In fact, this allocation is feasible in $\mathcal{E}_1$. To see this, note first that because $\mathbf{x}$ is a core allocation in $\mathcal{E}_r$, it must be feasible in $\mathcal{E}_r$. Therefore, we have

$$r\sum_{i\in\mathcal{I}} \mathbf{x}^i = r\sum_{i\in\mathcal{I}} \mathbf{e}^i,$$

which, dividing by r, shows that the allocation in (5.10) is feasible in the basic economy $\mathcal{E}_1$.

Altogether then, we have shown that every core allocation of the r-fold replica economy is simply an r-fold copy of some feasible allocation in the basic economy $\mathcal{E}_1$. Consequently, we can keep track of how the core changes as we replicate the economy simply by keeping track of those allocations in $\mathcal{E}_1$ corresponding to the core of each r-fold replica. With this in mind, define C_r as follows:

$$C_r \equiv \left\{ \mathbf{x} = (\mathbf{x}^1, \ldots, \mathbf{x}^I) \in F(\mathbf{e}) \mid \Big(\underbrace{\mathbf{x}^1, \ldots, \mathbf{x}^1}_{r \text{ times}}, \ldots, \underbrace{\mathbf{x}^I, \ldots, \mathbf{x}^I}_{r \text{ times}} \Big) \text{ is in the core of } \mathcal{E}_r \right\}.$$

We can now describe formally the idea that the core 'shrinks' as the economy is replicated.

LEMMA 5.3 *The sequence of sets* $C_1, C_2, \ldots,$ *is decreasing. That is* $C_1 \supseteq C_2 \supseteq \ldots \supseteq C_r \supseteq \ldots$.

Proof: It suffices to show that for $r > 1$, $C_r \subseteq C_{r-1}$. So, suppose that $\mathbf{x} = (\mathbf{x}^1, \ldots, \mathbf{x}^I) \in C_r$. This means that its r-fold copy cannot be blocked in the r-fold replica economy. We must show that its $(r-1)$-fold copy cannot be blocked in the $(r-1)$-fold replica economy. But a moment's thought will convince you of this once you realise that any coalition that blocks the $(r-1)$-fold copy in $\mathcal{E}_{r-1}$ could also block the r-fold copy in $\mathcal{E}_r$ – after all, all the members of that coalition are present in $\mathcal{E}_r$ as well, and their endowments have not changed. ∎

So, by keeping track of the allocations in the basic economy whose r-fold copies are in the core of the r-fold replica, Lemma 5.3 tells us that this set will get no larger as r increases. To see how the core actually shrinks as the economy is replicated, we shall look again at economies with just two types of consumers. Because we are only concerned with core allocations in these economies, we can exploit the equal-treatment property and illustrate our arguments in an Edgeworth box like Fig. 5.10. This time, we think of the preferences and endowments in the box as those of a *representative* consumer of each type.

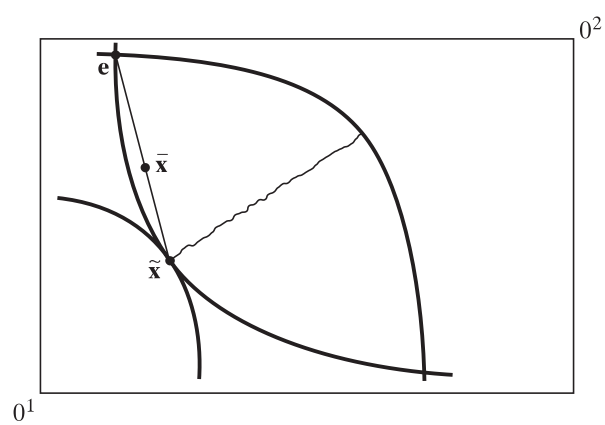


Figure 5.10. An Edgeworth box for a two-type replica economy.

In the basic economy with one consumer of each type, the core of $\mathcal{E}_1$ is the squiggly line between the two consumers' respective indifference curves through their endowments at $\mathbf{e}$. The core of $\mathcal{E}_1$ contains some allocations that are WEA and some that are not. The allocation marked $\tilde{\mathbf{x}}$ is *not* a WEA because the price line through $\tilde{\mathbf{x}}$ and $\mathbf{e}$ is not tangent to the consumers' indifference curves at $\tilde{\mathbf{x}}$. Note that $\tilde{\mathbf{x}}$ is on consumer 11's indifference curve through his endowment. If we now replicate this economy once, can the replication of this allocation be in the core of the larger four-consumer economy?

The answer is no; and to see it, first notice that any point along the line joining $\mathbf{e}$ and $\tilde{\mathbf{x}}$ is preferred to both $\mathbf{e}$ and $\tilde{\mathbf{x}}$ by both (there are now two) type 1's because their preferences are strictly convex. In particular, the midpoint $\bar{\mathbf{x}}$ has this property. Now consider the three-consumer coalition, $S = \{11, 12, 21\}$, consisting of both type 1's and one type 2 consumer (either one will do). Let each type 1 consumer have a bundle corresponding to the type 1 bundle at $\bar{\mathbf{x}}$ and let the lone type 2 consumer have a type 2 bundle like that at $\tilde{\mathbf{x}}$. We know that each type 1 strictly prefers this to the type 1 bundle at $\tilde{\mathbf{x}}$, and the type 2 consumer is just as well off. Specifically, we know

$$\begin{aligned}\bar{\mathbf{x}}^{11} &\equiv \tfrac{1}{2}(\mathbf{e}^1 + \tilde{\mathbf{x}}^{11}) \succ^1 \tilde{\mathbf{x}}^{11}, \\ \bar{\mathbf{x}}^{12} &\equiv \tfrac{1}{2}(\mathbf{e}^1 + \tilde{\mathbf{x}}^{12}) \succ^1 \tilde{\mathbf{x}}^{12}, \\ \tilde{\mathbf{x}}^{21} &\sim^2 \tilde{\mathbf{x}}^{21}.\end{aligned}$$

Are bundles $\{\bar{\mathbf{x}}^{11}, \bar{\mathbf{x}}^{12}, \tilde{\mathbf{x}}^{21}\}$ feasible for S? From the definitions, and noting that $\tilde{\mathbf{x}}^{11} = \tilde{\mathbf{x}}^{12}$, we have

$$\begin{aligned}\bar{\mathbf{x}}^{11} + \bar{\mathbf{x}}^{12} + \tilde{\mathbf{x}}^{21} &= 2\big(\tfrac{1}{2}\mathbf{e}^1 + \tfrac{1}{2}\tilde{\mathbf{x}}^{11}\big) + \tilde{\mathbf{x}}^{21} \\ &= \mathbf{e}^1 + \tilde{\mathbf{x}}^{11} + \tilde{\mathbf{x}}^{21}.\end{aligned} \tag{5.11}$$

Next recall that $\tilde{\mathbf{x}}$ is in the core of $\mathcal{E}_1$, so it must be feasible in the two-consumer economy. This implies

$$\tilde{\mathbf{x}}^{11} + \tilde{\mathbf{x}}^{21} = \mathbf{e}^1 + \mathbf{e}^2. \tag{5.12}$$

Combining (5.11) and (5.12) yields

$$\bar{\mathbf{x}}^{11} + \bar{\mathbf{x}}^{12} + \tilde{\mathbf{x}}^{21} = 2\mathbf{e}^1 + \mathbf{e}^2,$$

so the proposed allocation is indeed feasible for the coalition S of two type 1's and one type 2. Because we have found a coalition and an allocation they can achieve that makes two of them strictly better off and the other no worse off than their assignments under $\tilde{\mathbf{x}}$, that coalition blocks $\tilde{\mathbf{x}}$ in the four-consumer economy, ruling it out of the core of $\mathcal{E}_2$.

If we continue to replicate the economy, so that more consumers can form more coalitions, can we 'shrink' the core even further? If so, are there any allocations that are never ruled out and so belong to the core of every replica economy? The answer to both questions is yes, as we now proceed to show in the general case.

We would like to demonstrate that the set of core allocations for $\mathcal{E}_r$ converges to its set of Walrasian equilibrium allocations as r increases. Through the equal treatment property, we have been able to describe core allocations for $\mathcal{E}_r$ as r-fold copies of allocations in the basic economy. We now do the same for $\mathcal{E}_r$'s set of Walrasian equilibria.

LEMMA 5.4 *An allocation* $\mathbf{x}$ *is a WEA for* $\mathcal{E}_r$ *if and only if it is of the form*

$$\mathbf{x} = \Bigg(\underbrace{\mathbf{x}^1, \ldots, \mathbf{x}^1}_{r\ times}, \underbrace{\mathbf{x}^2, \ldots, \mathbf{x}^2}_{r\ times}, \ldots, \underbrace{\mathbf{x}^I, \ldots, \mathbf{x}^I}_{r\ times}\Bigg),$$

and the allocation $(\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^I)$ *is a WEA for* $\mathcal{E}_1$.

Proof: If $\mathbf{x}$ is a WEA for $\mathcal{E}_r$, then by Theorem 5.5, it is in the core of $\mathcal{E}_r$, so that by Theorem 5.16 it must satisfy the equal treatment property. Hence, it must be an r-fold copy of some allocation in $\mathcal{E}_1$. We leave it as an exercise for you to show that this allocation in $\mathcal{E}_1$ is a WEA for $\mathcal{E}_1$. In addition, we leave the converse as an exercise. ∎

Lemma 5.4 says that as we replicate the economy, the set of Walrasian equilibria remains 'constant' in the sense that it consists purely of copies of Walrasian equilibria of the basic economy. Consequently, the set of Walrasian equilibria of the basic economy keeps track, exactly, of the set of Walrasian equilibria of the r-fold replicas.

We can now compare the set of core allocations for $\mathcal{E}_r$ with its set of Walrasian equilibrium allocations by comparing the set C_r – whose members are allocations for $\mathcal{E}_1$ – with the set of Walrasian equilibrium allocations for $\mathcal{E}_1$.

Because $C_1 \supset C_2 \supset \ldots$, the core is shrinking, as we have already seen. Moreover, $C_1 \supset C_2 \supset \ldots . \supset W_1(\mathbf{e})$, the set of WEAs for $\mathcal{E}_1$. To see this, note that by Lemma 5.4, the r-fold copy of a WEA for $\mathcal{E}_1$ is in the core of $\mathcal{E}_r$, which by the definition of C_r means that the original WEA for $\mathcal{E}_1$ is in C_r.

Now, as we replicate the economy and consider C_r, in the limit only those allocations satisfying $\mathbf{x} \in C_r$ for every $r = 1, 2, \ldots$ will remain. Thus, to say that the core shrinks to the set of competitive equilibria is to say that if $\mathbf{x} \in C_r$ for every r, then $\mathbf{x}$ is a competitive equilibrium allocation for $\mathcal{E}_1$. This is precisely what Debreu and Scarf have shown.

THEOREM 5.17 ***(Edgeworth-Debreu-Scarf) A Limit Theorem on the Core***

If $\mathbf{x} \in C_r$ *for every* $r = 1, 2, \ldots,$ *then* $\mathbf{x}$ *is a Walrasian equilibrium allocation for* $\mathcal{E}_1$.

Before presenting the general argument, we will sharpen our intuition by considering the two-type Edgeworth box case. So, consider Fig. 5.11. Let us suppose, by way of contradiction, that some non-Walrasian equilibrium allocation, $\tilde{\mathbf{x}}$, is in C_r for every r. In particular, then, $\tilde{\mathbf{x}}$ is in the core of the basic two consumer economy consisting of one consumer of each type. In Fig. 5.11, this means that $\tilde{\mathbf{x}}$ must be within the lens and on the contract curve. That is, it must be on the squiggly line, and the consumers' indifference curves through $\tilde{\mathbf{x}}$ must be tangent.

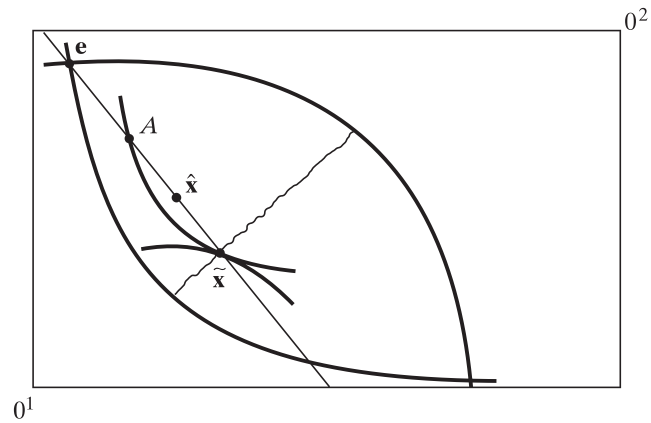


Figure 5.11. Illustration for the proof of Theorem 5.17.

Now consider the line joining the endowment point, **e**, and $\tilde{\mathbf{x}}$. This corresponds to a budget line for both consumers and an associated pair of prices p_1, p_2 for the two goods. Because $MRS^1_{12}(\tilde{\mathbf{x}}^1) = MRS^2_{12}(\tilde{\mathbf{x}})$, either $p_1/p_2 > MRS^1_{12}(\tilde{\mathbf{x}}^1)$, or $p_2/p_1 > MRS^2_{12}(\tilde{\mathbf{x}}^2)$. Note that equality cannot hold; otherwise, these prices would constitute a Walrasian equilibrium, and **x** would be a Walrasian equilibrium allocation. Fig. 5.11 depicts the first case. The second is handled analogously by reversing the roles of types 1 and 2.

As shown, the line from **e** to $\tilde{\mathbf{x}}$ therefore cuts the type 1's indifference curve at point A, and by strict convexity, lies entirely above it between A and $\tilde{\mathbf{x}}$. Thus, there exists some point like $\hat{\mathbf{x}}$ on the segment from A to $\tilde{\mathbf{x}}$, which a type 1 consumer strictly prefers to his bundle at $\tilde{\mathbf{x}}$. Because $\hat{\mathbf{x}}$ lies on the chord from **e** to $\tilde{\mathbf{x}}$, it can be expressed as a convex combination of **e** and $\tilde{\mathbf{x}}$. Thinking ahead a little, let us then write the type 1 bundle at $\hat{\mathbf{x}}$ as follows:

$$\hat{\mathbf{x}}^1 \equiv \frac{1}{r}\mathbf{e}^1 + \frac{r-1}{r}\tilde{\mathbf{x}}^1 \tag{5.13}$$

for some $r > 1$. Notice first that this is indeed a convex combination of the sort described because $1/r + (r-1)/r = 1$. For the record, let us recall that

$$\hat{\mathbf{x}}^1 \succ^1 \tilde{\mathbf{x}}^1. \tag{5.14}$$

Suppose, as can always be arranged, that r is an integer, and consider $\mathcal{E}_r$, the economy with r consumers of each type. Because we are assuming $\tilde{\mathbf{x}} \in C_r$, this means that the r-fold copy of $\tilde{\mathbf{x}}$ is in the core of $\mathcal{E}_r$. But can this be so? Not if we can find a coalition and an allocation that blocks it, and that is just what we will do.

This time, our coalition S consists of all r type 1 consumers and $r-1$ of the type 2 consumers. If we give each type 1 the bundle $\hat{\mathbf{x}}^1$, then from (5.14), each would prefer it to his assignment under $\tilde{\mathbf{x}}$. If we give each type 2 in the coalition a bundle $\tilde{\mathbf{x}}^2$ identical to her assignment under $\tilde{\mathbf{x}}$, each type 2 of course would be indifferent. Thus, we would have

$$\begin{aligned} \hat{\mathbf{x}}^1 &\succ^1 \tilde{\mathbf{x}}^1 && \text{for each of the } r \text{ type 1 consumers,} \\ \tilde{\mathbf{x}}^2 &\sim^2 \tilde{\mathbf{x}}^2 && \text{for each of the } (r-1) \text{ type 2 consumers.} \end{aligned} \tag{5.15}$$

Is such an allocation feasible for S? Summing over the consumers in S, their aggregate allocation is $r\hat{\mathbf{x}}^1 + (r-1)\tilde{\mathbf{x}}^2$. From the definition of $\hat{\mathbf{x}}^1$ in (5.13),

$$\begin{aligned} r\hat{\mathbf{x}}^1 + (r-1)\tilde{\mathbf{x}}^2 &= r\left[\frac{1}{r}\mathbf{e}^1 + \frac{r-1}{r}\tilde{\mathbf{x}}^1\right] + (r-1)\tilde{\mathbf{x}}^2 \\ &= \mathbf{e}^1 + (r-1)(\tilde{\mathbf{x}}^1 + \tilde{\mathbf{x}}^2). \end{aligned} \tag{5.16}$$

Now recall that $\tilde{\mathbf{x}}^1$ and $\tilde{\mathbf{x}}^2$ are, by assumption, in the core of the basic two-consumer economy. They therefore must be feasible for the two-consumer economy, so we know

$$\tilde{\mathbf{x}}^1 + \tilde{\mathbf{x}}^2 = \mathbf{e}^1 + \mathbf{e}^2. \tag{5.17}$$

Using (5.16) and (5.17), we find that

$$\begin{aligned} r\hat{\mathbf{x}}^1 + (r-1)\tilde{\mathbf{x}}^2 &= \mathbf{e}^1 + (r-1)(\tilde{\mathbf{x}}^1 + \tilde{\mathbf{x}}^2) \\ &= \mathbf{e}^1 + (r-1)(\mathbf{e}^1 + \mathbf{e}^2) \\ &= r\mathbf{e}^1 + (r-1)\mathbf{e}^2, \end{aligned}$$

confirming that the proposed allocation in (5.15) is indeed feasible for the coalition of r type 1's and $(r-1)$ type 2's. Because that allocation is feasible and strictly preferred by some members of S, and no worse for every member of S than the r-fold copy of $\tilde{\mathbf{x}}$, S blocks the r-fold copy of $\tilde{\mathbf{x}}$ and so it is not in the core of $\mathcal{E}_r$. We conclude that if $\mathbf{x} \in C_r$ for every r, then it must be a Walrasian equilibrium allocation in the basic economy.

We now give the general argument under two additional hypotheses. The first is that if $\mathbf{x} \in C_1$, then $\mathbf{x} \gg \mathbf{0}$. The second is that for each $i \in \mathcal{I}$, the utility function u^i representing $\succsim^i$ is differentiable on $\mathbb{R}^n_{++}$ with a strictly positive gradient vector there.

Proof: Suppose that $\tilde{\mathbf{x}} \in C_r$ for every r. We must show that $\tilde{\mathbf{x}}$ is a WEA for $\mathcal{E}_1$.

We shall first establish that

$$u^i((1-t)\tilde{\mathbf{x}}^i + t\mathbf{e}^i) \le u^i(\tilde{\mathbf{x}}^i), \qquad \forall\, t \in [0,1], \text{ and } \forall\, i \in \mathcal{I}. \tag{P.1}$$

To see that this inequality must hold, let us suppose that it does not and argue to a contradiction. So, suppose that for some $\bar{t} \in [0,1]$, and some $i \in \mathcal{I}$,

$$u^i((1-\bar{t})\tilde{\mathbf{x}}^i + \bar{t}\mathbf{e}^i) > u^i(\tilde{\mathbf{x}}^i).$$

By the strict quasiconcavity of u^i, this implies that

$$u^i((1-t)\tilde{\mathbf{x}}^i + t\mathbf{e}^i) > u^i(\tilde{\mathbf{x}}^i), \qquad \text{for all } t \in (0, \bar{t}].$$

Consequently, by the continuity of u^i, there is a positive integer, r, large enough such that

$$u^i\left(\left(1-\frac{1}{r}\right)\tilde{\mathbf{x}}^i + \frac{1}{r}\mathbf{e}^i\right) > u^i(\tilde{\mathbf{x}}^i).$$

But we can now use precisely the same argument that we gave in the discussion preceding the proof to show that the r-fold copy of $\tilde{\mathbf{x}}$ is then not in the core of $\mathcal{E}_r$. But this contradicts the fact that $\tilde{\mathbf{x}} \in C_r$. We therefore conclude that (P.1) must hold.

Now, look closely at (P.1). Considering the left-hand side as a real-valued function of t on [0, 1], it says that this function achieves a maximum at $t = 0$. Because this is on the lower boundary of [0, 1] it implies that the derivative of the left-hand side is non-positive when evaluated at $t = 0$. Taking the derivative and evaluating it at $t = 0$ then gives

$$\nabla u^i(\tilde{\mathbf{x}}^i) \cdot (\mathbf{e}^i - \tilde{\mathbf{x}}^i) \leq 0, \qquad \text{for all } i \in I. \tag{P.2}$$

Now, because $\tilde{\mathbf{x}}$ is in the core of $\mathcal{E}_1$, it is Pareto efficient. Moreover, by our additional hypotheses, $\tilde{\mathbf{x}} \gg \mathbf{0}$, and each $\nabla u^i(\tilde{\mathbf{x}}^i) \gg \mathbf{0}$. Consequently, as you are asked to show in Exercise 5.27, the strictly positive gradient vectors, $\nabla u^1(\tilde{\mathbf{x}}^1), \ldots, \nabla u^I(\tilde{\mathbf{x}}^I)$, are proportional to one another and so to a common vector $\tilde{\mathbf{p}} \gg \mathbf{0}$. Consequently, there are strictly positive numbers, $\lambda_1, \ldots, \lambda_I$ such that

$$\nabla u^i(\tilde{\mathbf{x}}^i) = \lambda_i \tilde{\mathbf{p}}, \qquad \text{for all } i \in \mathcal{I}. \tag{P.3}$$

Together, (P.2), (P.3), and the positivity of each of the λ_i's give

$$\tilde{\mathbf{p}} \cdot \tilde{\mathbf{x}}^i \geq \tilde{\mathbf{p}} \cdot \mathbf{e}^i \qquad \text{for all } i \in \mathcal{I}. \tag{P.4}$$

Note that we would be finished if each inequality in (P.4) were an equality. For in this case, $\tilde{\mathbf{x}}^i$ would satisfy the first-order conditions for a maximum of the consumer's utility-maximisation problem subject to the budget constraint at prices $\tilde{\mathbf{p}}$. Moreover, under the hypotheses we have made, the first-order conditions are sufficient for a utility-maximising solution as well (see Theorem 1.4). That is, $\tilde{\mathbf{x}}^i$ would be a Walrasian equilibrium allocation for $\mathcal{E}_1$.

We now show that indeed each inequality in (P.4) must be an equality. Note that because $\tilde{\mathbf{x}} \in C_r$, it must be feasible in $\mathcal{E}_1$. Therefore,

$$\sum_{i \in \mathcal{I}} \tilde{\mathbf{x}}^i = \sum_{i \in \mathcal{I}} \mathbf{e}^i,$$

so that

$$\tilde{\mathbf{p}} \cdot \sum_{i \in \mathcal{I}} \tilde{\mathbf{x}}^i = \tilde{\mathbf{p}} \cdot \sum_{i \in \mathcal{I}} \mathbf{e}^i.$$

However, this equality would fail if for even one consumer *i*, the inequality in (P.4) were strict. ∎

We have shown that for large enough economies, only WEAs will be in the core. This astonishing result really does point towards some unique characteristics of large market economies and suggests itself as a sort of ultimate 'proof' of Adam Smith's intuitions about the efficacy of competitive market systems. The result does bear some scrutiny, however. First of all, it was obtained within the rather rigid context of replica economies with equal numbers of each type of consumer. Second, we cannot lose sight of the fact that the core itself is a very weak solution concept with arguable equity properties. To the extent that a 'good' solution to the distribution problem from society's point of view includes considerations of equity, even the broadest interpretation of this result does not provide support to arguments for pure laissez-faire. The 'equity' of *any* core allocation, and so of any WEA, depends on what the initial endowments are.

The first of these objections can be, and has been, addressed. Abandoning the rigid world of replica economies in favour of more flexible 'continuum economies', Aumann (1964), Hildenbrand (1974), and others have proved even stronger results without the assumption of equal numbers of each type. What then of the second objection cited? Well, if we want to use the market system to achieve the 'good society', the Second Welfare Theorem tells us that we can. All we need to do is decide where in the core we want to be and then redistribute 'endowments' or 'income' and use the market to 'support' that distribution. Ah, but there's the rub. How do we decide where we want to be? How does 'society' decide which distribution in the core it 'prefers'? This is the kind of question we take up in the next chapter.

5.6 Exercises

5.1 In an Edgeworth box economy, do the following:

(a) Sketch a situation where preferences are neither convex nor strictly monotonic and there is no Walrasian equilibrium.

(b) Sketch a situation where preferences are neither convex nor strictly monotonic yet a Walrasian equilibrium exists nonetheless.

(c) Repeat parts (a) and (b), but this time assume preferences are not continuous.

5.2 Let some consumer have endowments $\mathbf{e}$ and face prices $\mathbf{p}$. His indirect utility function is thus $v(\mathbf{p}, \mathbf{p} \cdot \mathbf{e})$. Show that whenever the price of a good rises by a sufficiently small amount, the consumer will be made worse off if initially he was a net demander of the good (i.e., his demand exceeded his endowment) and made better of if he was initially a net supplier of the good. What can you say if the price of the good rises by a sufficiently *large* amount?

5.3 Consider an exchange economy. Let $\mathbf{p}$ be a vector of prices in which the price of at least one good is non-positive. Show that if consumers' utility functions are strongly increasing, then aggregate excess demand cannot be zero in every market.

5.4 Derive the excess demand function $\mathbf{z}(\mathbf{p})$ for the economy in Example 5.1. Verify that it satisfies Walras' law.

5.5 In Example 5.1, calculate the consumers' Walrasian equilibrium allocations and illustrate in an Edgeworth box. Sketch in the contract curve and identify the core.

5.6 Prove Lemma 5.1 and complete the proof of Lemma 5.2.

5.7 Consider an exchange economy with two goods. Suppose that its aggregate excess demand function is $\mathbf{z}(p_1, p_2) = (-1, p_1/p_2)$ for all $(p_1, p_2) \gg (0, 0)$.

(a) Show that this function satisfies conditions 1 and 2 of Theorem 5.3, but not condition 3.

(b) Show that the conclusion of Theorem 5.3 fails here. That is, show that there is no $(p_1^*, p_2^*) \gg (0, 0)$ such that $\mathbf{z}(p_1^*, p_2^*) = (0, 0)$.

5.8 Let $\mathbf{p}^m$ be a sequence of strictly positive prices converging to $\bar{\mathbf{p}}$, and let a consumer's endowment vector be $\mathbf{e}$. Show that the sequence $\{\mathbf{p}^m \cdot \mathbf{e}\}$ of the consumer's income is bounded. Indeed, show more generally that if a sequence of real numbers converges, then it must be bounded.

5.9 Prove the corollary to Theorem 5.8. Extend the argument to show that, under the same assumptions, any Pareto-efficient allocation can be supported as a WEA for some Walrasian equilibrium $\bar{\mathbf{p}}$ and some distribution of *income*, $(R^1, \ldots, R^I)$, where R^i is the income distributed to consumer i.

5.10 In a two-person, two-good exchange economy with strictly increasing utility functions, it is easy to see that an allocation $\bar{\mathbf{x}} \in F(\mathbf{e})$ is Pareto efficient if and only if $\bar{\mathbf{x}}^i$ solves the problem

$$\max_{\mathbf{x}^i} \ u^i(\mathbf{x}^i) \qquad \text{s.t.} \qquad \begin{aligned} u^j(\mathbf{x}^j) &\geq u^j(\bar{\mathbf{x}}^j), \\ x_1^1 + x_1^2 &= e_1^1 + e_1^2, \\ x_2^1 + x_2^2 &= e_2^1 + e_2^2 \end{aligned}$$

for $i = 1, 2$ and $i \neq j$.

(a) Prove the claim.

(b) Generalise this equivalent definition of a Pareto-efficient allocation to the case of n goods and I consumers. Then prove the general claim.

5.11 Consider a two-consumer, two-good exchange economy. Utility functions and endowments are

$$u^1(x_1, x_2) = (x_1 x_2)^2 \qquad \text{and} \qquad \mathbf{e}^1 = (18, 4),$$
$$u^2(x_1, x_2) = \ln(x_1) + 2\ \ln(x_2) \qquad \text{and} \qquad \mathbf{e}^2 = (3, 6).$$

(a) Characterise the set of Pareto-efficient allocations as completely as possible.

(b) Characterise the core of this economy.

(c) Find a Walrasian equilibrium and compute the WEA.

(d) Verify that the WEA you found in part (c) is in the core.

5.12 There are two goods and two consumers. Preferences and endowments are described by

$$u^1(x_1, x_2) = \min(x_1, x_2) \quad \text{and} \quad \mathbf{e}^1 = (30, 0),$$
$$v^2(\mathbf{p}, y) = y/2\sqrt{p_1 p_2} \quad \text{and} \quad \mathbf{e}^2 = (0, 20),$$

respectively.

(a) Find a Walrasian equilibrium for this economy and its associated WEA.

(b) Do the same when 1's endowment is $\mathbf{e}^1 = (5, 0)$ and 2's remains $\mathbf{e}^2 = (0, 20)$.

5.13 An exchange economy has two consumers with expenditure functions:

$$e^1(\mathbf{p}, u) = \left(3(1.5)^2 p_1^2 p_2 \exp(u)\right)^{1/3},$$
$$e^2(\mathbf{p}, u) = \left(3(1.5)^2 p_2^2 p_1 \exp(u)\right)^{1/3}.$$

If initial endowments are $\mathbf{e}^1 = (10, 0)$ and $\mathbf{e}^2 = (0, 10)$, find the Walrasian equilibrium.

5.14 Suppose that each consumer i has a strictly positive endowment vector, $\mathbf{e}^i$, and a Cobb-Douglas utility function on $\mathbb{R}^n_+$ of the form $u^i(\mathbf{x}) = x_1^{\alpha_1^i} x_2^{\alpha_2^i} \cdots x_n^{\alpha_n^i}$, where $\alpha_k^i > 0$ for all consumers i, and goods k, and $\sum_{k=1}^n \alpha_k^i = 1$ for all i.

(a) Show that no consumer's utility function is strongly increasing on $\mathbb{R}^n_+$, so that one cannot apply Theorem 5.5 to conclude that this economy possesses a Walrasian equilibrium.

(b) Show that conditions 1, 2, and 3 of Theorem 5.3 are satisfied so that one can nevertheless use Theorem 5.3 directly to conclude that a Walrasian equilibrium exists here.

(c) Prove that a Walrasian equilibrium would also exist with Cobb-Douglas utilities when production is present and each production set satisfies Assumption 5.2. Use the same strategy as before.

5.15 There are 100 units of x_1 and 100 units of x_2. Consumers 1 and 2 are each endowed with 50 units of each good. Consumer 1 says, 'I love x_1, but I can take or leave x_2'. Consumer 2 says, 'I love x_2, but I can take or leave x_1'.

(a) Draw an Edgeworth box for these traders and sketch their preferences.

(b) Identify the core of this economy.

(c) Find all Walrasian equilibria for this economy.

5.16 Consider a simple exchange economy in which consumer 1 has expenditure function

$$e^1(\mathbf{p}, u) = \begin{cases} \frac{1}{3}(p_1 + p_2)u & \text{for } p_2/2 < p_1 < 2p_2, \\ up_2 & \text{for } p_1 \geq 2p_2, \\ up_1 & \text{for } p_1 \leq p_2/2, \end{cases}$$

and consumer 2 has expenditure function

$$e^2(\mathbf{p}, u) = (p_1 + p_2)u \quad \text{for all } (p_1, p_2).$$

(a) Sketch the Edgeworth box for this economy when aggregate endowments are $(1,\ 1)$. Identify the set of Pareto-efficient allocations.

(b) Sketch the Edgeworth box for this economy when aggregate endowments are $(2,\ 1)$. Identify the set of Pareto-efficient allocations.

5.17 Consider an exchange economy with two identical consumers. Their common utility function is $u^i(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$ for $0 < \alpha < 1$. Society has 10 units of x_1 and 10 units of x_2 in all. Find endowments $\mathbf{e}^1$ and $\mathbf{e}^2$, where $\mathbf{e}^1 \neq \mathbf{e}^2$, and Walrasian equilibrium prices that will 'support' as a WEA the equal-division allocation giving both consumers the bundle $(5,\ 5)$.

5.18 In a two-good, two-consumer economy, utility functions are

$$u^1(x_1, x_2) = x_1(x_2)^2,$$
$$u^2(x_1, x_2) = (x_1)^2 x_2.$$

Total endowments are $(10, 20)$.

(a) A social planner wants to allocate goods to maximise consumer 1's utility while holding consumer 2's utility at $u^2 = 8000/27$. Find the assignment of goods to consumers that solves the planner's problem and show that the solution is Pareto efficient.

(b) Suppose, instead, that the planner just divides the endowments so that $\mathbf{e}^1 = (10, 0)$ and $\mathbf{e}^2 = (0, 20)$ and then lets the consumers transact through perfectly competitive markets. Find the Walrasian equilibrium and show that the WEAs are the same as the solution in part (a).

5.19 (Scarf) An exchange economy has three consumers and three goods. Consumers' utility functions and initial endowments are as follows:

$$u^1(x_1, x_2, x_3) = \min(x_1, x_2) \qquad e^1 = (1, 0, 0),$$
$$u^2(x_1, x_2, x_3) = \min(x_2, x_3) \qquad e^2 = (0, 1, 0),$$
$$u^3(x_1, x_2, x_3) = \min(x_1, x_3) \qquad e^1 = (0, 0, 1).$$

Find a Walrasian equilibrium and the associated WEA for this economy.

5.20 In an exchange economy with two consumers, total endowments are $(e_1, e_2) \equiv (e_1^1 + e_1^2, e_2^1 + e_2^2)$. Consumer i requires s_j^i units of good j to survive, but consumers differ in that $(s_1^1, s_2^1) \neq (s_1^2, s_2^2)$. Consumers are otherwise identical, with utility functions $u^i = (x_1^i - s_1^i)^\alpha + (x_2^i - s_2^i)^\alpha$ for $0 < \alpha < 1$ and $i = 1, 2$.

(a) Suppose now that there is a single hypothetical consumer with initial endowments (e_1, e_2) and utility function $u = (x_1 - s_1)^\alpha + (x_2 - s_2)^\alpha$, where $s_j \equiv s_j^1 + s_j^2$ for $j = 1, 2$. Calculate $(\partial u/\partial x_1)/(\partial u/\partial x_2)$ for this consumer and evaluate it at $(x_1, x_2) = (e_1, e_2)$. Call what you've obtained p^*.

(b) Show that p^* obtained in part (a) must be an equilibrium relative price for good x_1 in the exchange economy previously described.

5.21 Consider an exchange economy with the two consumers. Consumer 1 has utility function $u^1(x_1, x_2) = x_2$ and endowment $e^1 = (1, 1)$ and consumer 2 has utility function $u^2(x^1, x^2) = x^1 + x^2$ and endowment $e^2 = (1, 0)$.

(a) Which of the hypotheses of Theorem 5.4 fail in this example?

(b) Show that there does not exist a Walrasian equilibrium in this exchange economy.

5.22 This exercise will guide you through a proof of a version of Theorem 5.4 when the consumer's utility function is quasiconcave instead of strictly quasiconcave and strictly increasing instead of strongly increasing.

(a) If the utility function $u\colon \mathbb{R}^n_+ \to \mathbb{R}$ is continuous, quasiconcave and strictly increasing, show that for every $\varepsilon \in (0, 1)$ the approximating utility function $v_\varepsilon : \mathbb{R}^n_+ \to \mathbb{R}$ defined by

$$v_\varepsilon(\mathbf{x}) = u\left(x_1^\varepsilon + (1-\varepsilon)\sum_{i=1}^{n} x_i^\varepsilon, \ldots, x_n^\varepsilon + (1-\varepsilon)\sum_{i=1}^{n} x_i^\varepsilon\right),$$

is continuous, strictly quasiconcave and strongly increasing. Note that the approximation to $u(\cdot)$ becomes better and better as $\varepsilon \to 1$ because $v_\varepsilon(\mathbf{x}) \to u(\mathbf{x})$ as $\varepsilon \to 1$.

(b) Show that if in an exchange economy with a positive endowment of each good, each consumer's utility function is continuous, quasiconcave and strictly increasing on $\mathbb{R}^n_+$, there are approximating utility functions as in part (a) that define an exchange economy with the same endowments and possessing a Walrasian equilibrium. If, in addition, each consumer's endowment gives him a positive amount of each good, show that any limit of such Walrasian equilibria, as the approximations become better and better (e.g., as $\varepsilon \to 1$ in the approximations in part (a)) is a Walrasian equilibrium of the original exchange economy.

(c) Show that such a limit of Walrasian equilibria as described in part (b) exists. You will then have proven the following result.
If each consumer in an exchange economy is endowed with a positive amount of each good and has a continuous, quasiconcave and strictly increasing utility function, a Walrasian equilibrium exists.

(d) Which hypotheses of the Walrasian equilibrium existence result proved in part (b) fail to hold in the exchange economy in Exercise 5.21?

5.23 Show that if a firm's production set is strongly convex and the price vector is strictly positive, then there is at most one profit-maximising production plan.

5.24 Provide a proof of Theorem 5.10.

5.25 Complete the proof of Theorem 5.13 by showing that $\mathbf{z}(\mathbf{p})$ in the economy with production satisfies all the properties of Theorem 5.3.

5.26 Suppose that in a single-consumer economy, the consumer is endowed with none of the consumption good, y, and 24 hours of time, h, so that $\mathbf{e} = (24, 0)$. Suppose as well that preferences are defined over $\mathbb{R}^2_+$ and represented by $u(h, y) = hy$, and production possibilities are $Y = \{(-h, y) \mid 0 \le h \le b$ and $0 \le y \le \sqrt{h}\}$, where b is some large positive number. Let p_y and p_h be prices of the consumption good and leisure, respectively.

(a) Find relative prices p_y/p_h that clear the consumption and leisure markets simultaneously.

(b) Calculate the equilibrium consumption and production plans and sketch your results in $\mathbb{R}^2_+$.

(c) How many hours a day does the consumer work?

5.27 Consider an exchange economy $(u^i, \mathbf{e}^i)_{i\in\mathcal{I}}$ in which each u^i is continuous and quasiconcave on $\mathbb{R}^n_+$. Suppose that $\bar{\mathbf{x}} = (\bar{\mathbf{x}}^1, \bar{\mathbf{x}}^2, \ldots, \bar{\mathbf{x}}^I) \gg \mathbf{0}$ is Pareto efficient, that each u^i is continuously differentiable in an open set containing $\bar{\mathbf{x}}^i$, and that $\nabla u^i(\bar{\mathbf{x}}^i) \gg \mathbf{0}$. Under these conditions, which differ somewhat from those of Theorem 5.8, follow the steps below to derive another version of the Second Welfare Theorem.

(a) Show that for any two consumers i and j, the gradient vectors $\nabla u^i(\bar{\mathbf{x}}^i)$ and $\nabla u^j(\bar{\mathbf{x}}^j)$ must be proportional. That is, there must exist some $\alpha > 0$ (which may depend on i and j) such that $\nabla u^i(\bar{\mathbf{x}}^i) = \alpha \nabla u^j(\bar{\mathbf{x}}^j)$. Interpret this condition in the case of the Edgeworth box economy.

(b) Define $\bar{\mathbf{p}} = \nabla u^1(\bar{\mathbf{x}}^1) \gg \mathbf{0}$. Show that for every consumer i, there exists $\lambda_i > 0$ such that $\nabla u^i(\bar{\mathbf{x}}^i) = \lambda_i \bar{\mathbf{p}}$.

(c) Use Theorem 1.4 to argue that for every consumer i, $\bar{\mathbf{x}}^i$ solves

$$\max_{\mathbf{x}^i} u^i(\mathbf{x}^i) \qquad \text{s.t.} \qquad \bar{\mathbf{p}} \cdot \mathbf{x}^i \leq \bar{\mathbf{p}} \cdot \bar{\mathbf{x}}^i.$$

5.28 Suppose that all of the conditions in Exercise 5.27 hold, except the strict positivity of $\bar{\mathbf{x}}$ and the consumers' gradient vectors. Using an Edgeworth box, provide an example showing that in such a case, it may not be possible to support $\bar{\mathbf{x}}$ as a Walrasian equilibrium allocation. Because Theorem 5.8 does not require $\bar{\mathbf{x}}$ to be strictly positive, which hypothesis of Theorem 5.8 does your example violate?

5.29 Consider an exchange economy $(u^i, \mathbf{e}^i)_{i\in I}$ in which each u^i is continuous and quasiconcave on $\mathbb{R}^n_+$. Suppose that $\bar{\mathbf{x}} = (\bar{\mathbf{x}}^1, \bar{\mathbf{x}}^2, \ldots, \bar{\mathbf{x}}^I) \gg \mathbf{0}$ is Pareto efficient. Under these conditions, which differ from those of both Theorem 5.8 and Exercise 5.27, follow the steps below to derive yet another version of the Second Welfare Theorem.

(a) Let $C = \{\mathbf{y} \in \mathbb{R}^n \mid \mathbf{y} = \sum_{i\in I} \mathbf{x}^i$, some $\mathbf{x}^i \in \mathbb{R}^n$ such that $u^i(\mathbf{x}^i) \geq u^i(\bar{\mathbf{x}}^i)$ for all $i \in I$, with at least one inequality strict$\}$, and let $Z = \{\mathbf{z} \in \mathbb{R}^n \mid \mathbf{z} \leq \sum_{i\in I} \mathbf{e}^i\}$. Show that C and Z are convex and that their intersection is empty.

(b) Appeal to Theorem A2.24 to show that there exists a non-zero vector $\mathbf{p} \in \mathbb{R}^n$ such that

$$\mathbf{p} \cdot \mathbf{z} \leq \mathbf{p} \cdot \mathbf{y}, \quad \text{for every } \mathbf{z} \in Z \text{ and every } \mathbf{y} \in C.$$

Conclude from this inequality that $\mathbf{p} \geq \mathbf{0}$.

(c) Consider the same exchange economy, except that the endowment vector is $\bar{\mathbf{x}} = (\bar{\mathbf{x}}^1, \bar{\mathbf{x}}^2, \ldots, \bar{\mathbf{x}}^I)$. Use the inequality in part (b) to show that in this new economy, $\mathbf{p}$ is a Walrasian equilibrium price supporting the allocation $\bar{\mathbf{x}}$.

5.30 Suppose that $\mathbf{y} = \mathbf{0}$ solves

$$\max_{\mathbf{y}} \ \mathbf{p} \cdot \mathbf{y} \qquad \text{s.t.} \qquad \mathbf{y} \in Y - \mathbf{y}^0.$$

Show that $\mathbf{y}^0$ solves

$$\max_{\mathbf{y}} \ \mathbf{p} \cdot \mathbf{y} \qquad \text{s.t.} \qquad \mathbf{y} \in Y.$$

5.31 Consider an economy with production in which there are many goods produced by the production sector, but each firm produces only one of them. Suppose also that each firm's output is given by a differentiable production function and that each consumer's utility function is differentiable as well.

Assume that this economy is in a Walrasian equilibrium with strictly positive prices and that all consumer's marginal utilities (of consumption goods) and all firm's marginal products (of inputs) are also strictly positive.

(a) Show that the *MRS* between any two consumption goods is the same for each consumer, and that it is equal to the ratio of their prices.

(b) Show that the *MRTS* between any two inputs is the same for every firm and equal to the ratio of their prices.

(c) What does this tell you about the information content of Walrasian equilibrium prices?

5.32 Consider a simple economy with two consumers, a single consumption good x, and two time periods. Consumption of the good in period t is denoted x_t for $t = 1, 2$. Intertemporal utility functions for the two consumers are,

$$u_i(x_1, x_2) = x_1 x_2, \ \ i = 1, 2,$$

and endowments are $e^1 = (19, 1)$ and $e^2 = (1, 9)$. To capture the idea that the good is perfectly storable, we introduce a firm producing storage services. The firm can transform one unit of the good in period one into one unit of the good in period 2. Hence, the production set Y is the set of all vectors $(y_1, y_2) \in \mathbb{R}^2$ such that $y_1 + y_2 \leq 0$ and $y_1 \leq 0$. Consumer 1 is endowed with a 100 per cent ownership share of the firm.

(a) Suppose the two consumers cannot trade with one another. That is, suppose that each consumer is in a Robinson Crusoe economy and where consumer 1 has access to his storage firm. How much does each consumer consume in each period? How well off is each consumer? How much storage takes place?

(b) Now suppose the two consumers together with consumer 1's storage firm constitute a competitive production economy. What are the Walrasian equilibrium prices, p_1 and p_2? How much storage takes place now?

(c) Interpret p_1 as a spot price and p_2 as a futures price.

(d) Repeat the exercise under the assumption that storage is costly, i.e., that Y is the set of vectors $(y_1, y_2) \in \mathbb{R}^2$ such that $\delta y_1 + y_2 \leq 0$ and $y_1 \leq 0$, where $\delta \in [0, 1)$. Show that the existence of spot and futures markets now makes both consumers strictly better off.

5.33 The contingent-commodity interpretation of our general equilibrium model permits us to consider time (as in the previous exercise) as well as uncertainty and more (e.g. location). While the trading of contracts nicely captures the idea of futures contracts and prices, one might wonder about the role that spot markets play in our theory. This exercise will guide you through thinking about this. The main result is that once the date zero contingent-commodity contracts market has cleared at Walrasian prices, there is no remaining role for spot markets. Even if spot markets were to open up for some or all goods in some or all periods and in some or all states of the world, no additional trade would take place. All agents would simply exercise the contracts they already have in hand.

(a) Consider an exchange economy with I consumers, N goods, and $T = 2$ dates. There is no uncertainty. We will focus on one consumer whose utility function is $u(\mathbf{x}_1, \mathbf{x}_2)$, where $\mathbf{x}_t \in \mathbb{R}^N_+$ is a vector of period-t consumption of the N goods.

Suppose that $\hat{\mathbf{p}} = (\hat{\mathbf{p}}_1, \hat{\mathbf{p}}_2)$ is a Walrasian equilibrium price vector in the contingent-commodity sense described in Section 5.4, where $\hat{\mathbf{p}}_t \in \mathbb{R}^N_{++}$ is the price vector for period-t

contracts on the N goods. Let $\hat{\mathbf{x}} = (\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2)$ be the vector of contracts that our consumer purchases prior to date 1 given the Walrasian equilibrium price-vector $\hat{\mathbf{p}} = (\hat{\mathbf{p}}_1, \hat{\mathbf{p}}_2)$.

Suppose now that at each date t, spot-markets open for trade.

(i) Because all existing contracts are enforced, argue that our consumer's available endowment in period t is $\hat{\mathbf{x}}_t$.

(ii) Show that if our consumer wishes to trade in some period t spot-market and if all goods have period t spot-markets and the period t spot-prices are $\hat{\mathbf{p}}_t$, then our consumer's period t budget constraint is,

$$\hat{\mathbf{p}}_t \cdot \mathbf{x}_t \leq \hat{\mathbf{p}}_t \cdot \hat{\mathbf{x}}_t.$$

(iii) Conclude that our consumer can ultimately choose any $(\mathbf{x}_1, \mathbf{x}_2)$ such that

$$\hat{\mathbf{p}}_1 \cdot \mathbf{x}_1 \leq \hat{\mathbf{p}}_1 \cdot \hat{\mathbf{x}}_1 \quad \text{and} \quad \hat{\mathbf{p}}_2 \cdot \mathbf{x}_2 \leq \hat{\mathbf{p}}_2 \cdot \hat{\mathbf{x}}_2.$$

(iv) Prove that the consumer can do no better than to choose $\mathbf{x}_1 = \hat{\mathbf{x}}_1$ in period $t = 1$ and $\mathbf{x}_2 = \hat{\mathbf{x}}_2$ in period $t = 2$ by showing that any bundle that is feasible through trading in spot-markets is feasible in the contingent-commodity contract market. You should assume that in period 1 the consumer is forward-looking, knows the spot-prices he will face in period 2, and that he wishes to behave so as to maximise his lifetime utility $u(\mathbf{x}_1, \mathbf{x}_2)$. Further, assume that if he consumes $\bar{\mathbf{x}}_1$ in period $t = 1$, his utility of consuming any bundle $\mathbf{x}_2$ in period $t = 2$ is $u(\bar{\mathbf{x}}_1, \mathbf{x}_2)$.

Because the consumer can do no better if there are fewer spot-markets open, parts (i)–(iv) show that if there is a period t spot-market for good k and the period t spot-price of good k is $\hat{p}_{kt}$, then our consumer has no incentive to trade. Since this is true for all consumers, this shows that spot-markets clear at prices at which there is no trade.

(b) Repeat the exercise with uncertainty instead of time. Assume N goods and two states of the world, $s = 1, 2$. What is the interpretation of the assumption (analogous to that made in part (iv) of (a)) that if the consumer would have consumed bundle $\bar{\mathbf{x}}_1$ had state $s = 1$ occurred, his utility of consuming any bundle $\mathbf{x}_2$ in state $s = 2$ is $u(\bar{\mathbf{x}}_1, \mathbf{x}_2)$?

The next question shows that spot-markets nevertheless have a role.

5.34 (Arrow Securities) Exercise 5.33 shows that when there are opportunities to trade a priori in any commodity contingent on any date, state, etc., there is no remaining role for spot-markets. Here we show that if not all commodities can be traded contingent on every date and state, then spot-markets do have a role. We will in fact suppose that there is only one 'commodity' that can be traded a priori, an *Arrow security* (named after the Nobel prize winning economist Kenneth Arrow). An Arrow security for date t and state s entitles the bearer to one dollar at date t and in state s and nothing otherwise.

We wish to guide you towards showing that if $\hat{\mathbf{p}} \gg \mathbf{0}$ is a Walrasian equilibrium price in the contingent-commodity sense of Section 5.4 when there are N goods as well as time and uncertainty, and $\hat{\mathbf{x}} \geq \mathbf{0}$ is the corresponding Walrasian allocation, then the same prices and allocation arise when only Arrow securities can be traded a priori and all other goods must be traded on spot-markets. This shows that as long as there is a contingent-commodity market for a unit of account (money), the full contingent-commodity Walrasian equilibrium can be implemented with the aid of spot-markets. We will specialise our attention to exchange economies. You are invited to conduct the same analysis for production economies.

Consider then the following market structure and timing. At date zero, there is a market for trade in Arrow securities contingent on any date and any state. The price of each Arrow security is one dollar, and each date t and state s security entitles the bearer to one dollar at date t and in state s, and nothing otherwise. Let a^i_{ts} denote consumer i's quantity of date t and state s Arrow securities. No consumer is endowed with any Arrow securities. Hence, consumer i's budget constraint for Arrow securities at date zero is,

$$\sum_{t,s} a^i_{ts} = 0.$$

At each date $t \geq 1$, the date-t event s_t is realised and all consumers are informed of the date-t state of the world $s = (s_1, \ldots, s_t)$. Each consumer i receives his endowment $\mathbf{e}^i_{st} \in \mathbb{R}^N_+$ of the N goods. Spot-markets open for each of the N goods. If the spot-price of good k is p_{kts}, then consumer i's date-t state-s budget constraint is,

$$\sum_k p_{kts} x^i_{kts} = \sum_k p_{kts} e^i_{kts} + a^i_{ts}.$$

Each consumer i is assumed to know all current and future spot prices for every good in every state (a strong assumption!). Consequently, at date zero consumer i can decide on the trades he will actually make in each spot-market for each good at every future date and in every state. At date zero consumer i therefore solves,

$$\max_{(a^i_{ts}),(x^i_{kts})} \quad u^i((x^i_{kts}))$$

subject to the Arrow security budget constraint,

$$\sum_{t,s} a^i_{ts} = 0,$$

and subject to the spot-market budget constraint,

$$\sum_k p_{kts} x^i_{kts} = \sum_k p_{kts} e^i_{kts} + a^i_{ts} \geq 0,$$

for each date t and state s. (Note the inequality in the date-t state-s constraints. This ensures that there is no bankruptcy.)

(a) Argue that the above formulation implicitly assumes that at any date t, current and future utility in any state is given by $u^i(\cdot)$ where past consumption is fixed at actual levels and consumption in states that did not occur are fixed at the levels that would have been chosen had they occurred.

(b) The consumer's budget constraint in the contingent-commodity model of Section 5.4 specialised to exchange economies is,

$$\sum_{k,t,s} p_{kts} x^i_{kts} = \sum_{k,t,s} p_{kts} e^i_{kts}.$$

Show that (x^i_{kts}) satisfies this budget constraint if and only if there is a vector of Arrow securities (a^i_{st}) such that (x^i_{kts}) and (a^i_{st}) together satisfy the Arrow security budget constraint and each of the spot-market budget constraints.

(c) Conclude from (b) that any Walrasian equilibrium price and allocation of the contingent-commodity model of Section 5.4 can be implemented in the spot-market model described here and that there will typically be trade in the spot-markets. Show also the converse.

(d) Explain why the price of each Arrow security is one. For example, why should the price of a security entitling the bearer to a dollar today be equal to the price of a security entitling the bearer to a dollar tomorrow when it is quite possible that consumers prefer consumption today to the same consumption tomorrow? (Hint: Think about what a dollar will buy.)

(e) Repeat the exercise when, instead of paying the bearer in a unit of account, one date-t state-s Arrow security pays the bearer one unit of good 1 at date t in state s and nothing otherwise. What prices must be set for Arrow securities now in order to obtain the result in part (c)? How does this affect the consumer's Arrow security and spot-market budget constraints?

5.35 (Asset Pricing) We can use our general equilibrium Walrasian model to think about asset pricing. We do this in the simplest possible manner by considering a setting with $N = 1$ good, $T = 1$ period, and finitely many states, $s = 1, 2, \ldots, S$. Thus a consumption bundle $\mathbf{x} = (x_1, x_2, \ldots, x_S) \in \mathbb{R}^S_+$ describes the quantity of the good consumed in each state. Once again, we restrict attention to an exchange economy. There are I consumers and consumer i's utility function is $u^i(x_1, x_2, \ldots, x_S)$ and his endowment vector is $\mathbf{e}^i = (e^i_1, \ldots, e^i_S)$. Note that one unit of commodity s yields one unit of the good in state s. Hence, we can think of commodity s as an Arrow security for the good in state s. Because all Arrow securities are tradeable here, the market is said to be *complete*.

Before thinking about asset pricing, let us consider this simply as an exchange economy and suppose that $\hat{\mathbf{p}} \gg 0$ is a Walrasian equilibrium price vector and that $\hat{\mathbf{x}} = (\hat{\mathbf{x}}^1, \hat{\mathbf{x}}^2, \ldots, \hat{\mathbf{x}}^I)$ is the associated Walrasian equilibrium allocation. Therefore, for each consumer i, $\hat{\mathbf{x}}^i = (\hat{x}^i_1, \hat{x}^i_2, \ldots, \hat{x}^i_S)$ maximises $u^i(x_1, x_2, \ldots, x_S)$ subject to

$$\hat{p}_1 x_1 + \ldots + \hat{p}_S x_S = \hat{p}_1 e^i_1 + \ldots + \hat{p}_S e^i_S,$$

and markets clear. That is,

$$\sum_i \hat{x}^i_s = \sum_i e^i_s,$$

for every state $s = 1, 2, \ldots, S$.

It is convenient to normalise prices throughout this exercise so that they sum to one, i.e., so that $\hat{p}_1 + \ldots + \hat{p}_S = 1$. Then, because $(1, 1, \ldots, 1)$ is the bundle guaranteeing one unit of the good regardless of the state, $\hat{p}_k$ has the interpretation that it is the number of units of the good (i.e., the number of units of the bundle $(1, 1, \ldots, 1)$) that must be paid in order to receive one unit of the good in state k. Thus, each $\hat{p}_k$ is a real, as opposed to a nominal, price.

An *asset* yields in each state s some non-negative amount of the good. Thus an asset is a vector, $\alpha = (\alpha_1, \ldots, \alpha_S) \in \mathbb{R}^S_+$, where α_s denotes the amount of the good the asset yields in state s.

(a) Suppose that the Walrasian equilibrium prices $\hat{\mathbf{p}}$ are in effect and that in addition to markets for each Arrow security, a spot-market opens for trade in an asset $\alpha = (\alpha_1, \ldots, \alpha_S)$. There is zero aggregate supply of asset α but consumers are permitted to purchase both positive and negative quantities of it (negative demand is sometimes called taking a 'short position' in the asset) so long as bankruptcy can be avoided in every state. Argue that consumers would be indifferent to trading in this asset if its price were set equal to $\hat{\mathbf{p}} \cdot \alpha$ and hence that this price is consistent with zero excess demand for the asset. Show also that, given the price vector $\hat{\mathbf{p}}$ for the Arrow securities, $\hat{\mathbf{p}} \cdot \alpha$ is the only price consistent with market-clearing and the occurrence of trade in the asset α.

(b) Suppose that π_s is the probability that state s occurs and that all consumers agree on this. Further, suppose that each consumer's preferences are represented by a von Neumann-Morgenstern utility function, $v_i(x)$, assigning VNM utility to any quantity $x \geq 0$ of the good and that $v_i' > 0$. Further, assume that each consumer is strictly risk averse, i.e., that $v_i'' < 0$. Consequently, for each consumer i,

$$u^i(x_1, \ldots, x_S) = \sum_{s=1}^{S} \pi_s v_i(x_s).$$

(i) Suppose the total endowment of the good is constant across states, i.e., suppose that

$$\sum_i e_s^i = \sum_i e_{s'}^i, \text{ for all states, } s, s'.$$

Show that $\hat{\mathbf{p}} = (\pi_1, \ldots, \pi_S)$ is a Walrasian equilibrium in which each consumer's consumption is constant across all states and in which the equilibrium price of any traded asset $\alpha = (\alpha_1, \ldots, \alpha_S) \in \mathbb{R}_+^S$ is simply its expected value. Thus, when consumers are able to fully diversify their risk, no asset receives a premium over and above its expected value.

(ii) Suppose the total endowment of the good is not constant across states.

(1) Prove that $\hat{\mathbf{p}} \neq (\pi_1, \ldots, \pi_S)$ and, assuming $\hat{\mathbf{x}} \gg \mathbf{0}$, prove that no consumer's consumption is constant across all states.

(2) Argue that the price of any traded asset $\alpha = (\alpha_1, \ldots, \alpha_S) \in \mathbb{R}_+^S$ must be equal to,

$$\frac{E(v_1'(\tilde{x}^1)\tilde{\alpha})}{E(v_1'(\tilde{x}^1))} = \ldots = \frac{E(v_I'(\tilde{x}^I)\tilde{\alpha})}{E(v_I'(\tilde{x}^I))},$$

where E denotes mathematical expectation, $\tilde{x}^i$ is the random variable describing the amount of the good consumed by consumer i in equilibrium ($\tilde{x}^i = \hat{x}_s^i$ in state s), and $\tilde{\alpha}$ is the random variable describing the amount of the good the asset yields ($\tilde{\alpha} = \alpha_s$ in state s). Conclude, at least roughly, that the real price of an asset is higher the more negatively correlated are its returns with consumption – it is then more useful for diversifying risk. In particular, conclude that an asset whose returns are independent of any consumer's marginal utility of consumption has a price equal to its expected value. Thus, the price of an asset is not so much related to its variance but rather the extent to which it is correlated with consumption.

5.36 (Arbitrage Pricing) We shift gears slightly in this question by considering an arbitrage argument that delivers the same pricing of assets as derived in Exercise 5.35. Suppose once again that there is one good and S states. Suppose also that there are N assets, $\alpha^1, \alpha^2, \ldots, \alpha^N$, that can be traded, each being a vector in $\mathbb{R}_+^S$. Let the price of asset k be q_k. We shall normalise prices so that they are real prices. That is, q_k is the number of units of the good that must be given up to purchase one unit of asset k. Suppose an investor purchases x_k units of each asset k.

(a) Show that the (column) vector

$$A\mathbf{x} \in \mathbb{R}_+^S$$

is the induced asset held by the investor subsequent to his purchase, where A is the $S \times N$ matrix whose kth column is α^k, and $\mathbf{x} = (x_1, \ldots, x_N)$ is the vector of the investor's asset purchases.

(b) Argue that the vector

$$A\mathbf{x} - \mathbf{1}(\mathbf{q} \cdot \mathbf{x}) \in \mathbb{R}^S_+$$

describes the real net gain to the investor in every state, where $\mathbf{1}$ is the column vector of S 1's.

(c) Suppose that every coordinate of the real net gain vector

$$A\mathbf{x} - \mathbf{1}(\mathbf{q} \cdot \mathbf{x})$$

is strictly positive. Argue that the investor can earn arbitrarily large profits with an initial outlay of a single unit of the good by repurchasing $\mathbf{x}$ (or an affordable fraction of it) again and again using short sales to cover his expenses, and always guaranteeing against bankruptcy in any state.

(d) Conclude from (c) that for markets to clear, there can be no $\mathbf{x} \in \mathbb{R}^N$ such that every coordinate of the real net gain vector is strictly positive. (Parts (c) and (d) constitute an 'arbitrage-pricing' argument. We next turn to its consequences.)

(e) Let $C = \{\mathbf{y} \in \mathbb{R}^N : \mathbf{y} = A\mathbf{x} - \mathbf{1}(\mathbf{q} \cdot \mathbf{x})$ for some $\mathbf{x} \in \mathbb{R}^N\}$. Conclude from part (d) that

$$C \cap \mathbb{R}^N_{++} = \emptyset,$$

and use the separating hyperplane theorem, Theorem A2.24, to conclude that there is a non-zero vector, $\hat{\mathbf{p}} \in \mathbb{R}^N$ such that

$$\hat{\mathbf{p}} \cdot \mathbf{y} \leq \hat{\mathbf{p}} \cdot \mathbf{z},$$

for all $\mathbf{y} \in C$ and all $\mathbf{z} \in \mathbb{R}^N_{++}$. Show further that $\hat{\mathbf{p}} \geq \mathbf{0}$ because otherwise the right-hand side of the previous inequality could be made arbitrarily negative and therefore for any $\mathbf{y}$, the inequality would fail for some $\mathbf{z}$. Finally, normalise $\hat{\mathbf{p}} \geq \mathbf{0}$ so that its coordinates sum to one.

(f) Using the definition of C and the results from part (e), show that,

$$\left(\hat{\mathbf{p}}^T A - \mathbf{q}\right) \mathbf{x} \leq 0, \text{ for all } \mathbf{x} \in \mathbb{R}^N.$$

Argue that the inequality cannot be strict for any $\mathbf{x} \in \mathbb{R}^N$ because the inequality would then fail for $-\mathbf{x}$. Conclude that,

$$\left(\hat{\mathbf{p}}^T A - \mathbf{q}\right) \mathbf{x} = 0, \text{ for all } \mathbf{x} \in \mathbb{R}^N,$$

and therefore that,

$$\mathbf{q} = \hat{\mathbf{p}}^T A,$$

i.e., that for each asset k,

$$q_k = \hat{\mathbf{p}} \cdot \alpha^k.$$

(g) Compare the result in part (f) with the pricing of the asset that arose from the general equilibrium model considered in part (a) of Exercise 5.35. In that exercise, we assumed that all Arrow securities were tradeable, i.e., we assumed that the market was *complete*. Conclude from the current exercise that if there are no opportunities for profitable arbitrage among the assets that are available for trade, then even if markets are incomplete there are implicit prices, given by $\hat{\mathbf{p}}$, for all Arrow securities. Moreover, the prices of all tradeable assets are derived from these underlying Arrow security prices.

5.37 Complete the proof of Lemma 5.4.

(a) Show that if an allocation $\mathbf{x}$ is an r-fold copy of the allocation $(\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^I)$ in $\mathcal{E}_1$, and $\mathbf{x}$ is a WEA in $\mathcal{E}_r$, then $(\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^I)$ is a WEA in $\mathcal{E}_1$.

(b) Show that if $(\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^I)$ is a WEA in $\mathcal{E}_1$, then its r-fold copy is a WEA in $\mathcal{E}_r$.

5.38 Give a general proof of Theorem 5.16 that is valid for an arbitrary number I of consumer types and an arbitrary number r of consumers of each type.

5.39 (Cornwall) In an economy with two types of consumer, each type has the respective utility function and endowments:

$$u^{1q}(x_1, x_2) = x_1 x_2 \qquad \text{and} \qquad \mathbf{e}^1 = (8, 2),$$
$$u^{2q}(x_1, x_2) = x_1 x_2 \qquad \text{and} \qquad \mathbf{e}^2 = (2, 8).$$

(a) Draw an Edgeworth box for this economy when there is one consumer of each type.

(b) Characterise as precisely as possible the set of allocations that are in the core of this two-consumer economy.

(c) Show that the allocation giving $\mathbf{x}^{11} = (4, 4)$ and $\mathbf{x}^{21} = (6, 6)$ is in the core.

(d) Now replicate this economy once so there are two consumers of each type, for a total of four consumers in the economy. Show that the double copy of the previous allocation, giving $\mathbf{x}^{11} = \mathbf{x}^{12} = (4, 4)$ and $\mathbf{x}^{21} = \mathbf{x}^{22} = (6, 6)$, is *not* in the core of the replicated economy.

5.40 In a pure exchange economy, consumer i *envies* consumer j if $\mathbf{x}^j \succ^i \mathbf{x}^i$. (Thus, i envies j if i likes j's bundle better than his own.) An allocation $\mathbf{x}$ is therefore *envy free* if $\mathbf{x}^i \succsim^i \mathbf{x}^j$ for all i and j. We know that envy-free allocations will always exist, because the equal-division allocation, $\bar{\mathbf{x}} = (1/I)\mathbf{e}$, must be envy free. An allocation is called **fair** if it is both envy free *and* Pareto efficient.

(a) In an Edgeworth box, demonstrate that envy-free allocations need not be fair.

(b) Under Assumption 5.1 on utilities, prove that every exchange economy having a strictly positive aggregate endowment vector possesses at least one fair allocation.

5.41 There are two consumers with the following characteristics:

$$u^1(x_1, x_2) = e^{x_1} x_2 \qquad \text{and} \qquad \mathbf{e}^1 = (1, 1),$$
$$u^2(x_1, x_2) = e^{x_1} x_2^2 \qquad \text{and} \qquad \mathbf{e}^2 = (5, 5).$$

(a) Find the equation for the contract curve in this economy, and carefully sketch it in the Edgeworth box.

(b) Find a fair allocation of goods to consumers in this economy.

(c) Now suppose that the economy is replicated *three* times. Find a fair allocation of goods to consumers in this new economy.

5.42 There are two consumers with the following characteristics:

$$u^1(x_1, x_2) = 2x_1 + x_2 \quad \text{and} \quad \mathbf{e}^1 = (1, 6),$$
$$u^2(x_1, x_2) = x_1 + x_2 \quad \text{and} \quad \mathbf{e}^2 = (3, 4).$$

Find a fair allocation of goods to consumers.

5.43 Throughout, we have assumed that a consumer's utility depends only on his own consumption. Suppose, however, that consumers' utilities are *interdependent*, depending on their own consumption and that of everyone else as well. For example, in a two-good, two-person economy with total endowments $\mathbf{e}$, suppose that $u^1 = u^1(x_1^1, x_2^1, x_1^2, x_2^2)$ and $u^2 = u^2(x_1^2, x_2^2, x_1^1, x_2^1)$, where $\partial u^i/\partial x_1^j \neq 0$ and $\partial u^i/\partial x_2^j \neq 0$ for $i, j = 1, 2$ and $i \neq j$.

(a) What are the necessary conditions for a Pareto-efficient distribution of goods to consumers?

(b) Are the WEAs Pareto efficient in an economy like this? Why or why not?

5.44 In the text, we have called an allocation $\bar{\mathbf{x}}$ Pareto efficient if there exists no other feasible allocation $\mathbf{x}$ such that $\mathbf{x}^i \succsim^i \bar{\mathbf{x}}^i$ for all i and $\mathbf{x}^j \succ^j \bar{\mathbf{x}}^j$ for at least one j. Sometimes, an allocation $\bar{\mathbf{x}}$ is called Pareto efficient if there exists no other feasible allocation $\mathbf{x}$ such that $\mathbf{x}^i \succ^i \bar{\mathbf{x}}^i$ for all i.

(a) Show that when preferences are continuous and strictly monotonic, the two definitions are equivalent.

(b) Construct an example where the two definitions are *not* equivalent, and illustrate in an Edgeworth box.

5.45 (Eisenberg's Theorem) Ordinarily, a system of *market* demand functions need not satisfy the properties of an individual consumer's demand system, such as the Slutsky restrictions, negative semidefiniteness of the substitution matrix, and so forth. Sometimes, however, it is useful to know when the market demand system *does* behave as though it were generated from a single, hypothetical consumer's utility-maximisation problem. Eisenberg (1961) has shown that this will be the case when consumers' preferences can be represented by linear homogeneous utility functions (not necessarily identical), and when the distribution of income is fixed and independent of prices.

In particular, let $\mathbf{x}^i(\mathbf{p}, y^i)$ solve $\max_{\mathbf{x}^i \in \mathbb{R}^n_+} u^i(\mathbf{x}^i)$ subject to $\mathbf{p} \cdot \mathbf{x}^i = y^i$ for $i \in \mathcal{I}$. Let $\mathbf{x}(\mathbf{p}, y^*)$ solve $\max_{\mathbf{x} \in \mathbb{R}^n_+} U(\mathbf{x})$ subject to $\mathbf{p} \cdot \mathbf{x} = y^*$. If (1) $u^i(\mathbf{x}^i)$ is linear homogeneous for all $i \in \mathcal{I}$; (2) y^* is aggregate income and income shares are fixed so that $y^i = \delta^i y^*$ for $0 < \delta^i < 1$ and $\sum_{i \in \mathcal{I}} \delta^i = 1$; and (3)

$$U(\mathbf{x}) = \max \prod_{i \in \mathcal{I}} (u^i(\mathbf{x}^i))^{\delta^i} \quad \text{s.t.} \quad \mathbf{x} = \sum_{i \in \mathcal{I}} \mathbf{x}^i,$$

then $\mathbf{x}(\mathbf{p}, y^*) = \sum_{i \in \mathcal{I}} \mathbf{x}^i(\mathbf{p}, y^i)$, so the system of market demand functions behaves as though generated from a single utility-maximisation problem.

(a) Consider a two-good, two-person exchange economy with initial endowments $\mathbf{e}^1 = (\delta^1, \delta^1)$ and $\mathbf{e}^2 = (\delta^2, \delta^2)$, where $0 < \delta^1 < 1$ and $\delta^1 + \delta^2 = 1$. Verify that income shares are fixed and independent of prices $\mathbf{p} = (p_1, p_2)$.

(b) Solve for $U(\mathbf{x})$ in the economy of part (a) when

$$u^1(\mathbf{x}^1) = \left(x_1^1\right)^{\alpha}\left(x_2^1\right)^{1-\alpha},$$
$$u^2(\mathbf{x}^2) = \left(x_1^2\right)^{\beta}\left(x_2^2\right)^{1-\beta}$$

for $0 < \alpha < 1$ and $0 < \beta < 1$.

(c) Verify Eisenberg's theorem for this economy.

5.46 In an exchange economy with initial endowments $\mathbf{e}$, prove that the aggregate excess demand vector, $\mathbf{z}(\mathbf{p})$, is independent of the initial distribution of endowments if and only if preferences are identical and homothetic.