CHAPTER 21

EQUILIBRIUM ANALYSIS

In this chapter we discuss some topics in general equilibrium analysis that don't conveniently fit in the other chapters. Our first topic concerns the core, a generalization of the Pareto set, and its relationship to Walrasian equilibrium. We follow this by a brief discussion of the relationship between convexity and size. Following this we discuss conditions under which there will be only one Walrasian equilibrium. Finally, the chapter ends with a discussion of the stability of general equilibrium.

21.1 The core of an exchange economy

We have seen that Walrasian equilibria will generally exist and that they will generally be Pareto efficient. But the use of a competitive market mechanism system is only one way to allocate resources. What if we used some other social institution to facilitate trade? Would we still end up with an allocation that was "close to" a Walrasian equilibrium?

In order to examine this question we consider a "market game" where each agent i comes to the market with an initial endowment of $\boldsymbol{\omega}_i$. Instead of using a price mechanism, the agents simply wander around and make tentative arrangements to trade with each other. When all agents have made the best arrangement possible for themselves, the trades are carried out.

As described so far the game has very little structure. Instead of specifying the game in sufficient detail to calculate an equilibrium we ask a more general question. What might be a "reasonable" set of outcomes for this game? Here is a set of definitions that may be useful in thinking about this question.

Improve upon an allocation. *A group of agents S is said to* **improve upon** *a given allocation* $\mathbf{x}$ *if there is some allocation* $\mathbf{x}'$ *such that*

$$\sum_{i \in S} \mathbf{x}_i' = \sum_{i \in S} \boldsymbol{\omega}_i,$$

and

$$\mathbf{x}_i' \succ_i \mathbf{x}_i \quad \text{for all } i \in S.$$

If an allocation $\mathbf{x}$ can be improved upon, then there is some group of agents that can do better by not engaging in the market at all; they would do better by only trading among themselves. An example of this might be a group of consumers who organize a cooperative store to counteract high prices at the grocery store. It seems that any allocation that can be improved upon does not seem like a reasonable equilibrium—some group would always have an incentive to split off from the rest of the economy.

Core of an economy. *A feasible allocation* $\mathbf{x}$ *is in the* **core** *of the economy if it cannot be improved upon by any coalition.*

Notice that, if $\mathbf{x}$ is in the core, $\mathbf{x}$ must be Pareto efficient. For if $\mathbf{x}$ were not Pareto efficient, then the coalition consisting of the entire set of agents could improve upon $\mathbf{x}$. In this sense the core is a generalization of the idea of the Pareto set. If an allocation is in the core, every group of agents gets some part of the gains from trade—no group has an incentive to defect.

One problem with the concept of the core is that it places great informational requirements on the agents—the people in the dissatisfied coalition have to be able to find each other. Furthermore, it is assumed that there are no costs to forming coalitions so that, even if only very small gains can be made by forming coalitions, they will nevertheless be formed.

A geometrical picture of the core can be obtained from the standard Edgeworth box diagram for the two-person, two-good case. See Figure 21.1. In this case the core will be the subset of the Pareto set at which each agent does better than by refusing to trade.

Will the core of an economy generally be nonempty? If we continue to make the assumptions that ensure the existence of a market equilibrium, it will, since the market equilibrium is always contained in the core.

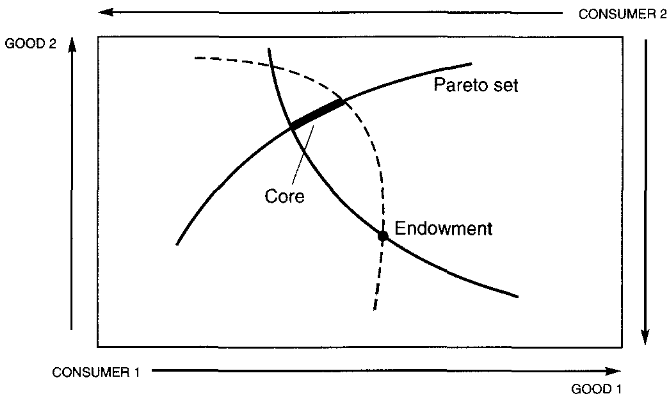


Figure 21.1 **Core in an Edgeworth box.** In the Edgeworth box diagram, the core is simply that segment of the Pareto set that lies between the indifference curves that pass through the initial endowment.

Walrasian equilibrium is in core. *If* $(\mathbf{x}^*, \mathbf{p})$ *is a Walrasian equilibrium with initial endowments* $\boldsymbol{\omega}_i$, *then* $\mathbf{x}^*$ *is in the core.*

Proof. Assume not; then there is some coalition S and some feasible allocation $\mathbf{x}'$ such that all agents i in S strictly prefer $\mathbf{x}'_i$ to $\mathbf{x}^*_i$ and furthermore

$$\sum_{i \in S} \mathbf{x}'_i = \sum_{i \in S} \boldsymbol{\omega}_i.$$

But the definition of the Walrasian equilibrium implies

$$\mathbf{p}\mathbf{x}'_i > \mathbf{p}\boldsymbol{\omega}_i \quad \text{for all } i \text{ in } S$$

so

$$\mathbf{p} \sum_{i \in S} \mathbf{x}'_i > \mathbf{p} \sum_{i \in S} \boldsymbol{\omega}_\mathbf{i}$$

which contradicts the first equality. ∎

We can see from the Edgeworth box diagram that generally there will be other points in the core than just the market equilibrium. However, if we allow our 2-person economy to grow we will have more possible coalitions and hence more opportunities to improve upon any given allocation. Therefore, one might suspect that the core might shrink as the economy grows. One problem with formalizing this idea is that the core is a subset of the allocation space and thus as the economy grows the core keeps changing dimension. Thus we want to limit ourselves to a particularly simple type of growth.

We will say two agents are of the same **type** if both their preferences and their initial endowments are the same. We will say that one economy is a **replica** of another if there are r times as many agents of each type in one economy as in the other. This means that if a large economy replicates a smaller one, it is just a "scaled up" version of the small one. For simplicity we will limit ourselves to only two types of agents, type A and type B. Consider a fixed 2-person economy; by the r-core of this economy, we mean the core of the r^{th} replication of the original economy.

It turns out that all agents of the same type must receive the same bundle at any core allocation. This result makes for a much simpler analysis.

Equal treatment in the core. *Suppose agents' preferences are strictly convex, strongly monotonic, and continuous. Then if* $\mathbf{x}$ *is an allocation in the r-core of a given economy, then any two agents of the same type must receive the same bundle.*

Proof. Let $\mathbf{x}$ be an allocation in the core and index the $2r$ agents using subscripts $A1, \ldots, Ar$ and $B1, \ldots, Br$. If all agents of the same type do not get the same allocation, there will be one agent of each type who is most poorly treated. We will call these two agents the "type-A underdog" and the "type-B underdog." If there are ties, select any of the tied agents.

Let $\overline{\mathbf{x}}_A = \frac{1}{r}\sum_{j=1}^{r} \mathbf{x}_{A_j}$ and $\overline{\mathbf{x}}_B = \frac{1}{r}\sum_{j=1}^{r} \mathbf{x}_{B_j}$ be the average bundle of the type-A and type-B agents. Since the allocation $\mathbf{x}$ is feasible, we have

$$\begin{aligned} \frac{1}{r}\sum_{j=1}^{r} \mathbf{x}_{A_j} + \frac{1}{r}\sum_{j=1}^{r} \mathbf{x}_{B_j} &= \frac{1}{r}\sum_{j=1}^{r} \boldsymbol{\omega}_{A_j} + \frac{1}{r}\sum_{j=1}^{r} \boldsymbol{\omega}_{B_j} \\ &= \frac{1}{r} r\boldsymbol{\omega}_A + \frac{1}{r} r\boldsymbol{\omega}_B. \end{aligned}$$

It follows that

$$\overline{\mathbf{x}}_A + \overline{\mathbf{x}}_B = \boldsymbol{\omega}_A + \boldsymbol{\omega}_B,$$

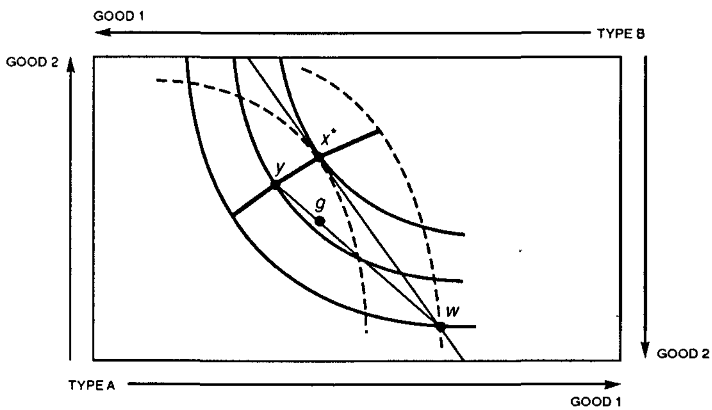
so that $(\overline{\mathbf{x}}_A, \overline{\mathbf{x}}_B)$ is feasible for the coalition consisting of the two underdogs. We are assuming that at least for one type, say type A, two of the type-A agents receive different bundles. Hence, the A underdog will strictly prefer $\overline{\mathbf{x}}_A$ to his present allocation by strict convexity of preferences (since it is a weighted average of bundles that are at least as good as $\mathbf{x}_A$), and the B underdog will think $\overline{\mathbf{x}}_B$ is at least as good as his present bundle. Strong monotonicity and continuity allows A to remove a little from $\overline{\mathbf{x}}_A$, and bribe the type-B underdog, thus forming a coalition that can improve upon the allocation. ∎

Since any allocation in the core must award agents of the same type with the same bundle, we can examine the cores of replicated two-agent economies by use of the Edgeworth box diagram. Instead of a point $\mathbf{x}$ in the core representing how much A gets and how much B gets, we think of $\mathbf{x}$ as telling us how much *each* agent of type A gets and how much *each* agent of type B gets. The above lemma tells us that all points in the r-core can be represented in this manner.

The following proposition shows that any allocation that is not a market equilibrium allocation must eventually not be in the r-core of the economy. This means that core allocations in large economies look just like Walrasian equilibria.

Shrinking core. *Assume that preferences are strictly convex and strongly monotonic, and that there is a unique market equilibrium* $\mathbf{x}^*$ *from initial endowment* $\boldsymbol{\omega}$. *Then if* $\mathbf{y}$ *is not the market equilibrium, there is some replication* r *such that* $\mathbf{y}$ *is not in the* r*-core.*

Proof. Refer to the Edgeworth box in Figure 21.2. We want to show that a point like $\mathbf{y}$ can eventually be improved upon. Since $\mathbf{y}$ is not a Walrasian equilibrium, the line segment through $\mathbf{y}$ and $\boldsymbol{\omega}$ must cut at least one agent's indifference curve through $\mathbf{y}$. Thus it is possible to choose a point such as $\mathbf{g}$ which, for example, agent A prefers to $\mathbf{y}$. There are several cases to treat, depending on the location of $\mathbf{g}$; however, the arguments are essentially the same, so we treat only the case depicted.



The shrinking core. As the economy replicates, a point like y will eventually not be in the core. **Figure 21.2**

Since $\mathbf{g}$ is on the line segment connecting $\mathbf{y}$ and $\boldsymbol{\omega}$, we can write

$$\mathbf{g} = \theta\boldsymbol{\omega}_A + (1-\theta)\mathbf{y}_A$$

for some $\theta > 0$. By continuity of preference, we can also suppose that $\theta = T/V$ for some integers T and V. Hence,

$$\mathbf{g}_A = \frac{T}{V}\boldsymbol{\omega}_A + \left(1 - \frac{T}{V}\right)\mathbf{y}_A.$$

Suppose the economy has replicated V times. Then form a coalition consisting of V consumers of type A and $V - T$ consumers of type B, and consider the allocation $\mathbf{z}$ where agents of type A in the coalition receive $\mathbf{g}_A$ and agents of type B receive $\mathbf{y}_B$. This allocation is preferred to $\mathbf{y}$ by all members of the coalition (we can remove a little from the A agents and give it to the B agents to get strict preference). We will show that it is feasible for the members of the coalition. This follows from the following calculation:

$$\begin{aligned}
V\mathbf{g}_A + &(V-T)\mathbf{y}_B \\
&= V\left[\frac{T}{V}\boldsymbol{\omega}_A + \left(1 - \frac{T}{V}\right)\mathbf{y}_A\right] + (V-T)\mathbf{y}_B \\
&= T\boldsymbol{\omega}_A + (V-T)\mathbf{y}_A + (V-T)\mathbf{y}_B \\
&= T\boldsymbol{\omega}_A + (V-T)[\mathbf{y}_A + \mathbf{y}_B] \\
&= T\boldsymbol{\omega}_A + (V-T)[\boldsymbol{\omega}_A + \boldsymbol{\omega}_B] \\
&= T\boldsymbol{\omega}_A + V\boldsymbol{\omega}_A - T\boldsymbol{\omega}_A + (V-T)\boldsymbol{\omega}_B \\
&= V\boldsymbol{\omega}_A + (V-T)\boldsymbol{\omega}_B.
\end{aligned}$$

This is exactly the endowment of our coalition since it has V agents of type A and $(V - T)$ agents of type B. Thus, this coalition can improve upon $\mathbf{y}$, proving the proposition. ∎

Many of the restrictive assumptions in this proposition can be relaxed. In particular we can easily get rid of the assumptions of strong monotonicity and uniqueness of the market equilibrium. Convexity appears to be crucial to the proposition, but, as in the existence theorem, that assumption is unnecessary for large economies. Of course, we can also allow for there to be more than only two types of agents.

In the study of Walrasian equilibrium we found that the price mechanism leads to a well-defined equilibrium. In the study of Pareto efficient allocations we found that nearly all Pareto efficient allocations can be obtained through a suitable reallocation of endowments and a price mechanism. And here, in the study of a general pure exchange economy, prices appear in a third and different light: the only allocations that are in the core of a large economy are market equilibrium allocations. The shrinking core theorem shows that Walrasian equilibria are robust: even very weak equilibrium concepts, like that of the core, tend to yield allocations that are close to Walrasian equilibria for large economies.

21.2 Convexity and size

Convexity of preference has come up in several general equilibrium models. Usually, the assumption of strict convexity has been used to assure that the demand function is well-defined—that there is only a single bundle demanded at each price—and that the demand function be continuous—that small changes in prices give rise to small changes in demand. The convexity assumption appears to be necessary for the existence of an equilibrium allocation since it is easy to construct examples where nonconvexities cause discontinuities of demand and thus nonexistence of equilibrium prices.

Consider, for example, the Edgeworth box diagram in Figure 21.3. Here agent A has nonconvex preferences while agent B has convex preferences. At the price p^*, there are two points that maximize utility; but supply is not equal to demand at either point.

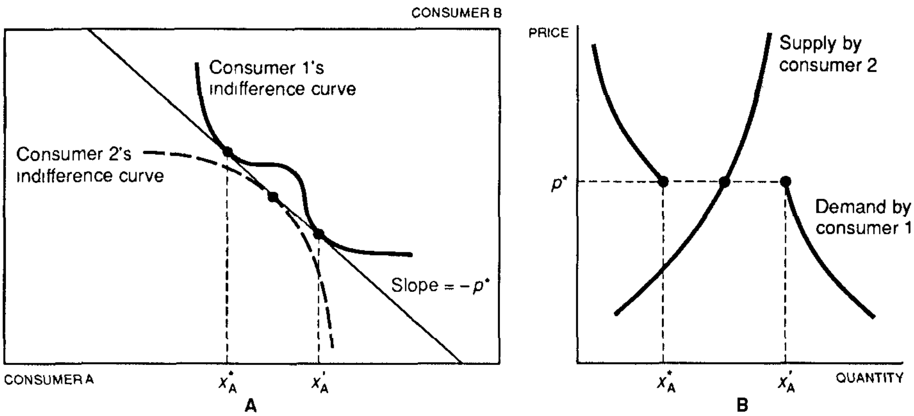


Figure 21.3 **Nonexistence of an equilibrium with nonconvex preferences.** Panel A depicts an Edgeworth box example in which one agent has nonconvex preferences. Panel B shows the associated aggregate demand curve, which will be discontinuous.

However, perhaps equilibrium is not so difficult to achieve as this example suggests. Let us consider a specific example. Suppose that the total supply of the good is just halfway between the two demands at p^* as in Figure 21.3B. Now think what would happen if the economy would replicate once so that there were two agents of type A and two agents of type B. Then at the price p^*, one type-A agent could demand x_A^* and the other type-A agent could demand x'_A. In that case, the total demand by the agents would in fact be equal to the total amount of the good supplied. A Walrasian equilibrium exists for the replicated economy.

It is not hard to see that a similar construction will work no matter where the supply curve lies: if it were two-thirds of the way between x_A^* and x'_A, we would just replicate three times, and so on. We can get aggregate demand arbitrarily close to aggregate supply just by replicating the economy a sufficient number of times.

This argument suggests that in a large economy in which the scale of nonconvexities is small relative to the size of the market, there will generally be a price vector that results in demand being close to supply. For a large enough economy small nonconvexities do not cause serious difficulties.

This observation is closely related to the replication argument described in our discussion of competitive firms behavior. Consider a classic model of firms with fixed costs and U-shaped average cost functions. The supply functions of individual firms will typically be discontinuous, but these discontinuities will be irrelevant if the scale of the market is sufficiently large.

21.3 Uniqueness of equilibrium

We know from the section on existence of general equilibrium that under appropriate conditions a price vector will exist that clears all markets; i.e., there exists a $\mathbf{p}^*$ such that $\mathbf{z}(\mathbf{p}^*) \leq \mathbf{0}$. The question we ask in this section is that of uniqueness: when is there only one price vector that clears all markets?

The free goods case is not of great interest here, so we will rule it out by means of the desirability assumption: we will assume that the excess demand for each good is strictly positive when its relative price is zero. Economically this means that, when the price of a good goes to zero, everyone demands a lot of it, which seems reasonable enough. This has the obvious consequence that at all equilibrium price vectors the price of each good must be strictly positive.

As before, we will want to assume $\mathbf{z}$ is continuous, but now we need even more than that—we want to assume continuous differentiability. The reasons for this are fairly clear; if indifference curves have kinks in them, we can find whole ranges of prices that are market equilibria. Not only are the equilibria not unique, they aren't even *locally* unique.

Given these assumptions, we have a purely mathematical problem: given a smooth mapping $\mathbf{z}$ from the price simplex to R^k, when is there a unique point that maps into zero? It is too much to hope that this will occur in general, since one can construct easy counterexamples, even in the two-dimensional case. Hence, we are interested in finding restrictions on the excess demand functions that ensure uniqueness. We will then be interested in whether these restrictions are strong or weak, what their economic meaning is, and so on.

We will here consider two restrictions on $\mathbf{z}$ that ensure uniqueness. The first case, that of **gross substitutes**, is interesting because it has clear economic meaning and allows a simple, direct proof of uniqueness. The second case, that of **index analysis**, is interesting because it is very general. In fact it contains almost all other uniqueness results as special cases. Unfortunately, the proof utilizes a rather advanced theorem from differential topology.

Gross substitutes

Roughly speaking, two goods are gross substitutes if an increase in the price of one of the goods causes an increase in the demand for the other good. In elementary courses, this is usually the definition of substitutes. In more advanced courses, it is necessary to distinguish between the idea of **net substitutes**—when the price of one good increases, the Hicksian demand for the other good increases—and **gross substitutes**—which replaces "Hicksian" with "Marshallian" in this definition.

Gross substitutes. *Two goods, i and j, are gross substitutes at a price vector* $\mathbf{p}$ *if* $\frac{\partial z_j(\mathbf{p})}{\partial p_i} > 0$ *for* $i \neq j$.

This definition says that two goods are gross substitutes if an increase in price i brings about an increase in the excess demand for good j. If all goods are gross substitutes, the Jacobian matrix of $\mathbf{z}$, $\mathbf{Dz}(\mathbf{p})$, will have all positive off-diagonal terms.

Gross substitutes implies unique equilibrium. *If all goods are gross substitutes at all prices, then if* $\mathbf{p}^*$ *is an equilibrium price vector, it is the unique equilibrium price vector.*

Proof. Suppose $\mathbf{p}'$ is some other equilibrium price vector. Since $\mathbf{p}^* \gg 0$ we can define $m = \max p'_i/p^*_i \neq 0$. By homogeneity and the fact that $\mathbf{p}^*$ is an equilibrium, we know that $\mathbf{z}(\mathbf{p}^*) = \mathbf{z}(m\mathbf{p}^*) = \mathbf{0}$. We know that for some price, p_k, we have $mp^*_k = p'_k$ by the definition of m. We now lower each price mp^*_i other than p_k successively to p'_i. Since the price of each good other than k goes down in the movement from $m\mathbf{p}^*$ to $\mathbf{p}'$, we must have the demand for good k going down. Thus $z_k(\mathbf{p}') < 0$ which implies $\mathbf{p}'$ cannot be an equilibrium. ∎

Index analysis

Consider an economy with only two goods. Choose the price of good 2 as the numeraire, and draw the excess demand curve for the good 1 as a function of its own price. Walras' law implies that, when the excess demand for good 1 is zero, we have an equilibrium. The desirability assumption we have made implies that, when the relative price of good 1 is large, the excess demand for good 1 is negative; and when the relative price of good 1 is small, the excess demand for good 1 is positive.

Refer to Figure 21.4, where we have drawn some examples of what can happen. Note that (1) the equilibria are usually isolated; (2) and (3) the cases where they are not isolated are not "stable" with respect to minor perturbations; (4) there is usually an odd number of equilibria; (5) if the excess demand curve is downward sloping at all equilibria, there can be only one equilibrium, and if there is only one equilibrium, the excess demand curve must be downward sloping at the equilibrium.

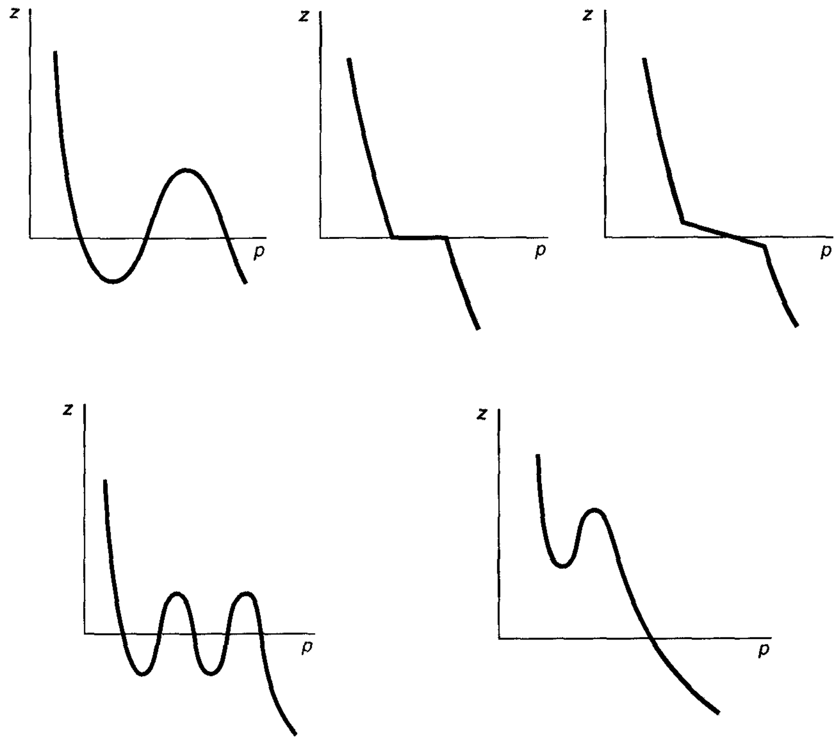


Figure 21.4 **Uniqueness and local uniqueness of equilibrium.** These panels depict some examples used in the discussion of uniqueness of equilibrium.

In the above one-dimensional case note that if $dz(p)/dp < 0$ at all equilibria, then there can be only one equilibrium. Index analysis is a way of generalizing this result to k dimensions so as to give us a simple necessary and sufficient condition for uniqueness.

Given an equilibrium $\mathbf{p}^*$, define the **index** of $\mathbf{p}^*$ in the following way: write down the negative of the Jacobian matrix of the excess supply function $-\mathbf{Dz}(\mathbf{p}^*)$, drop the last row and column, and take the determinant of the resulting matrix. Assign the point $\mathbf{p}^*$ an index $+1$, if the determinant is positive, and assign $\mathbf{p}^*$ an index -1 if the determinant is negative. (Removing the last row and column is equivalent to choosing the last good to be numeraire just as in our simple one-dimensional example.)

We also need a boundary condition; there are several general possibilities, but the simplest is to assume $z_i(\mathbf{p}) > 0$ when $p_i = 0$. In this case, a fundamental theorem of differential topology states that, if all equilibria have positive index, there can be only one of them. This immediately gives us a uniqueness theorem.

Uniqueness of equilibrium. *Suppose* $\mathbf{z}$ *is a continuously differentiable aggregate excess demand function on the price simplex with* $z_i(\mathbf{p}) > 0$ *when* p_i *equals zero. If the* $(k-1)$ *by* $(k-1)$ *matrix* $(-\mathbf{Dz}(\mathbf{p}^*))$ *has positive determinant at all equilibria, then there is only one equilibrium.*

This uniqueness theorem is a purely mathematical result. It has the advantage that the theorem can be applied to a number of different equilibrium problems. If an equilibrium existence theorem can be formulated as a fixed point problem, then we can generally use an index theorem to find conditions under which that equilibrium is unique. However, the theorem has the disadvantage that it is hard to interpret what it means in economic terms.

In the case we are examining here, we are interested in the determinant of the aggregate excess supply function. We can use Slutsky's equation to write the derivative of the aggregate excess supply function as

$$-\mathbf{Dz}(\mathbf{p}) = -\sum_{i=1}^{n} \mathbf{D_p h}_i(\mathbf{p}, u_i) - \sum_{i=1}^{n} \mathbf{D}_m \mathbf{x}_i(\mathbf{p}, \mathbf{p}\boldsymbol{\omega}_i)[\boldsymbol{\omega}_i - \mathbf{x}_i].$$

When will the matrix on the left-hand side have a positive determinant? Let's look at the right-hand side of the expression. The first term on the right-hand side works out nicely; the substitution matrix is a negative semidefinite matrix, so the (negative) of the $(k-1) \times (k-1)$ principal minor of that matrix will typically be a positive definite matrix. The sum of positive definite matrices is positive definite, and will therefore have a positive determinant.

The second term is more problematic. This term is essentially the covariance of the excess supplies of the goods with the marginal propensity to consume the goods. There is no reason to think that it would have any particular structure in general. All we can say is that if these income effects are small relative to the substitution effects, so that the first term dominates, it is reasonable to expect that equilibrium will be unique.

21.4 General equilibrium dynamics

We have shown that under plausible assumptions on the behavior of economic agents there will always exist a price vector that equates demand and supply. But we have given no guarantee that the economy will actually operate at this "equilibrium" point. What forces exist that might tend to move prices to a market-clearing price vector? In this section we will examine some of the problems encountered in trying to model the price adjustment mechanism in a competitive economy.

The biggest problem is one that is the most fundamental, namely the paradoxical relationship between the idea of competition and price adjustment: if all economic agents take market prices as given and outside their control, how can prices move? Who is left to adjust prices?

This puzzle has led to the erection of an elaborate mythology which postulates the existence of a "Walrasian auctioneer" whose sole function is to search for the market clearing prices. According to this construction, a competitive market functions as follows:

> At time zero the Walrasian auctioneer calls out some vector of prices. All agents determine their demands and supplies of current and futures goods at those prices. The auctioneer examines the vector of aggregate excess demands and adjusts prices according to some rule, presumably raising the price of goods for which there is excess demand and lowering the price of goods for which there is excess supply. The process continues until an equilibrium price vector is found. At this point, all trades are made including the exchanges of contracts for future trades. The economy then proceeds through time, each agent carrying out the agreed upon contracts.

This is, of course, a very unrealistic model. However, the basic idea that prices move in the direction of excess demand seems plausible. Under what conditions will this sort of adjustment process lead one to an equilibrium?

21.5 Tatonnement processes

Let's consider an economy that takes place over time. Each day the market opens and people present their demands and supplies to the market. At an arbitrary price vector $\mathbf{p}$, there will in general be excess demands and supplies in some markets. We will assume that prices adjust according to the following rule, the so-called law of supply and demand.

Price adjustment rule. $\dot{p}_i = G_i(z_i(\mathbf{p}))$ *for* $i = 1, \ldots, k$ *where* G_i *is some smooth sign-preserving function of excess demand.*

It is convenient to make some sort of desirability assumption to rule out the possibility of equilibria at a zero price, so we will generally assume that $z_i(\mathbf{p}) > 0$ when $p_i = 0$.

It is useful to draw some pictures of the dynamical system defined by this price adjustment rule. Let's consider a special case where $G_i(z_i)$ equals the identity function for each $i = 1, \ldots, k$. Then, along with the boundary assumption, we have a system in R^k defined by:

$$\dot{\mathbf{p}} = \mathbf{z}(\mathbf{p})$$

From the usual considerations we know that this system obeys Walras' law, $\mathbf{pz}(\mathbf{p}) \equiv 0$. Geometrically, this means that $\mathbf{z}(\mathbf{p})$ will be orthogonal to the price vector $\mathbf{p}$.

Walras' law implies a very convenient property. Let's look at how the the Euclidean norm of the price vector changes over time:

$$\frac{d}{dt}\left(\sum_{i=1}^{k} p_i^2(t)\right) = \sum_{i=1}^{k} 2p_i(t)\dot{p}_i(t) = 2\sum_{i=1}^{k} p_i(t)z_i(\mathbf{p}(t)) = 0$$

by Walras' law. Hence, Walras' law requires that the sum-of-squares of the prices remains constant as the prices adjust. This means that the paths of prices are restricted on the surface of a k-dimensional sphere. Furthermore, since $z_i(\mathbf{p}) > 0$ where $p_i = 0$, we know that the paths of price movements always point inwards near the points where $p_i = 0$. In Figure 21.5 we have some pictures for $k = 2$ and $k = 3$.

The third picture is especially unpleasant. It depicts a situation where we have a unique equilibrium, but it is completely unstable. The adjustment process we have described will almost never converge to an equilibrium. This seems like a perverse case, but it can easily happen.

Debreu (1974) has shown essentially that *any* continuous function that satisfies Walras' law is an excess demand function for some economy; thus the utility maximization hypothesis places no restrictions on *aggregate* demand behavior, and any dynamical system on the price sphere can arise from our model of economic behavior. Clearly, to get global stability results one has to assume special conditions on demand functions. The value of the results will then depend on the economic naturalness of the conditions assumed.

We will sketch an argument of global stability for one such special assumption under a special adjustment process, namely the assumption that aggregate demand behavior satisfies the Weak Axiom of Revealed Preference described in Chapter 8, page 133. This says that if $\mathbf{px}(\mathbf{p}) \geq \mathbf{px}(\mathbf{p}^*)$ we must have $\mathbf{p}^*\mathbf{x}(\mathbf{p}) > \mathbf{p}^*\mathbf{x}(\mathbf{p}^*)$ for all $\mathbf{p}$ and $\mathbf{p}^*$. Since this condition holds for all $\mathbf{p}$ and $\mathbf{p}^*$, it certainly must hold for equilibrium values of $\mathbf{p}^*$. Let us derive the implications of this condition for the *excess* demand function.

Subtracting $\mathbf{p}\boldsymbol{\omega}$ and $\mathbf{p}^*\boldsymbol{\omega}$ from each of these inequalities yields the following implication:

$$\mathbf{px}(\mathbf{p}) - \mathbf{p}\boldsymbol{\omega} \geq \mathbf{px}(\mathbf{p}^*) - \mathbf{p}\boldsymbol{\omega} \text{ implies } \mathbf{p}^*\mathbf{x}(\mathbf{p}) - \mathbf{p}^*\boldsymbol{\omega} > \mathbf{p}^*\mathbf{x}(\mathbf{p}^*) - \mathbf{p}^*\boldsymbol{\omega}.$$

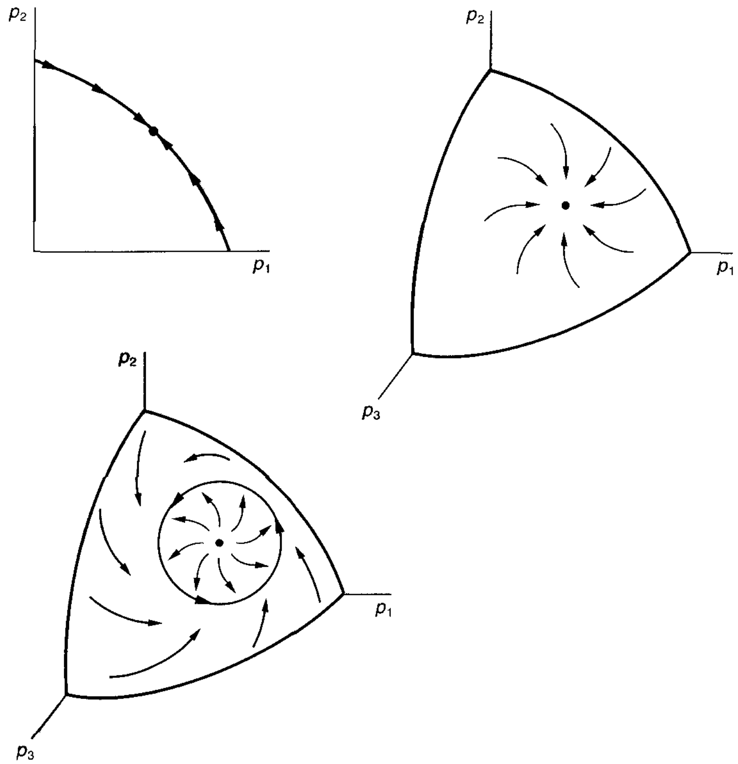


Figure 21.5 **Examples of price dynamics.** The first two examples show a stable equilibrium; the third example has a unique unstable equilibrium.

Using the definition of excess demand, we can write this expression as

$$\mathbf{pz(p)} \geq \mathbf{pz(p^*)} \text{ implies } \mathbf{p^*z(p)} > \mathbf{p^*z(p^*)}. \tag{21.1}$$

Now observe that the condition on the left side of (21.1) must be satisfied by any *equilibrium* price vector $\mathbf{p}^*$. To see this simply observe that Walras' law implies that $\mathbf{pz(p)} \equiv 0$, and the definition of equilibrium implies $\mathbf{pz(p^*)} = 0$. It follows that the right-hand side must hold for any equilibrium $\mathbf{p}^*$. Hence, we must have $\mathbf{p^*z(p)} > 0$ for all $\mathbf{p} \neq \mathbf{p}^*$.

WARP implies stability. *Suppose the adjustment rule is given by $\dot{p}_i = z_i(\mathbf{p})$ for $i = 1, \ldots, k$ and the excess demand function obeys the Weak Axiom of Revealed Preference; i.e., if $\mathbf{p}^*$ is an equilibrium of the economy, then $\mathbf{p^*z(p)} > 0$ for all $\mathbf{p} \neq \mathbf{p}^*$. Then all paths of prices following the above rule converge to $\mathbf{p}^*$.*

Proof. (Sketch) We will construct a Liaponov function for the economy. (See Chapter 26, page 485.) Let $V(\mathbf{p}) = \sum_{i=1}^{k}[(p_i - p_i^*)^2]$. Then

$$\frac{dV(\mathbf{p})}{dt} = \sum_{i=1}^{k} 2(p_i - p_i^*)\dot{p}_i(t) = 2\sum_{i=1}^{k}(p_i - p_i^*)z_i(\mathbf{p})$$
$$= 2\sum_{i=1}^{k}[p_i z_i(\mathbf{p}) - p_i^* z_i(\mathbf{p})] = 0 - 2\mathbf{p}^*\mathbf{z}(\mathbf{p}) < 0.$$

This implies that $V(\mathbf{p})$ is monotonically declining along solution paths for $\mathbf{p} \neq \mathbf{p}^*$. According to Liaponov's theorem we need only to show boundedness of $\mathbf{p}$ to conclude that $V(\mathbf{p})$ is a Liaponov function and that the economy is globally stable. We omit this part of the proof. ∎

21.6 Nontatonnement processes

The tatonnement story makes sense in two sorts of situations: either no trade occurs until equilibrium is reached, or no goods are storable so that each period the consumers have the same endowments. If goods can be accumulated, the endowments of consumers will change over time and this in turn will affect demand behavior. Models that take account of this change in endowments are known as **nontatonnement models.**

In such models, we must characterize the state of the economy at time t by the current vector of prices $\mathbf{p}(t)$ and the current endowments $(\boldsymbol{\omega}_i(t))$. We normally assume that the prices adjust according to the sign of excess demand, as before. But how should the endowments evolve?

We consider two specifications. The first specification, the **Edgeworth process**, says that the technology for trading among agents has the property that the utility of each agent must continually increase. This is based on the view that agents will not voluntarily trade unless they are made better off by doing so. This specification has the convenient property that it quickly leads to a stability theorem; we simply define the Liaponov function to be $\sum_{i=1}^{n} u_i(\boldsymbol{\omega}_i(t))$. By assumption, the sum of the utilities must increase over time, so a simple boundedness argument will give us a convergence proof.

The second specification is known as the **Hahn process.** For this process we assume that the trading rule has the property that there is no good in excess demand by some agent that is in excess supply by some other agent. That is, at any point in time, if a good is in excess demand by a particular agent, it is also in *aggregate* excess demand.

This assumption has an important implication. We have assumed that when a good is in excess demand its price will increase. This will make the *indirect* utility of agents who demand that good lower. Agents who have already committed themselves to supply the good at current prices are not affected by this price change. Hence, aggregate *indirect* utility should decline over time.

To make this argument rigorous, we need to make one further assumption about the change in endowments. The value of consumer i's endowment at time t is $m_i(t) = \sum_{j=1}^{k} p_j(t)\omega_i^j(t)$. Differentiating this with respect to t gives

$$\frac{dm_i(t)}{dt} = \sum_{j=1}^{k} p_j(t)\frac{d\omega_i^j(t)}{dt} + \sum_{j=1}^{k} \frac{dp_j(t)}{dt}\omega_i^j(t).$$

It is reasonable to suppose that the first term in this expression is zero. This means that the change in the endowment at any instant, valued at current prices, is zero. This is just saying that each agent will trade a dollar's worth of goods for a dollar's worth of goods. The value of the endowment will change over time due to changes in price, but not because agents managed to make profitable trades at constant prices.

Given this observation, it is easy to show that the sum of indirect utilities decreases with time. The derivative of agent i's indirect utility function is

$$\frac{dv_i(\mathbf{p}(t), \mathbf{p}(t)\boldsymbol{\omega}_i(t))}{dt} = \sum_{j=1}^{k} \frac{\partial v_i}{\partial p_j}\frac{dp_j}{dt} + \frac{\partial v_i}{\partial m_i}\left[\sum_{j=1}^{k} p_j \frac{d\omega_i^j}{dt} + \frac{dp_j}{dt}\omega_i^j\right].$$

Using Roy's law and the fact that the value of the change in the endowment at current prices must be zero, we have

$$\frac{dv_i(\mathbf{p}(t), \mathbf{p}(t)\boldsymbol{\omega}_i(t))}{dt} = -\frac{\partial v_i}{\partial m_i}\sum_{j=1}^{k}\left[x_i^j(\mathbf{p}, \mathbf{p}\boldsymbol{\omega}_i) - \omega_i^j\right]\frac{dp_j(t)}{dt}.$$

By assumption if good j is in excess demand by agent i, $dp_j/dt > 0$ and vice versa. Since the marginal utility of income is positive, the sign of the whole expression will be negative as long as aggregate demand is not equal to aggregate supply. Hence the indirect utility of each agent i must decrease when the the economy is not in equilibrium.

Notes

See Arrow & Hahn (1971) for a more elaborate discussion of these topics. The importance of the topological index to uniqueness was first recognized by Dierker (1972). The core convergence result was rigorously established by Debreu & Scarf (1963).

Exercises

21.1. There are two agents with identical, strictly convex preferences and equal endowments. Describe the core of this economy and illustrate it in an Edgeworth box.

21.2. Consider a pure exchange economy in which all consumers have differentiable quasilinear utility functions of the form $u(x_1, \ldots, x_n) + x_0$. Assume that $u(x_1, \ldots, x_n)$ is strictly concave. Show that equilibrium must be unique.

21.3. Suppose that the Walrasian auctioneer follows the price adjustment rule $\dot{p} = [\mathbf{Dz(p)}]^{-1}\mathbf{z(p)}$. Show that $V(\mathbf{p}) = -\mathbf{z(p)z(p)}$ is a Liaponov function for the dynamical system.