Deriving Kepler's Laws of Planetary Motion

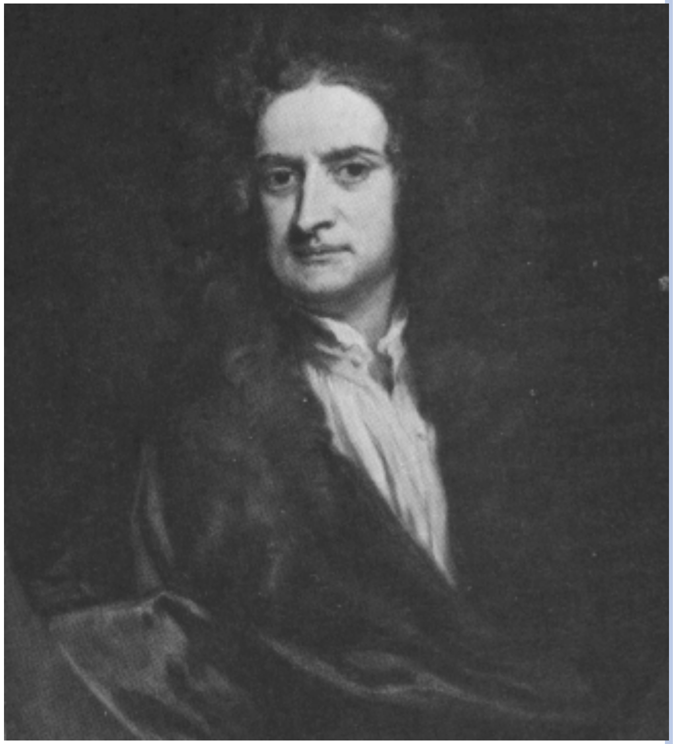
By: Emily Davis

Who is Johannes Kepler?

- German mathematician, physicist, and astronomer
- Worked under Tycho Brahe
- Observation alone
- Founder of celestial mechanics

What about Isaac Newton?

- "If I have seen further it is by standing on the shoulders of Giants."



- Laws of Motion
- Universal Gravitation
- Explained Kepler's laws
  - The laws could be explained mathematically if his laws of motion and universal gravitation were true.
- Developed calculus

Kepler's Laws of Planetary Motion

1. Planets move around the Sun in ellipses, with the Sun at one focus.

2. The line connecting the Sun to a planet sweeps equal areas in equal times.

3. The square of the orbital period of a planet is proportional to the cube of the semimajor axis of the ellipse.

INITIAL VALUES AND EQUATIONS

Unit vectors of polar coordinates

(1)
$$u_r = (\cos\theta)i + (\sin\theta)j$$
$$u_\theta = -(\sin\theta)i + (\cos\theta)j$$

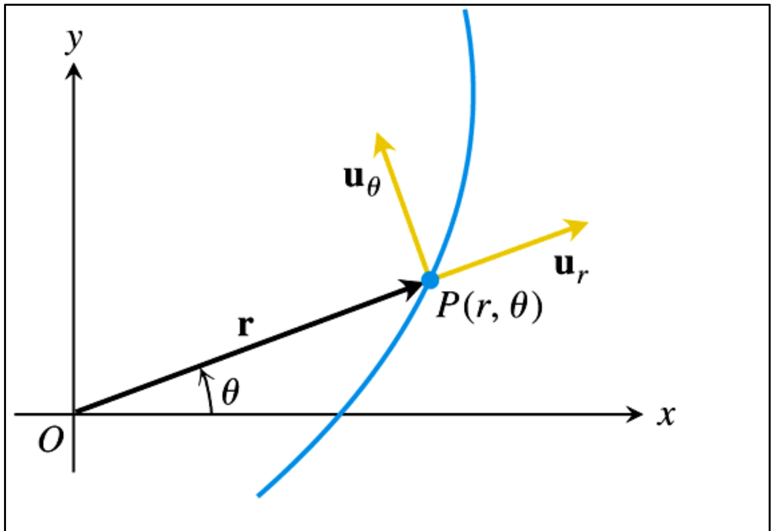


FIGURE 13.32 The length of $\mathbf{r}$ is the positive polar coordinate r of the point P. Thus, $\mathbf{u}_r$, which is $\mathbf{r}/|\mathbf{r}|$, is also $\mathbf{r}/r$. Equations (1) express $\mathbf{u}_r$ and $\mathbf{u}_\theta$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

Initial Values and Equations

From (1),

(2)

$$\frac{du_r}{d\theta} = -(\sin\theta)i + (\cos\theta)j = u_\theta$$

$$\frac{du_\theta}{d\theta} = -(\cos\theta)i - (\sin\theta)j = -u_r$$

Differentiate with respect to time ***t***

(3)

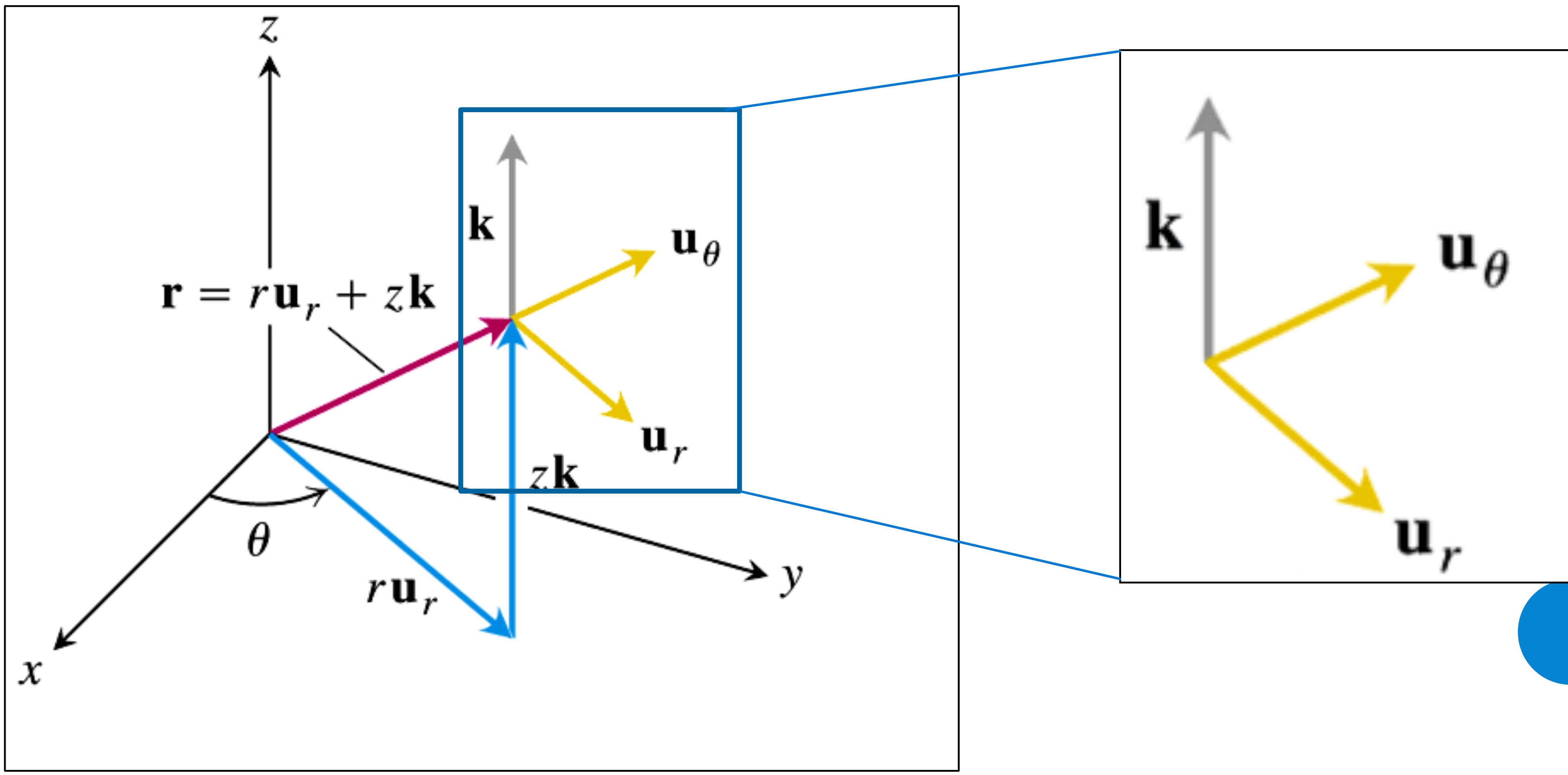
$$u_r{}' = \frac{du_r}{d\theta}\theta' = \theta' u_\theta$$

$$u_\theta{}' = \frac{du_\theta}{d\theta}\theta' = -\theta' u_r$$

Initial Values and Equations Continued…

Vectors follow the right-hand rule

$$(8) \qquad u_r \times u_\theta = k \qquad u_\theta \times k = u_r \qquad k \times u_r = u_\theta$$



Initial Values and Equations Continued…

Force between the sun and a planet

(9)
$$F = -\frac{GmM}{|r|^2}\frac{r}{|r|}$$

F-force
G-universal gravitational constant
M-mass of sun
m-mass of planet
r-radius from sun to planet

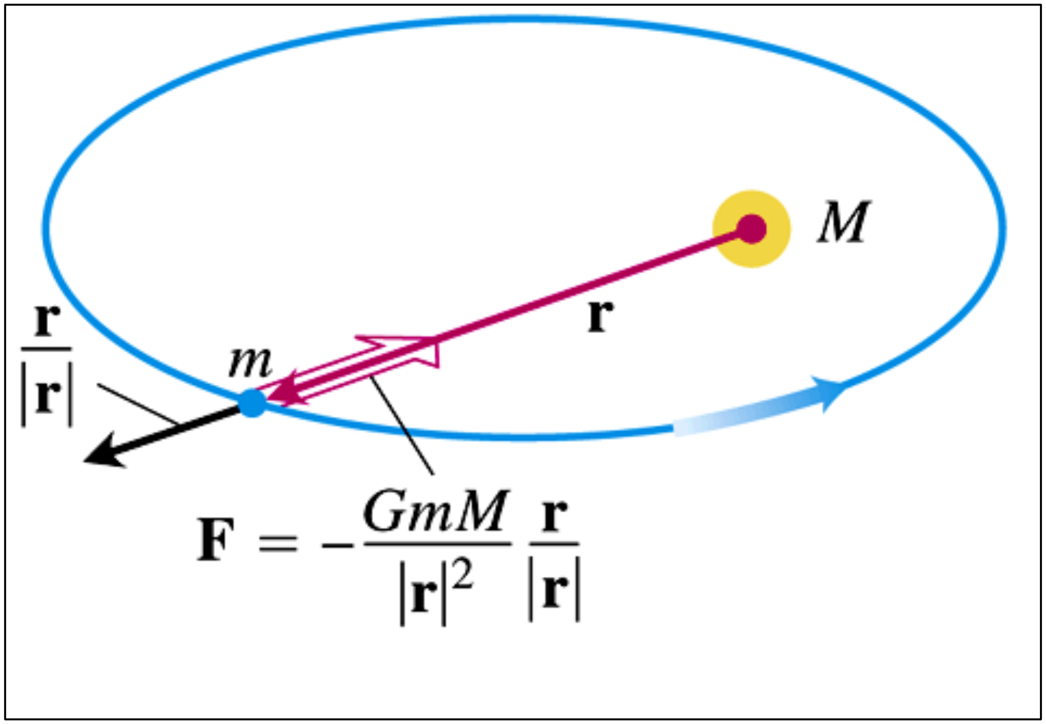
Newton's 2nd law of motion: $F=ma$

(10)
$$F = mr''$$
$$r'' = -\frac{GM}{|r|^2}\frac{r}{|r|}$$

Initial Values and Equations Continued…

Planets accelerate toward the sun, and a is a scalar multiple of r.

(11) $$r \times r'' = 0$$



Initial Values and Equations Continued…

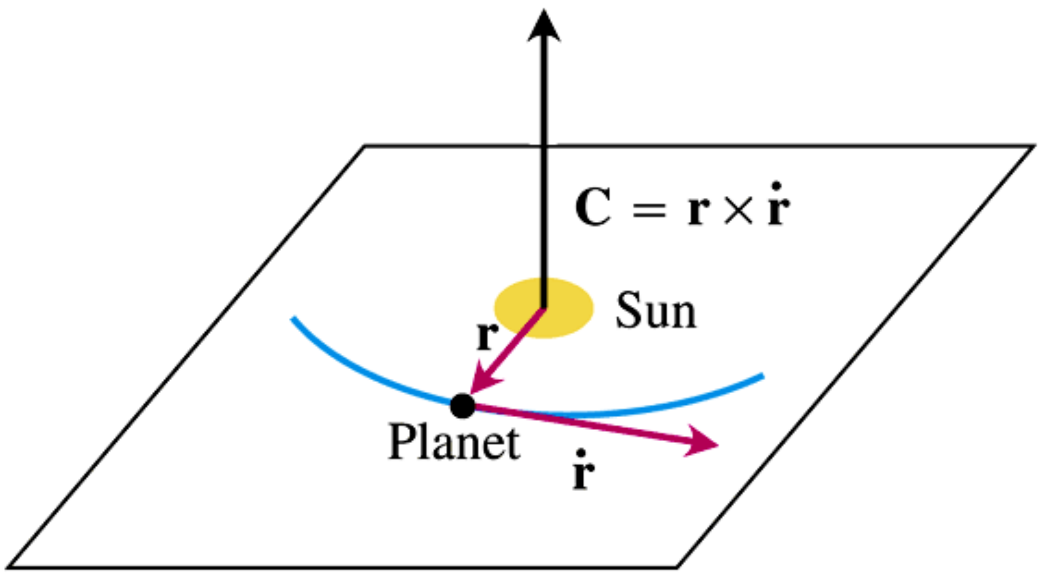
Derivative of $r \times r'$

(12) $$\frac{d}{dt}(r \times r') = r' \times r' + r \times r'' = r \times r''$$

(11) and (12) together

(13) $$\frac{d}{dt}(r \times r') = 0$$

INITIAL VALUES AND EQUATIONS CONTINUED…



Integrates to a constant

(14) $$r \times r' = C$$

Initial Values and Equations Continued...

When t=0,

1. $r = r_0$
2. $r' = 0$
3. $\theta = 0$
4. $|v| = v_0$
5. $r\theta' = v_0$

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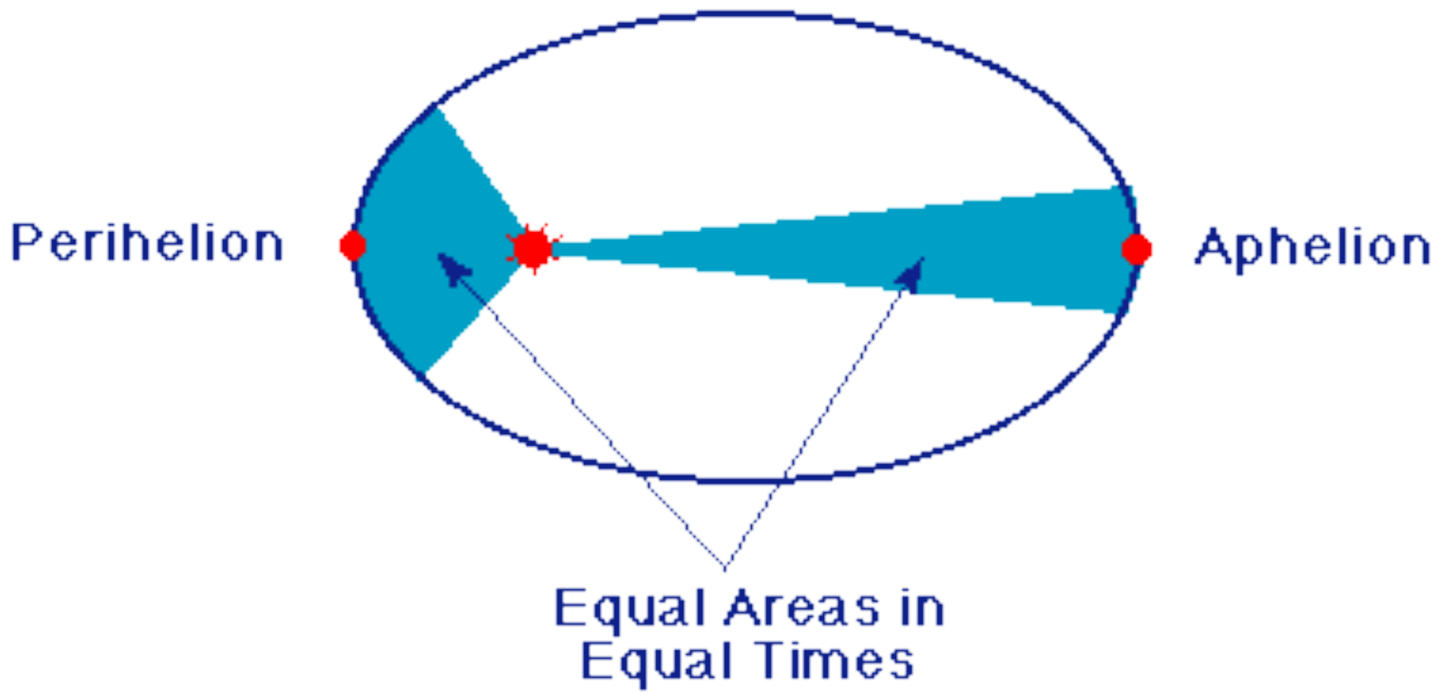
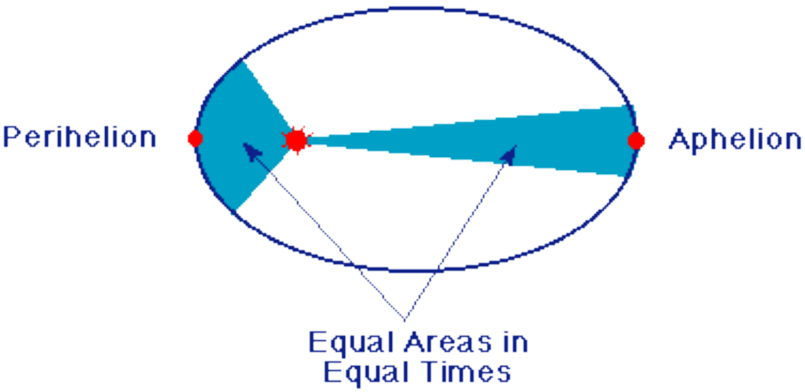
What does Law 2 mean?



The line connecting the Sun to a planet sweeps equal areas in equal times.

PROOF OF LAW 2 CONTINUED…



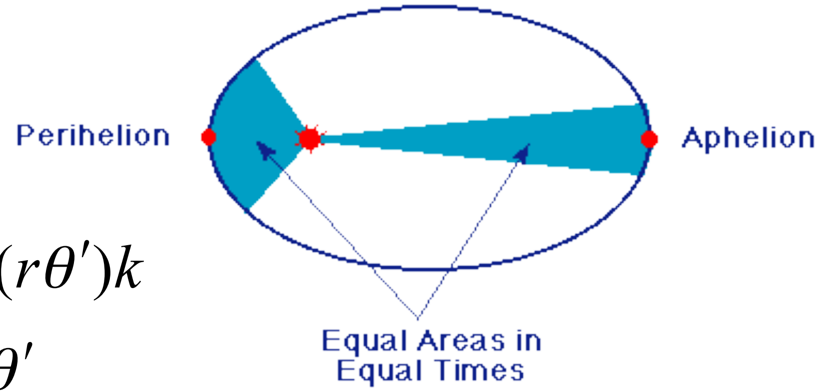
Insert (4) into (14)

(17)
$$C = r \times r' = r \times v$$
$$C = rr'(u_r \times u_r) + r(r\theta')(u_r \times u_\theta)$$
$$C = rr'(0) + r(r\theta')(k)$$
$$C = r(r\theta')k$$

(17) is simplified when t=0

(18)
$$C = r(r\theta')k = r_0 v_0 k$$

Proof of Law 2 Continued…



C is always constant so (17)=(18)

(19)
$$r_0 v_0 k = r(r\theta')k$$
$$r_0 v_0 = r^2 \theta'$$

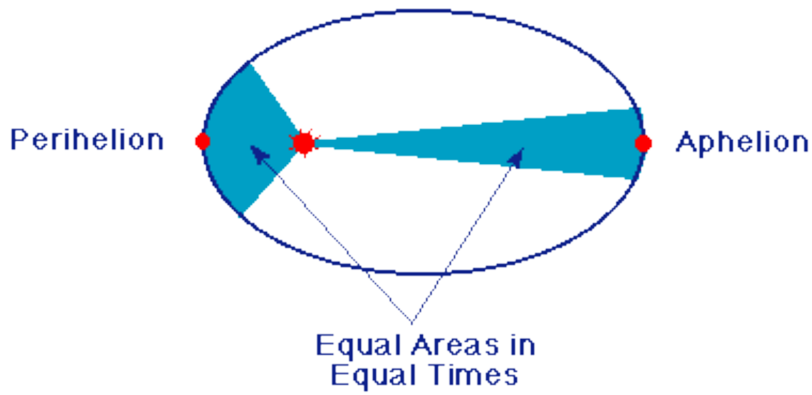
Differential area equation for polar coordinates

$$dA = \frac{1}{2} r^2 d\theta$$

Differential area equation is half of the constant function.

(20)
$$\frac{dA}{dt} = \frac{1}{2} r^2 \theta' = \frac{1}{2} r_0 v_0$$

Proof of Law 2 Continued…



LAW 2 is PROVED!!!!

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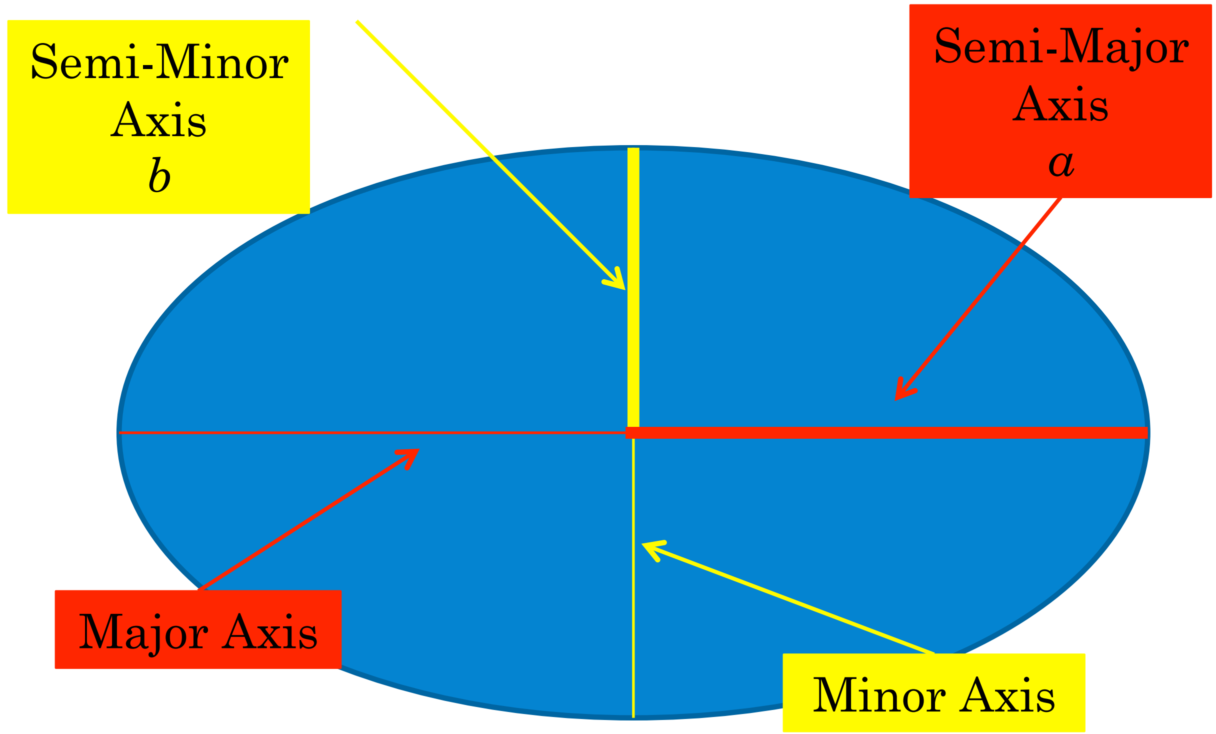
TERMINOLOGY OF LAW 3



a=semi-major axis
b=semi-minor axis
T=period

What does Law 3 mean?

The square of the orbital period of a planet is proportional to the cube of the semimajor axis of the ellipse.

$$T^2 = ka^3$$

Proof of Law 3

There is a concept from the proof of Law 1 that is needed…Eccentricity

Two formulas (31) and (32)

(32) $$e = \frac{1}{r_0 h} - 1 = \frac{r_0 v_0^2}{GM} - 1$$

(31) $$r = \frac{(1+e)r_0}{1+e\cos\theta}$$

Proof of Law 3 Continued…

Kepler's law says

(33) $$\frac{T^2}{a^3} = \frac{4\pi^2}{GM}$$

Formula 1 for area

$$Area = \pi ab$$

Formula 2 uses (20)

$$Area = \int_0^T \frac{1}{2} r_0 v_0 dt$$

$$Area = \frac{1}{2} T r_0 v_0$$

Proof of Law 3 Continued…

Combining Formula 1 and 2 and solving for T

$$\pi ab = \frac{1}{2} T r_0 v_0$$

(34)
$$T = \frac{2\pi a^2}{r_0 v_0}\sqrt{1-e^2}$$

r_{max} is equal to (31) when $\theta=\pi$

$$r_{\max} = r_0 \frac{1+e}{1-e}$$

Proof of Law 3 Continued…

a is found using the maximum position vector

$$2a = r_0 + r_{\max}$$

$$2a = r_0 + r_0 \frac{1+e}{1-e}$$

$$2a = \frac{2r_0}{1-e} \tag{35}$$

$$2a = \frac{2r_0 GM}{2GM - r_0 v_0^{\;2}}$$

Proof of Law 3 Continued…

a can be found from (35)

$$a = \frac{r_0 GM}{2GM - r_0 {v_0}^2}$$

$$\frac{1}{a} = \frac{2GM - r_0 {v_0}^2}{r_0 GM}$$

Proof of Law 3 Continued…

Square both sides of (34) and use (32) and (35)

$$(T)^2 = \left(\frac{2\pi a^2}{r_0 v_0}\sqrt{1-e^2}\right)^2$$

$$T^2 = \frac{4\pi^2 a^4}{r_0^{\,2} v_0^{\,2}}(1-e^2)$$

$$T^2 = \frac{4\pi^2 a^4}{r_0^{\,2} v_0^{\,2}}\left(\frac{2 r_0 v_0^{\,2}}{GM} - \frac{r_0^{\,2} v_0^{\,4}}{G^2 M^2}\right)$$

$$T^2 = \frac{4\pi^2 a^4}{GM}\left(\frac{2GM - r_0 v_0^{\,2}}{r_0 GM}\right)$$

Proof of Law 3 Continued…

Solve T^2

$$T^2 = \frac{4\pi^2 a^4}{GM}\left(\frac{1}{a}\right)$$

$$T^2 = \frac{4\pi^2}{GM}a^3$$

$$\frac{T^2}{a^3} = \frac{4\pi^2}{GM}$$

Proof of Law 3 Continued…

LAW 3 is PROVED!!!!

References

"Johannes Kepler: His Life, His Laws and Times." *Kepler Mission*. 4 Aug. 2009. Web. http://kepler.nasa.gov/johannes/.

"Johannes Kepler: The Laws of Planetary Motion." *Astronomy 161 The Solar System*. Department of Physics & Astronomy University of Tennessee, 10 Aug. 2000. Web. <http://csep10.phys.utk.edu/astr161/lect/history/kepler.htmal>.

"Newton's Three Laws of Motion." *Astronomy 161 The Solar System*. Department of Physics & Astronomy University of Tennessee, 10 Aug. 2000. Web. <http://csep10.phys.utk.edu/astr161/lect/history/newton3laws.html>.

Stern, David P. "Kepler's Three Laws of Planetary Motion." *NASA Polar, Wind, and Geotail Projects*. 7 Feb. 2008. Web. <http://www-istp.gsfc.nasa.gov/stargaze/Kep3laws.htm>.

Weir, Maurice D. "Planetary Motion and Satellites." *Thomas' calculus*. 11th ed. Boston: Pearson Addison Wesley, 2005. 950-56. Print.

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