Angular Momentum

- Vector cross products
- Torque
- Rotation
- Angular Momentum

Vector Cross Products

Recall from our discussion of vectors that
there are two ways to multiply them.

There's the scalar (or dot) product which results in a number

$$W = \vec{F} \cdot \vec{r}$$

work has magnitude, but no direction

There's also the vector (or cross) product which results in another vector

$$\vec{C} = \vec{A} \times \vec{B}$$

The result of **A** x **B** is another vector **C** which has both magnitude and direction

The direction of **C** is simultaneously perpendicular to *both* **A** & **B**

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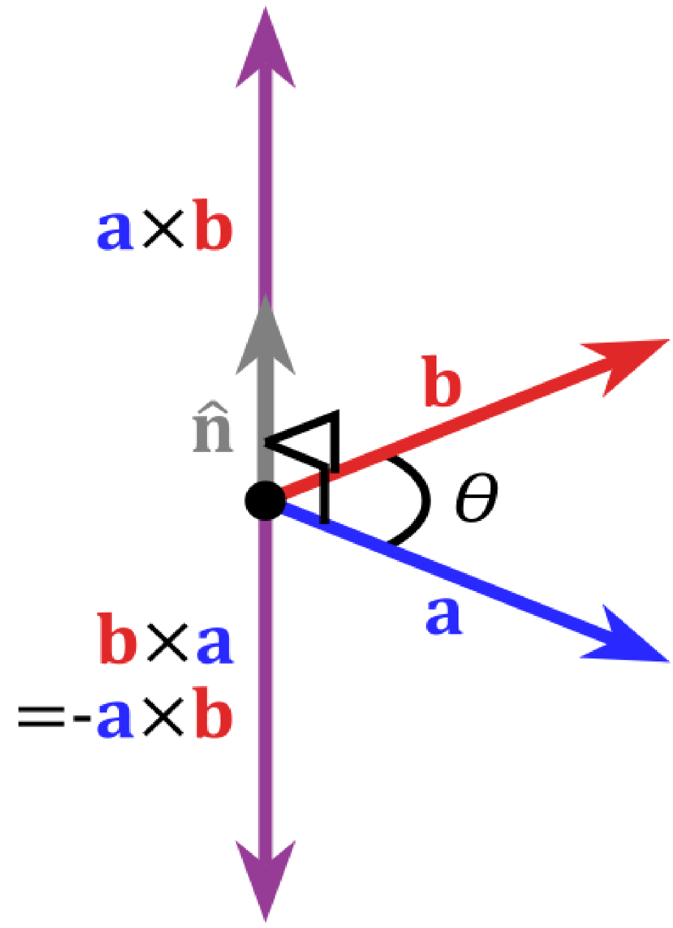
Vector Product

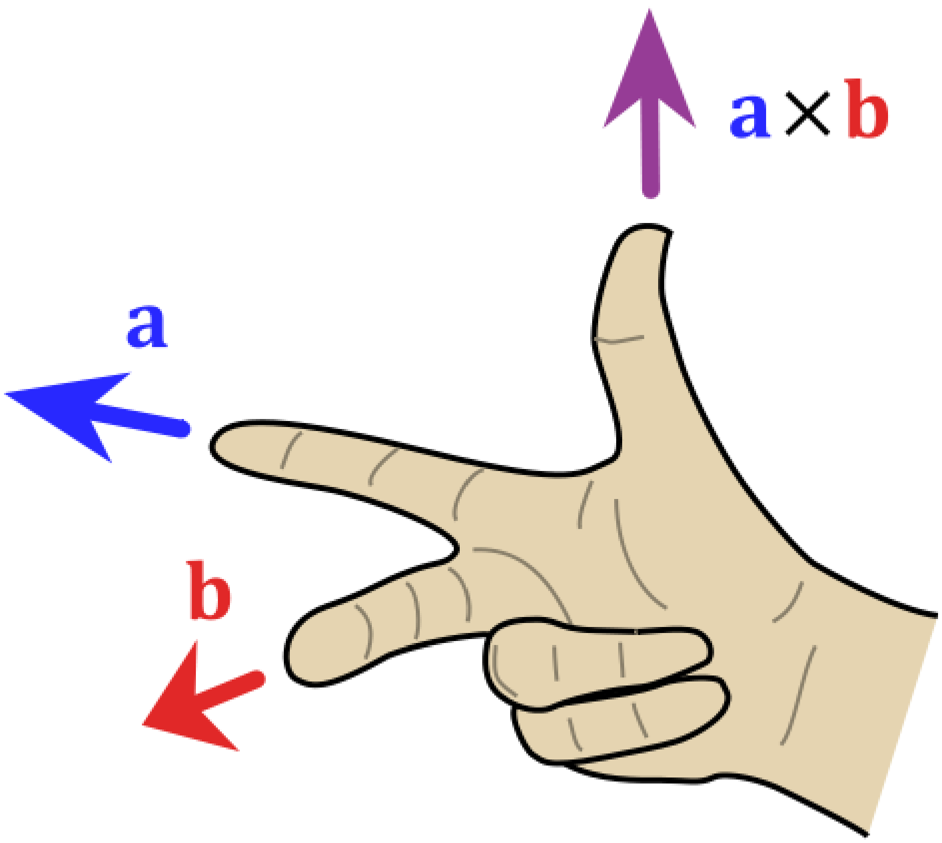
$$\vec{a}\times\vec{b}=ab\sin(\theta)\hat{n}$$



The magnitude of the cross product of **a** and **b** is the magnitudes of **a** & **b** and the sine of the angle between them

The direction you can get from our old friend ...

The Right-Hand Rule



Alternate RHR... lay your fingers along **a** and then curl them in the direction of **b**.

Either way, now do the same thing for $\vec{b} \times \vec{a}$

Notice that when you do that, your thumb points in the opposite direction

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Vector Product Properties

$$\vec{A}\times\vec{B}=-\vec{B}\times\vec{A}$$

the cross-product is non-commutative
Reversing the order yields the same magnitude in the opposite direction.

$$\vec{A}\times(\vec{B}+\vec{C})=\vec{A}\times\vec{B}+\vec{A}\times\vec{C}$$

the cross-product is distributive

$$\text{if } \vec{A}\|\vec{B} \text{ then } \vec{A}\times\vec{B}=\vec{0}\rightarrow\vec{A}\times\vec{A}=\vec{0}$$

$$\frac{d}{dt}(\vec{A}\times\vec{B})=\frac{d\vec{A}}{dt}\times\vec{B}+\vec{A}\times\frac{d\vec{B}}{dt}$$

Product rule for derivatives works on the cross product

Just for the record...

$$\vec{A}\times\vec{B}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}$$

$$\vec{A}=(a_x,a_y,a_z)$$

$$\vec{B}=(b_x,b_y,b_z)$$

$$=(a_y b_z-a_z b_y)\hat{i}-(a_x b_z-a_z b_x)\hat{j}+(a_x b_y-a_y b_x)\hat{z}$$

$$\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=\vec{0}$$

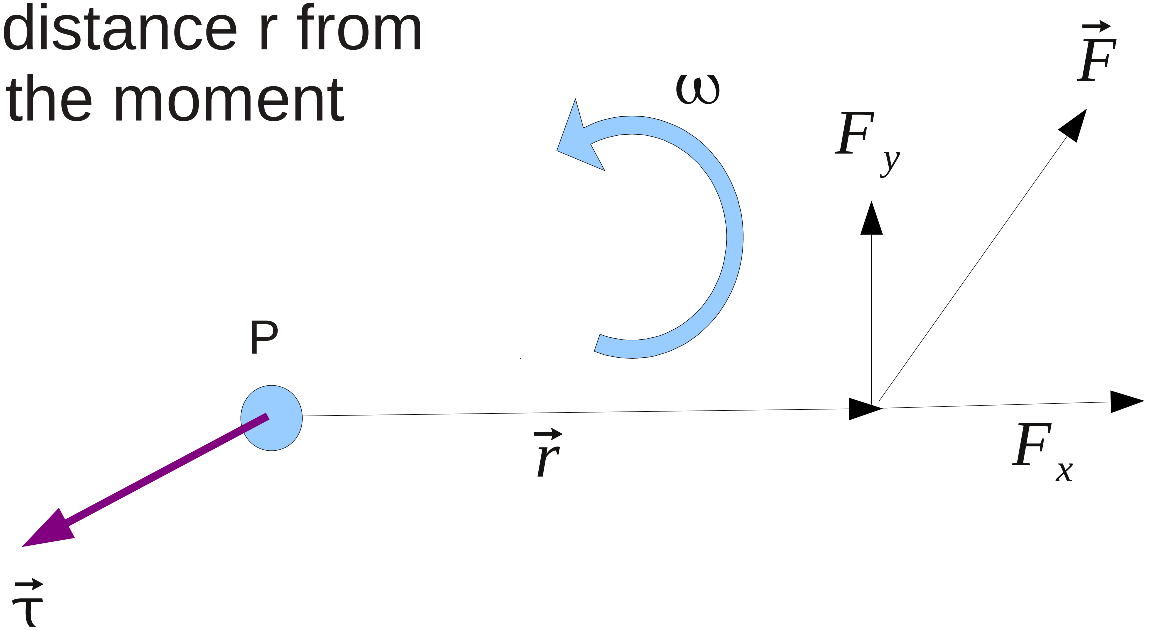
$$\hat{i}\times\hat{j}=-\hat{j}\times\hat{i}=\hat{k} \qquad \hat{j}\times\hat{k}=-\hat{k}\times\hat{j}=\hat{i} \qquad \hat{k}\times\hat{i}=-\hat{i}\times\hat{k}=\hat{j}$$

Torque

A force applied at some distance from a pivot point tends to produce a rotation about the pivot. This is torque. The distance r from the pivot point P is known as the moment arm or lever arm

$$\vec{\tau}=\vec{r}\times\vec{F}$$

$$|\vec{\tau}|=r\,F\sin(\theta)$$



direction of the torque can be obtained from the right-hand rule or computed directly

$$\vec{\tau}=\vec{r}\times\vec{F}=r\,\hat{i}\times(F_x\,\hat{i}+F_y\,\hat{j})=r\,F_y\,\hat{i}\times\hat{j}=r\,F_y\,\hat{k}$$

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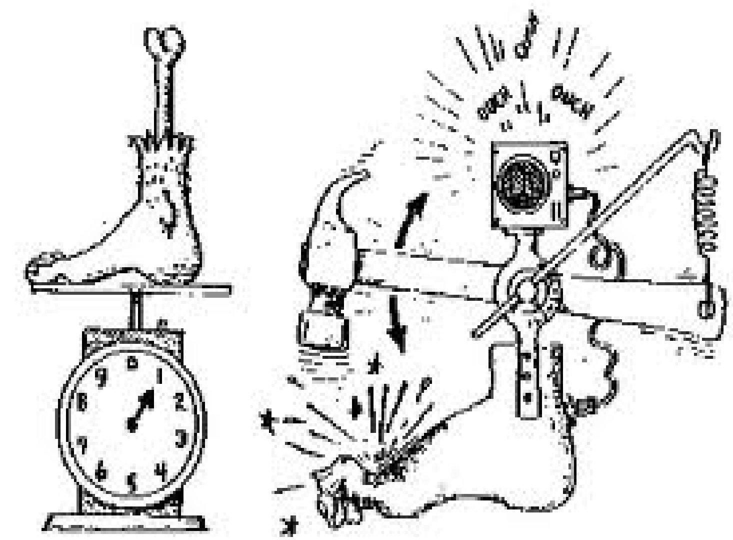
Torque

The units of torque are [length] x [force]

In english units this is a foot – pound

Not to be confused with the units of work which are pound – feet (ouch!)



The SI unit of torque is mN which is ambiguous with milli-Newtons or Nm (Newton – meters = J) which is ambiguous with work and energy. This is not a coincidence since 1 Nm of torque applied for one full rotation requires exactly 2π J of work. $E = \tau \theta$

SI recommendation is to use the Newton-meter for torque and the Joule for work/energy

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Torque

$$\Sigma\vec{\tau}=\vec{r}\times\Sigma\vec{F}$$

Newtons' 2nd Law $\Sigma\vec{F}=\frac{d\vec{p}}{dt}$

$$\frac{d\vec{r}}{dt}=\vec{v}\rightarrow\frac{d\vec{r}}{dt}\times\vec{p}=m(\vec{v}\times\vec{v})=\vec{0}$$

$$\Sigma\vec{\tau}=\vec{r}\times\frac{d\vec{p}}{dt}+\frac{d\vec{r}}{dt}\times\vec{p}=\frac{d}{dt}(\vec{r}\times\vec{p})$$

Angular Momentum

$$\Sigma \vec{\tau} = \frac{d}{dt}(\vec{r} \times \vec{p})$$

This looks very similar to Newtons' 2nd Law for translational motion

$$\Sigma \vec{F} = \frac{d\,\vec{p}}{dt}$$

$$\vec{L} \equiv \vec{r} \times \vec{p}$$

L is the angular momentum of an object moving along a curved path

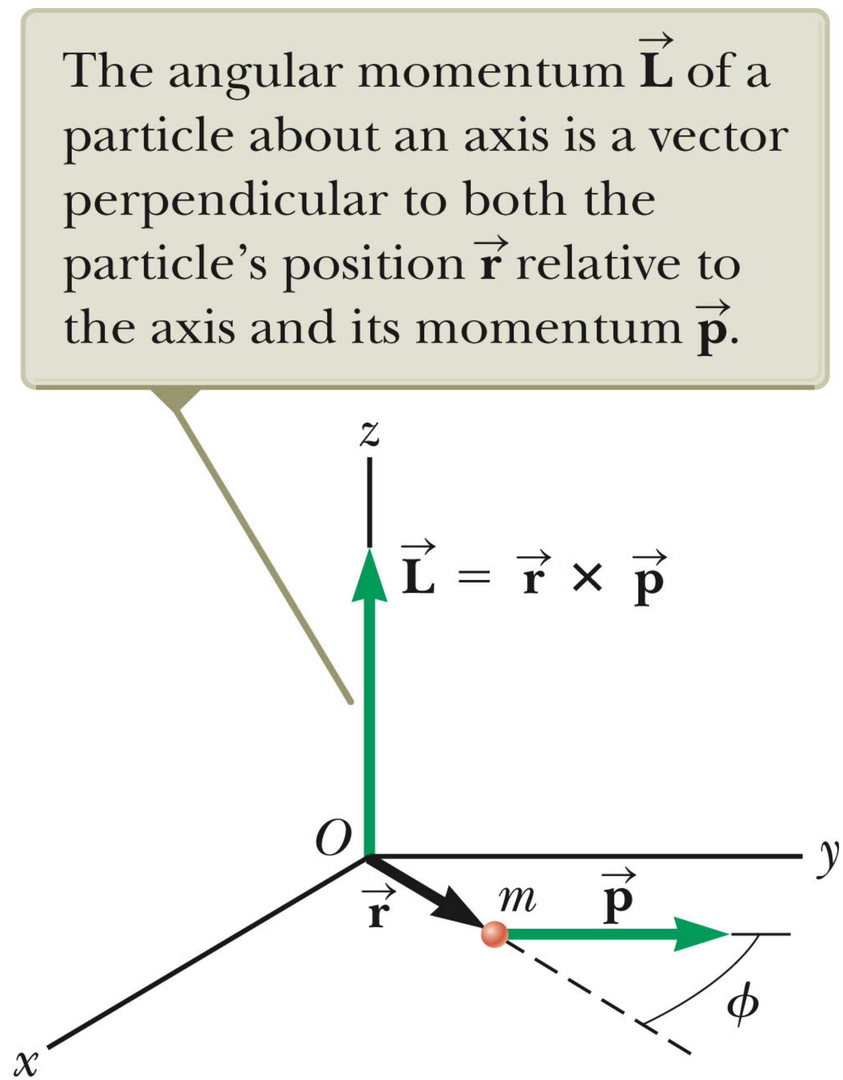
$$\Sigma \vec{\tau} = \frac{d\,\vec{L}}{dt}$$

Rotational form of Newtons' 2nd Law
The torque resultant on an object is equal to the rate of change of *angular* momentum in time.

Angular Momentum

$$\left|\vec{L}\right| = r\,p\sin(\phi) = mvr\sin(\phi)$$

The direction of the angular momentum comes from the right-hand rule, and will be in the same direction as the angular velocity



Angular Momentum

for a system of particles the total angular momentum of the system is the sum of the angular momenta of the particles

$$\vec{L}_T = \Sigma_i \vec{L}_i \qquad \frac{d\vec{L}_T}{dt} = \Sigma_i \frac{d\vec{L}_i}{dt} = \Sigma_i \vec{\tau}_i$$

$$\Sigma \vec{\tau}_{ext} = \frac{d\vec{L}_T}{dt}$$ which is the rotational analog of $$\Sigma \vec{F}_{ext} = \frac{d\vec{p}_T}{dt}$$

If we integrate the external torque over time

$$\int \Sigma \vec{\tau}_{ext} dt = \Delta \vec{L}_T$$

We get the impulse – angular momentum theorem which is analogous to the impulse – momentum theorem for translational motion

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Angular Momentum

$$\Sigma\vec{\tau}_{ext}=\frac{d\vec{L}_T}{dt}$$

is a statement of the conservation of angular momentum. That is, angular momentum does not change unless something (external) is done to make it change.

Conservation of angular momentum is a statement that is just as important to physics as conservation of energy & conservation of (linear) momentum

Notice that this is true regardless of what the center-of-mass of the body is doing.

Rigid Body Rotation

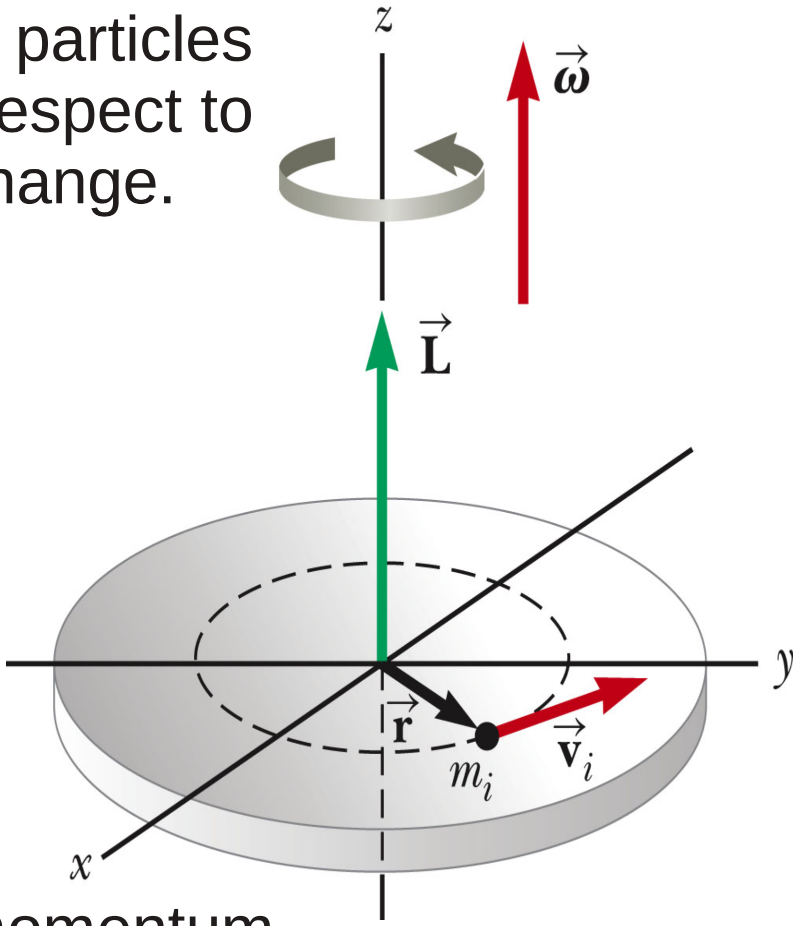
A rigid body can be treated as a system of particles of infinitesimal mass whose position with respect to the other particles in the system doesn't change.

for a particle in circular motion

$$\vec{L} = mvr\sin(90°)\hat{L} = mvr\hat{L} = mr^2\vec{\omega}$$

For a rotating body

$$\vec{L}_T = \Sigma_i \vec{L}_i = \left(\Sigma_i m_i r_i^2\right)\vec{\omega} = I\vec{\omega}$$



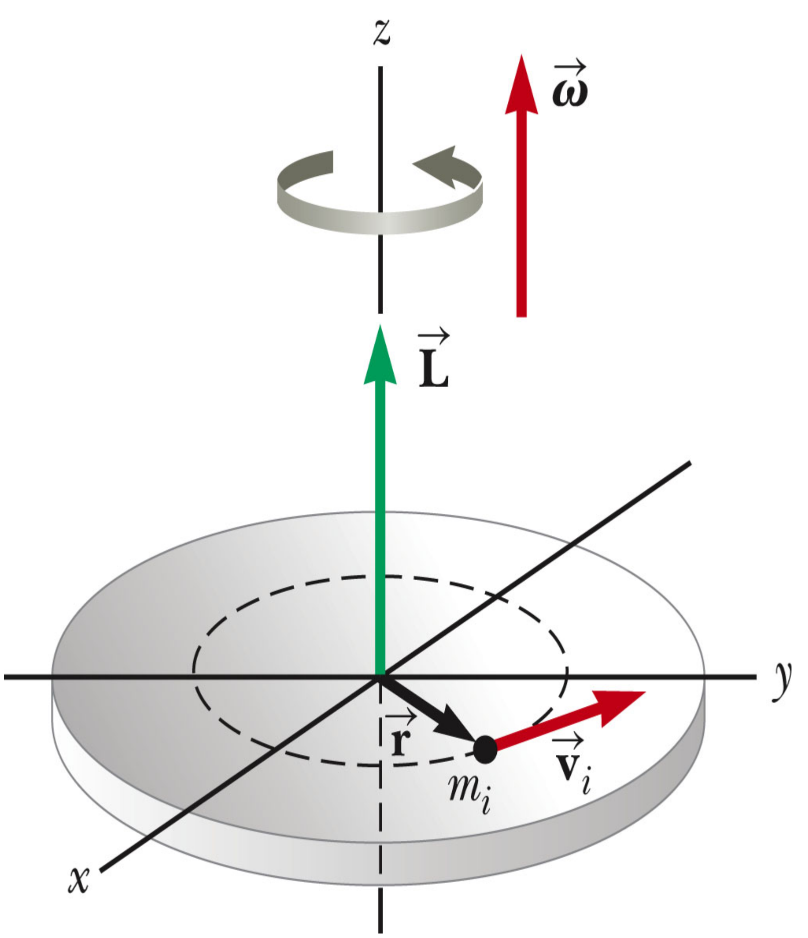
which confirms the assertion that angular momentum is in the same direction as angular velocity

Rigid Body Rotation

$$\vec{L}_T = I\,\vec{\omega}$$

$$\frac{d\,\vec{L}_T}{dt} = I\,\frac{d\,\vec{\omega}}{dt} = I\,\vec{\alpha} = \Sigma\,\vec{\tau}_{ext}$$

The angular acceleration of an object is the result of a net torque applied to it



So if we apply a torque to the disk in the picture in the direction that it's rotating (**F** is parallel to **v**), the angular acceleration tends to increase the angular velocity and the angular momentum. If **F** is anti-parallel to **v** then it will tend to slow down the rotation

Isolated System

If the net external torque acting on a system is zero the angular momentum is constant (in magnitude and direction)

$$\Sigma \vec{\tau}_{ext} = 0 = \frac{d\vec{L}_T}{dt} \rightarrow \vec{L}_f = \vec{L}_i$$

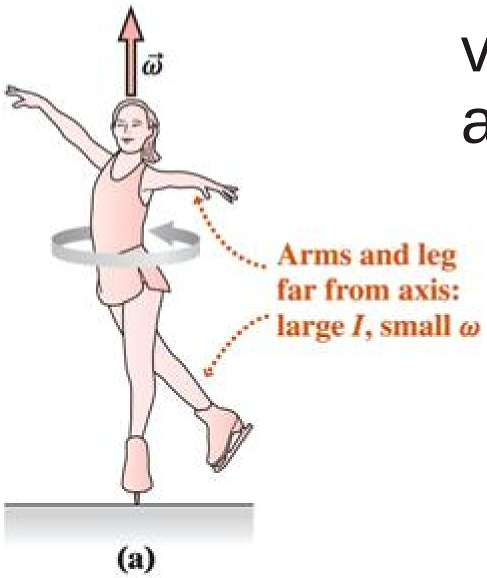
This is an alternate way to state conservation of angular momentum.

Implies that $I_f \omega_f = I_i \omega_i$

(so what! Moments of inertia are constant right?)

Conservation of Angular Momentum

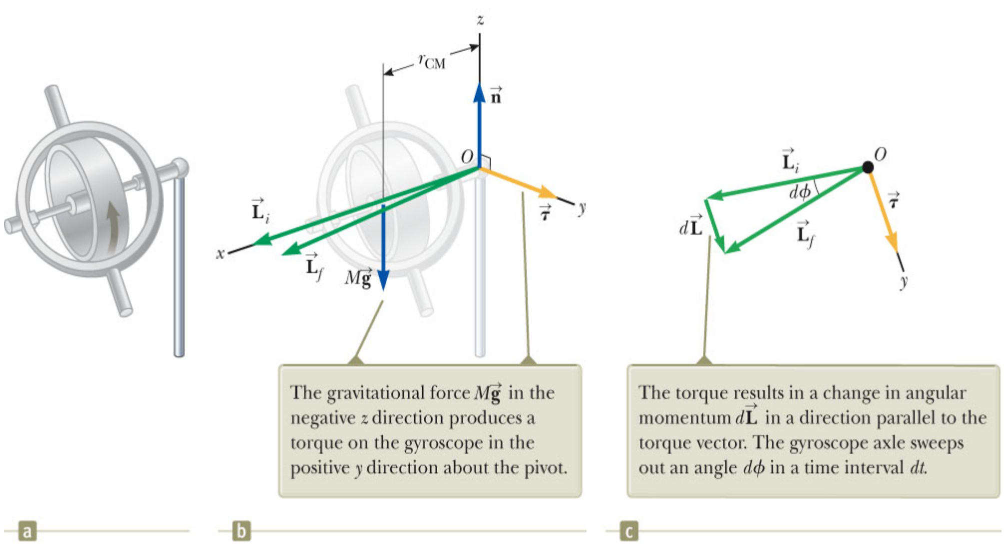
This is why an ice skater who brings in her arms while spinning spins faster. Her moment of inertia is dropping (reducing the moment arm) so her angular velocity increases to keep the angular momentum constant

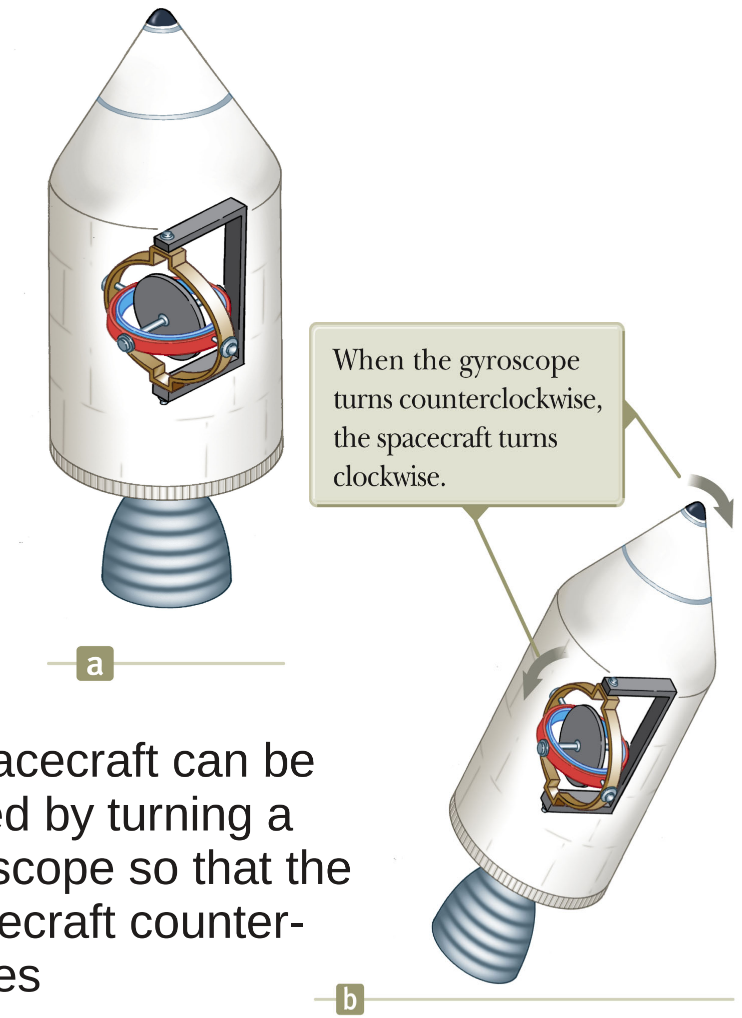


Conservation of Angular Momentum

It's also what makes a gyroscope function

A pilot can tell the attitude (pitch & yaw) of his airplane from a gyroscope even if he can't see outside. As the plane rotates on it's axis the gyroscope counter-rotates to compensate

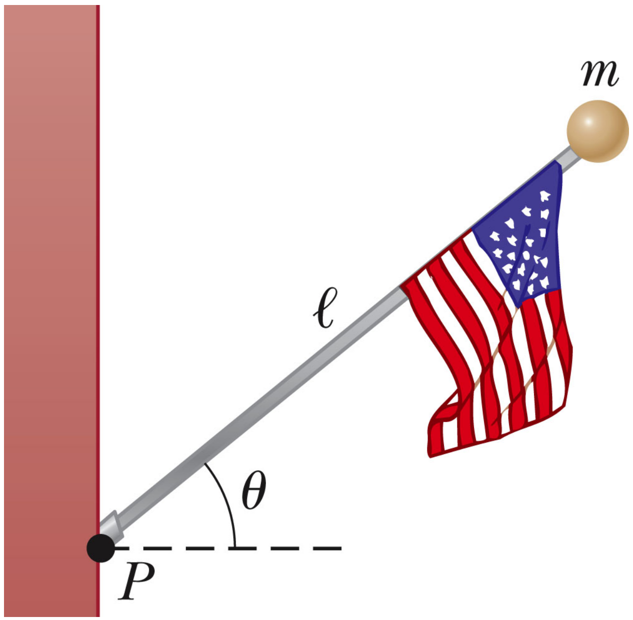




A spacecraft can be turned by turning a gyroscope so that the spacecraft counter-rotates

Example (Prob 11.21)

A ball of mass m is attached to the end of a flagpole which is connected to the side of a building. The length of the pole is l and it makes an angle θ with the horizontal. The ball becomes loose & begins to fall with an acceleration -g**j**.



a) What is the angular momentum of the ball about the point P as a function of time?

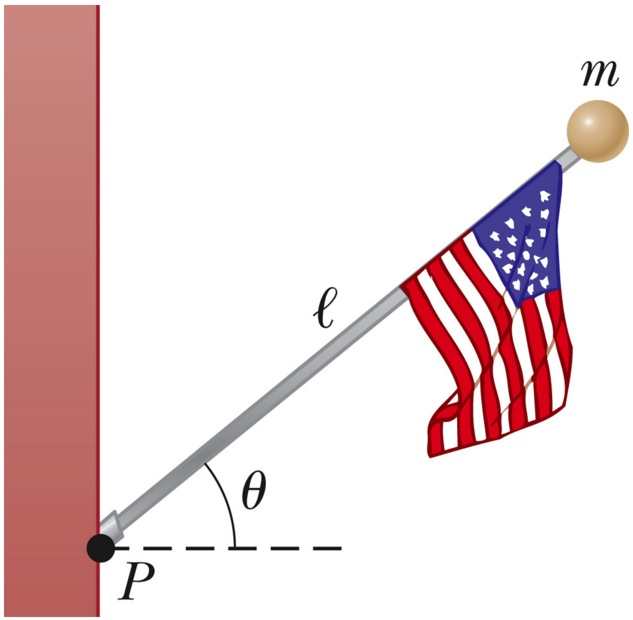
$$\vec{r}=\vec{r}_i+\vec{v}_i+\tfrac{1}{2}\vec{a}t^2=(l\cos(\theta)\hat{i}+l\sin(\theta)\hat{j})-\tfrac{1}{2}gt^2\hat{j}$$

$$\vec{v}=\vec{v}_i+\vec{a}t=-gt\hat{j}$$

$$\vec{L}=\vec{r}\times\vec{p}=m(l\cos(\theta))(-\tfrac{1}{2}gt)(\hat{i}\times\hat{j})=-mglt\cos(\theta)\hat{k}$$

Example (Prob 11.21)

A ball of mass m is attached to the end of a flagpole which is connected to the side of a building. The length of the pole is l and it makes an angle θ with the horizontal. The ball becomes loose & falls with an acceleration -g**j**.

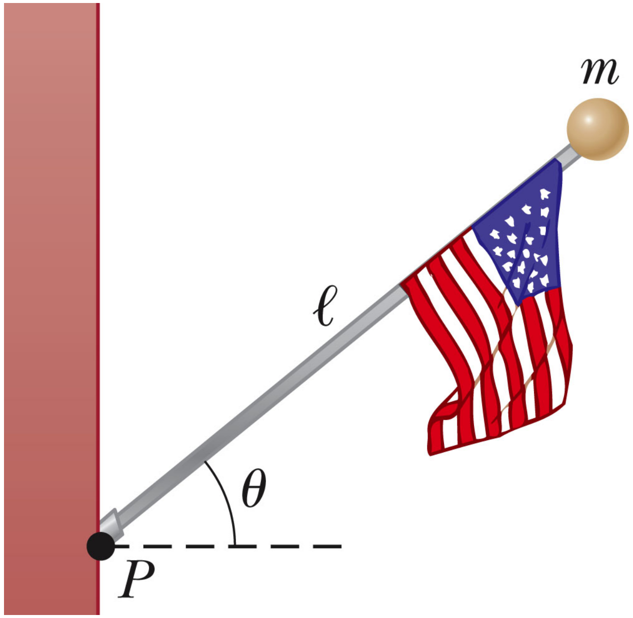


b) What physical reason is there for the balls angular momentum to be changing?

The earth is exerting a torque (through gravity) on the ball. Since the external torque is non-zero the rate of change of angular momentum is non-zero

Example (Prob 11.21)

A ball of mass m is attached to the end of a flagpole which is connected to the side of a building. The length of the pole is l and it makes an angle θ with the horizontal. The ball becomes loose & falls with an acceleration -g**j**.

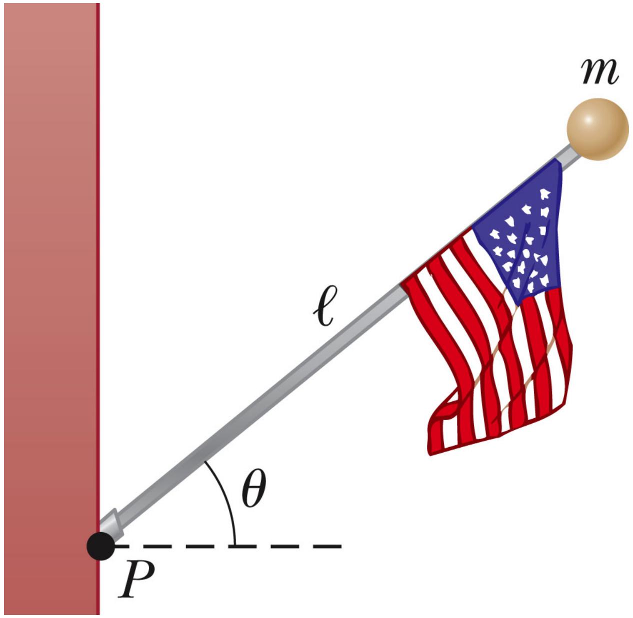


c) What is the rate of change of the angular momentum of the ball about the point P?

$$\frac{d\vec{L}}{dt}=\frac{d}{dt}(-mglt\cos(\theta)\hat{k})=-mgl\cos(\theta)\hat{k}$$

Example (Prob 11.21+)

A ball of mass m is attached to the end of a flagpole which is connected to the side of a building. The length of the pole is l and it makes an angle θ with the horizontal. The ball becomes loose & falls with an acceleration -g**j**.



d) What is the torque on the ball due to gravity about the point P?

$$\vec{\tau} = \vec{r} \times \vec{F} = (l\cos(\theta)\hat{i} + l\sin(\theta)\hat{k}) \times (-mg\,\hat{j})$$

$$= -l\cos(\theta) mg (\hat{i} \times \hat{j}) = -mgl\cos(\theta)\hat{k}$$

$$= \frac{d\vec{L}}{dt}$$

Objective Question 1

Is it possible to calculate the torque acting on an object without specifying an axis of rotation?

No.

Is the torque independent of the location of the axis of rotation?

No. An axis must be specified, and the moment arm is measured from the axis so the value of the torque depends on the choice of the axis

Objective Question 2a

Vector **A** is in the negative y direction, and vector **B** is in the negative x direction.

What is the direction of **A** x **B**?

a) No direction, it's a scalar
b) x
c) -y
d) z
e) -z

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Objective Question 2b

Vector **A** is in the negative y direction, and vector **B** is in the negative x direction.

What is the direction of **B** x **A**?

a) No direction, it's a scalar
b) x
c) -y
d) z
e) -z

Objective Question 3a

Consider three perpendicular directions. Right, up, & toward you, with unit vectors **r**, **u**, **t**, respectively. Consider the quantity (-3**u** x 2**t**)

The magnitude of this vector is

a) 6
b) 3
c) 2
d) 0

Objective Question 3b

Consider three perpendicular directions. Right, up, & toward you, with unit vectors **r**, **u**, **t**, respectively. Consider the quantity (-3**u** x 2**t**)

The direction of this vector is

a) down
b) toward you
c) up
d) away from you
e) left

Objective Question 5

An ice skater starts a spin with her arms stretched out to the sides. She balances on the tip of one skate to turn (without friction since she's a physics ice skater and knows how to do such things). She then pulls her arms in so that her moment of inertia decreases by a factor of 2. Her kinetic energy

a) increases by a factor of 4
b) increases by a factor of 2
c) remains constant
d) decreases by a factor of 2
e) decreases by a factor of 4

Her angular momentum remains constant. So as I decreases by 2 ω increases by 2 and ½ Iω^2 doubles

Objective Question 6a

A pet mouse sleeps near the eastern edge of a turntable* that is supported by a frictionless, vertical axle through its center. The mouse wakes up and starts to walk north on the turntable.

As the mouse takes its first steps, what is the direction of its displacement relative to the stationary ground below?

a) north
b) south
c) none

* A prehistoric type of over-sized CD player that your parents used

Objective Question 6b

A pet mouse sleeps near the eastern edge of a turntable* that is supported by a frictionless, vertical axle through its center. The mouse wakes up and starts to walk north on the turntable.

As the mouse takes its first steps, what is the direction of displacement of the spot on which it was sleeping relative to the stationary ground below?

a) north
b) south
c) none

The turntable counter-rotates clockwise underneath the mouses feet

* A prehistoric type of over-sized CD player that your parents used

Objective Question 6c

A pet mouse sleeps near the eastern edge of a turntable* that is supported by a frictionless, vertical axle through its center. The mouse wakes up and starts to walk north on the turntable.

In this process, is the mechanical energy of the system (mouse+turntable) conserved?

No. The mouse converts chemical energy into mechanical energy which drives both of the motions.

* A prehistoric type of over-sized CD player that your parents used

Objective Question 6d

A pet mouse sleeps near the eastern edge of a turntable* that is supported by a frictionless, vertical axle through its center. The mouse wakes up and starts to walk north on the turntable.

Is the momentum of the system constant?

No. The turntable has zero linear momentum while the mouse has a little bit of linear momentum northwards.

* A prehistoric type of over-sized CD player that your parents used

Objective Question 6e

A pet mouse sleeps near the eastern edge of a turntable* that is supported by a frictionless, vertical axle through its center. The mouse wakes up and starts to walk north on the turntable.

Is the angular momentum of the system constant?

Yes. Angular momentum is constant with a value of zero

* A prehistoric type of over-sized CD player that your parents used

Objective Question 7a

Two ponies of equal mass are initially at diametrically opposite points on the rim of a large horizontal turntable that is rotating freely on a frictionless, vertical axle through its center. They start walking toward each other across the turntable

As they walk, what happens to the angular speed of the turntable?

a) it increases
b) it decreases
c) it remains constant

Objective Question 7b

Two ponies of equal mass are initially at diametrically opposite points on the rim of a large horizontal turntable that is rotating freely on a frictionless, vertical axle through its center. They start walking toward each other across the turntable

Is the mechanical energy of the system (ponies+turntable) conserved?

No. The ponies must do work to push themselves inward toward the center, so the mechanical energy increases.

Objective Question 7c

Two ponies of equal mass are initially at diametrically opposite points on the rim of a large horizontal turntable that is rotating freely on a frictionless, vertical axle through its center. They start walking toward each other across the turntable

Is the momentum of the system (ponies+turntable) conserved?

Yes, the linear momentum stays constant with a value of zero.

Objective Question 7d

Two ponies of equal mass are initially at diametrically opposite points on the rim of a large horizontal turntable that is rotating freely on a frictionless, vertical axle through its center. They start walking toward each other across the turntable

Is the angular momentum of the system (ponies+turntable) conserved?

Yes, the angular momentum has a non-zero value, but there are no external torques so it remains constant.

Objective Question 8

Consider an isolated system moving through empty space. The system consists of objects that interact with each other and can change location with respect to one another. Which of the following quantities can change over time.

a) the angular momentum of the system
b) the linear momentum of the system
c) both the linear & angular momentum of the system
d) neither the linear & angular momentum of the system

as long as no net external forces or torques act on the system, linear and angular momentum are both conserved and do not change over time

Conceptual Question 1

If the torque acting on a particle about an axis through a certain origin is zero, what can you say about its angular momentum about that axis?

That the angular momentum about that axis is conserved. You can conclude nothing about the magnitude of the angular momentum about that axis

Conceptual Question 2

A ball is thrown in such a way that it does not spin about its own axis. Is the angular momentum of the ball zero about any arbitrary axis?

No. The angular momentum about any axis which is *not* along the path of the ball is non-zero.

Conceptual Question 3

Why does a tightrope walker use a long pole to stay balanced?

Since the pole has a very large moment of inertia about the tightrope, an imbalance will produced a much smaller angular velocity (rotation) allowing more time to regain balance and not fall screaming to his death.

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Conceptual Question 4a

Two children are playing with a roll of paper towels. One child holds the roll between the index fingers of her hands so that it is free to rotate, and the second child pulls at constant speed on the free end of the paper towels. As the child pulls the paper towels, the radius of the roll of remaining towels decreases.

How does the torque on the roll change with time?

This is a torque due to friction, and the magnitude depends on the normal force between the surfaces in contact. As the weight of the roll decreases the frictional force decreases leading to a decreasing torque.

Conceptual Question 4b

Two children are playing with a roll of paper towels. One child holds the roll between the index fingers of her hands so that it is free to rotate, and the second child pulls at constant speed on the free end of the paper towels. As the child pulls the paper towels, the radius of the roll of remaining towels decreases.

How does the angular speed of the roll change in time?

The child is pulling at constant speed, as the radius of the roll decreases the angular velocity will increase.

Conceptual Question 4c

Two children are playing with a roll of paper towels. One child holds the roll between the index fingers of her hands so that it is free to rotate, and the second child pulls at constant speed on the free end of the paper towels. As the child pulls the paper towels, the radius of the roll of remaining towels decreases.

If the child suddenly jerks the paper towels with a large force, is the towel more likely to break free from the others when pulled from a nearly full or a nearly empty roll?

Approximate the roll as a cylinder, the its moment of inertia is $I = MR^2$ and the mass is is proportional to the area. So the inertia is proportional to R^4. If the roll is given a sudden jerk, its angular acceleration may not be enough to get it moving with the paper so it breaks. This is more likely when the radius is large.

Conceptual Question 5

Both torque & work are products of force and displacement. How are they different. Do they have the same units?

Work done by a torque results in a change of rotational kinetic energy, work done by a force results in a change in translational kinetic energy. Both have similar units: Nm = J

Conceptual Question 6

In some motorcycle races the riders ride over small hills and the motorcycle becomes airborne for a short time. If the racer keeps the throttle open while airborne the motorcycle tends to nose up. Why?

As the motorcycle leaves the ground the drive wheel speeds up, and no external torque is acting on it in the air so angular momentum is conserved. As the angular momentum of the wheel increases the frame counter-rotates – front up tail down.


*

* This..... will not end well.

Conceptual Question 7

Stars originate as large bodies of slowly rotating gas. Because of gravity these clumps of gas slowly decrease in radius. What happens to the angular speed of a star as it shrinks?

Since the star is isolated from external torques, its angular momentum is conserved while it's compressing. As the radius decreases the moment of inertia decreases and the angular speed increases.

Conceptual Question 8

A scientist arriving at a hotel asks a bellhop to carry a heavy suitcase. When the bellhop rounds a corner the suitcase suddenly swings away from him for some unknown reason. The bellhop drops the suitcase & runs away. What might be in the suitcase?

It might contain a gyroscope* If the gyroscope is spinning about an axis passing horizontally through the bellhop the force he applies to turn a corner results in a torque that would make the suitcase swing away.

* @#$(@#%_@#(% scientists!

Conceptual Question 9

If global warming continues over the next hundred years it is possible that some polar ice could melt and the water will be distributed closer to the equator.

(a) How would that change the moment of inertia of the earth?
(b) Would the duration of a day increase or decrease?

a) If mass moved from the pole to the equator it would be moving *away* from the axis of rotation so the moment of inertia would increase.

b) Since the earths angular momentum is conserved, the day would be longer (by a couple of nanoseconds).

Conceptual Question 10

A cat usually lands on it feet regardless of how it was dropped. A slow motion film of a cat falling shows the upper half of its body twisting in one direction. Why would the cat do this?

Since the cats angular momentum is conserved rotating one half of it's body will cause the other to rotate in the opposite direction which allows the cat to get it's feet underneath it