ROTATIONAL DYNAMICS

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ABSTRACT

Lecture notes on what the title says and what the subject headings say.

Subject headings: rotational dynamics — cross product — angular momentum — torque — conservation of angular momentum — rigid-body rotation — rotational inertial — gravitational torque — physical pendulum — parallel-axis theorem — rotational work-kinetic-energy theorem — rotational kinetic energy — work-energy theorem — rollers — Atwood's machine

1. INTRODUCTION

Rotational dynamics are the dynamics of rotating systems.

It's a big subject in general.

There are lots of rotating systems—some of which are very complex. Rotating systems are not particles: they are made of particles exhibiting **COLLECTIVE MOTION**.

We will deal with some of the less complex rotating systems.

In fact, we specialize to rigid bodies rotating around a fixed axis most of the time.

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No new general principles of physics are needed in rotational dynamics.

It seems to me that this is often obscured in intro physics textbooks where the derivations seem to be omitted or skimped without commenting on the fact.

The rotational physics rules we introduce are all derived from basic Newtonian physics (Newton's three laws of motion, force laws, energy, etc.). Those are the general principles.

But applying general principles directly is often **NOT** convenient.

Table 1. Correspondence Between General Dynamics Rules and Formalisms and Rotational Dynamics Ones

General Item	Rotational Item
position coordinate $\vec{r}$	angular coordinate θ
velocity $\vec{v}$	angular velocity ω
acceleration $\vec{a}$	angular acceleration α
mass m	rotational inertia or moment of inertia I
momentum $\vec{p}$	angular momentum $\vec{L}$
force $\vec{F}$	torque $\vec{\tau}$
Newton's 2nd law $(d\vec{p}/dt) = \vec{F}_{\rm net}$	rotational 2nd law $(d\vec{L}/dt) = \vec{\tau}_{\rm net}$
for a system of particles	
Newton's 2nd law for 1-d $(dp/dt) = F_{\rm net}$	rotational 2nd law for a rigid body $\tau_{\rm net} = I\alpha$
for a system of particles	
kinetic energy $KE = (1/2)mv^2$	rotational kinetic energy for a rigid body $KE_{\rm rot} = (1/2)I\omega^2$

Note. — The table only gives the most obvious and important correspondences. Like Newton's laws, the rotational analogs are referenced to inertial reference frames. The symbols used for the quantities are pretty standard, but some variations do turn up. Generally, context must decide what the symbols mean and what qualifications apply to the symbols. In the table, the acceleration $\vec{a}$ and velocity $\vec{v}$ are center-of-mass quantities for the system described by Newton's 2nd law. Momentum $\vec{p}$ is the total momentum of the system. In the table, subscript "net" indicates net force or torque and subscript "rot" indicates rotational quantity.

Often there is a long and **NOT** obvious path from general principles to the special rules of some fields of application. The path is so long that often one can't even imagine how to follow it. It's one of the glories of physics that one can start from general (or basic) principles and go off on a path through formalism losing touch with intuitional understanding and touch down with results that apply in the real world. Great minds spent years finding those paths in some cases.

The paths to the rules for rotational dynamics are somewhat long.

Therefore, it's convenient to remember those rotational dynamics rules themselves and not refer back to general principles—except when one has to do that—or when it's more convenient to do that.

In fact, the special rules and formalisms of rotational dynamics have been set up to mimic the general principles and formalisms. This is convenient for remembering them and in knowing how to apply them.

For example, there is a rotational Newton's 2nd law. It's not a general principle like Newton's 2nd law, but it has an analogous formula and is applied in an analogous way.

Table 1 shows the correspondence between general dynamics rules and formalisms and rotational dynamics ones.

Those items you don't recognize in Table 1 will be elucidated in the following sections.

Be warned: to prevent a plethora special-case symbols and subscripts, generic symbols are often used for special cases—context must help you decide what is being meant in particular cases.

Be warned too. In the opinion of yours truly, this is the hardest material in the 1st semester of intro physics for students, instructors, and, evidently, the writers of textbooks.

The hardness is conceptual, not in the problems which are of much the same difficulty as other problems in the 1st semester of intro physics.

2. CROSS PRODUCT

The multiplication of vectors has to be defined by rules. There are several multiplication rules. From one point of view, these rules are artificial. Mathematicians created them since they lead to interesting and useful operations.

From a physics perspective, the common rules for multiplication of vectors turn out to describe certain physical quantities—and so don't seem so artificial, but things actually embedded in nature.

We have already considered the product of a scalar and vector and the **DOT OR SCALAR PRODUCT**.

Another rule is for the **CROSS OR VECTOR PRODUCT**.

Show of hands for those who are familiar with the cross product.

We use the cross product extensively in rotational dynamics.

So let's look at it.

The formula for the cross product of general vectors $\vec{A}$ and $\vec{B}$ is written

$$\vec{A} \times \vec{B} = AB \sin\theta \hat{n} \ , \tag{1}$$

where A and B are, respectively, the magnitudes of $\vec{A}$ and $\vec{B}$, θ is the angle between $\vec{A}$ and $\vec{B}$ (whose tails are always put at one point), and $\hat{n}$ is a unit vector normal to the plane defined by $\vec{A}$ and $\vec{B}$ whose sense is determined by a right-hand rule—one of many right-hand rules in physics. The cross product is always written with the multiplication sign shown explicitly.

The cross product operation gives a vector result. Actually a pseudovector result (e.g. Arfken 1970, p. 131), but that mathematical refinement is beyond the scope of this class.

This is unlike the dot product that gives a scalar result. A scalar recall is a quantity that is independent of coordinate transformation and is often just a simple real number. In the linear algebra branch of pure math, scalars are just real numbers. Physicists often use scalar and real number as synonyms. This is sloppy since they are not exact synonyms, but it's conventional and as long as one knows what one means it's OK.

The cross-product right-hand rule is evaluated by taking your right hand and sweeping your fingers from the first vector in the cross product to the second vector in the cross product and your right-hand thumb gives the sense of $\hat{n}$. ("Sense" in math is one of two opposite directions a vector can point.)

There are other hand-waving ways to evaluate the right-hand rule, but they all give the same result and the one given is the one yours truly remembers.

Here question for the class.

What is $\vec{B} \times \vec{A}$ in terms of $\vec{A} \times \vec{B}$? You have 30 seconds working individually or in groups. Go.

Fig. 1.— The cross product $\vec{A} \times \vec{B}$ of general vectors $\vec{A}$ and $\vec{B}$.

Yes, the right-hand rule implies that the cross product is **ANTICOMMUTATIVE** and **NOT** commutative. Thus,

$$\vec{B} \times \vec{A} = -\vec{A} \times \vec{B} \ . \tag{2}$$

What is the cross product if $\vec{A}$ and $\vec{B}$ are aligned: i.e., they point in the same direction or opposite directions. (Note we think of the anti-aligned case as just a special case of alignment.)

You have 30 seconds to write down the result. Go

Yes, their cross product is **ZERO**.

This is an important result that is needed several times in this lecture and is generally important in physics.

The zero result is just a consequence of the sine in the cross product rule:

$$\sin\theta = \begin{cases} 0 & \text{for } \theta = 0; \\ 0 & \text{for } \theta = \pi. \end{cases} \tag{3}$$

To summarize the cross product behavior, note

$$\vec{A} \times \vec{B} = \begin{cases} AB\sin\theta\hat{n} & \text{in general;} \\ 0 & \text{for } \theta = 0 \text{ or } \pi; \\ AB\hat{n} & \text{for } \theta = \pi/2; \\ -\vec{B} \times \vec{A} & \text{in general.} \end{cases} \tag{4}$$

The cross product has the distributive property: i.e.,

$$\vec{A} \times \left(\vec{B} + \vec{C}\right) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C} \ , \tag{5}$$

where $\vec{A}$, $\vec{B}$, and $\vec{C}$ are general vectors. The cross-product distributive property actually requires a proof, but leave that to some vector math course. We briefly describe how one would go about doing the proof in Appendix A. We will just assume the distributive property.

There is an alternative component-form formula for the cross product.

It is equivalent to the formula $\vec{A}\times\vec{B} = AB\sin\theta\hat{n}$. Equivalence means each formula can be derived from the other in math.

The component-form formula is pretty commonly in use—but we don't use it in this course—except for **ONE THING**—and so we relegate it to Appendix A.

The **ONE THING** is that the component-form formula can be used to prove the product rule for the cross product for differentiation with respect an independent variable. The independent variable of interest to us is time t. This cross-product product rule is

$$\frac{d(\vec{A}\times\vec{B}\,)}{dt} = \frac{d\vec{A}}{dt}\times\vec{B} + \vec{A}\times\frac{d\vec{B}}{dt} \tag{6}$$

which is just what you'd guess it had to be. But a proof is necessary to know it—that proof is given in Appendix A as aforesaid.

3. ANGULAR MOMENTUM AND TORQUE DEFINED

Angular momentum and torque are new dynamical quantities—relative to this course that is.

We define them using more basic dynamical quantities.

Thus, angular momentum and torque give us no new fundamental physics—they are a powerful way of organizing old physics (i.e., the general principles of Newtonian physics) to treat rotational dynamics.

They are vector quantities.

Angular momentum has the generic symbol $\vec{L}$ and torque, $\vec{\tau}$, where τ is the small Greek letter tau.

The angular momentum and torque definitions for a particle (a classical point particle) are, respectively,

$$\vec{L} = \vec{r} \times \vec{p} \qquad \text{and} \qquad \vec{\tau} = \vec{r} \times \vec{F} \ , \tag{7}$$

where $\vec{r}$ is the particle position relative to some origin, $\vec{p}$ is the particle momentum (i.e., linear momentum), $\vec{F}$ is a force on the particle ($\vec{\tau}$ is the torque of that force $\vec{F}$), and the definition makes use of the cross product (see § 2).

The reference frame for the definitions is a usually an inertial frame. One can use non-inertial frames by introducing **INERTIAL FORCES** which were treated in the lecture *NEWTONIAN PHYSICS II*. We do consider non-inertial frame cases in §§ 7 and 12.

The angular momentum and torque vectors point in real physical space, but their extent is in their own abstract angular momentum and torque spaces. Their tails are at the origins of those spaces.

We see that angular momentum and torque are dependent on the origin of the spatial coordinate system. An arbitrary origin for evaluated angular momentum and torque may not be useful in many cases. However, origins which are physically significant are often very useful for evaluating angular momentum and torque. Cases of physically significant origins are one that is the center of mass (particularly for objects floating in force-free space), one

Fig. 2.— Schematic diagram of angular momentum and torque formulae.

that is the center of a **CENTRAL FORCE**, and one that is on a rigid axis about which a body is constrained to rotate. The last case is that we mostly concentrate on in this lecture.

Because they are vectors, angular momentum and torque are **INDEPENDENT** of the choice of orientation of the axes for the coordinate system, but their components **ARE DEPENDENT** on the choice of orientation of the axes for the coordinate system. We take up the subject of components of angular momentum and torque in § 5.

We also see that because of the nature of cross product, angular momentum will tend to point along the axis about which one sees rotation and torque will tend to point along the axis about a force is try to cause rotational acceleration. These facts about angular momentum and torque take some getting used to.

The MKS units of angular momentum and torque are, respectively, given by

$$\text{unit}\left[\vec{L}\right] = \text{kg}\,\text{m}^2/\text{s} = \text{J}\,\text{s} \qquad \text{and} \qquad \text{unit}\left[\vec{\tau}\,\right] = \text{N}\,\text{m} = \text{kg}\,\text{m}^2/\text{s}^2 \ , \tag{8}$$

where unit[ ] is my idiosyncratic unit function. The units have no special names or symbols, but N m is usually referred to as the Newton-meter.

In macroscopic physics, one almost always uses MKS units for angular momentum and torque in actual calculations, and so no other units for these quantities are in wide use. American engineers may use US customary units for angular momentum and torque—but I've no idea what those are. In special fields, special convenient units for angular momentum and torque may turn (e.g., astronomy), but mostly for thinking about purposes and not calculations.

In fact when using MKS units for angular momentum and torque, people sometimes don't bother writing units down since they are just understood. But for **FINAL ANSWERS** on test problems, write down the units always.

Note the Newton-meter (N m) is also a joule. But torque is not an energy quantity—it just has the same physical dimensions and unit when the unit is expressed in terms of basic units—and we never give torque in joules, but always in Newton-meters.

Angular momentum and torque are, respectively, the rotational analogs of momentum and force—which are considered translational variables. The analog nature of angular momentum and torque are suggested by the definitions of angular momentum and torque. The roles that angular momentum and torque play in rotational dynamics verify their analog natures.

In fact, one can say that angular momentum and torque are, respectively, position-weighted analogs of momentum and force. When we introduce the rotational inertia in § 6, we will see that it is position-weighted analog of mass. So rotational dynamics uses position-weighted analogs to the dynamical variables of translational motion. This is part and parcel of how rotational dynamics deals with the internal (i.e., not center-of-mass) motions of systems of particles.

As we'll discuss in § 6, rotational inertia is a measure of resistance to angular acceleration just as mass is measure of resistance to acceleration. One can say that angular momentum is vaguely a measure of rotational motion just as momentum is a measure of motion. Torque as we'll see in a moment can cause a change in angular momentum (or vaguely in angular motion) just as force can cause a change of momentum. One can also think of torque as being able to cause an angular acceleration just can think of force being able to cause an acceleration.

Because torque can cause a change in angular momentum, one can vaguely think of a torque as twist just as one vaguely thinks of a force as a push or a pull. A twist might vaguely be described as force exerted on a body in order to rotate it about some origin. The displacement from the origin to where the twist force is applied is a crucial ingredient in the effect of the twist force. The twist force and displacement correspond, respectively, to the force and radius vectors in the torque definition.

To make more perfect the analogy of rotational dynamical variables to translational variables, there should be a rotational analog to Newton's 2nd law $\vec{F}_{\rm net} = m\vec{a}$ or $d\vec{p}/dt = \vec{F}_{\rm net}$ for a particle.

There is.

We will derive it.

First recall that Newton's laws must be referenced to inertial reference frames, unless inertial forces are introduced. Since we use those laws in our derivation, we will assume that angular momentum and torque evaluated in an inertial reference frame or one in which we do introduce inertial forces. We will assume inertial reference frames always hereafter, unless otherwise specified (as in §§ 7 and 12).

Given that $\vec{L}$ is the rotational analog of $\vec{p}$ and Newton's 2nd law is

$$\frac{d\vec{p}}{dt} = \vec{F}_{\rm net} \ , \tag{9}$$

what is the natural starting point for finding a rotational 2nd law?

You have 30 seconds to write down the natural starting point. Go.

Well it seems natural to take the time derivative of $\vec{L}$ and see where that leads us.

Behold:

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{v} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = 0 + \vec{r} \times \vec{F}_{\rm net} = \vec{r} \times \vec{F}_{\rm net} = \vec{\tau}_{\rm net} \ , \tag{10}$$

where we have used the facts that the product rule does apply to the cross product, that $\vec{v}$ (the particle velocity) is aligned with $\vec{p} = m\vec{v}$ and so $\vec{v} \times \vec{p} = 0$, and that $\vec{F}_{\rm net} = m\vec{a} = d\vec{p}/dt$ for a constant mass particle (which we assume). The net torque on the particle is naturally defined

$$\vec{\tau}_{\rm net} = \vec{r} \times \vec{F}_{\rm net} \ . \tag{11}$$

The fact that $\vec{F}_{\rm net} = m\vec{a}$ is used in equation (10) is the key ingredient of that equation.

The analog to $\vec{F}_{\rm net} = m\vec{a}$ is now seen to be

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{\rm net} \ . \tag{12}$$

The net torque (which is the analog of net force) causes angular momentum (the analog of momentum to change).

The formula equation (12) is obviously analogous to Newton's 2nd law for a particle by inspection especially when Newton's 2nd law is in the form

$$\frac{d\vec{P}}{dt} = \vec{F}_{\rm net} \ . \tag{13}$$

If one knows what one is talking about, one can, of course, drop the subscript "net" in equations (12) and (13). But "net" is one subscript that these lectures seldom drop.

Equation (12) is called the rotational Newton's 2nd law for a particle or, for short, the rotational 2nd law.

Note that if $\vec{\tau}_{\rm net} = 0$, then

$$\vec{L} = \text{constant} \ : \tag{14}$$

i.e., angular momentum is conserved. Note all the components of angular momentum are conserved: i.e., L_x, L_y, and L_z are conserved.

The conservation of angular momentum when net torque is zero is exactly analogous to the conservation of momentum when net force is zero.

4. SYSTEMS OF PARTICLES

Our focus of study is rotational motion and, in particular, rigid-body rotation about a fixed axis.

But for the moment, let's be general.

Say we had a general system of classical point particles: the general particle is particle i.

Actually, we could go to the continuum limit and smooth the particles out into a material continuum. We then would make use of integrations to get macroscopic quantities. However, for simplicity in our developments we just leave going the continuum limit implicit and deal with particles and use summations to get macroscopic quantities. We do go to the continuum limit for some cases.

The natural definitions (and nothing forbids us from making them) for the system total angular momentum and net torque are, respectively,

$$\vec{L} = \sum_i \vec{L}_i \qquad \text{and} \qquad \vec{\tau}_{\rm net} = \sum_i \vec{\tau}_{i,\rm net} \ , \tag{15}$$

where the sum is over all particles i, $\tau_{i,\rm net}$ is the net torque on particle i (not just any particular torque), and here the generic symbols $\vec{L}$ and $\vec{\tau}_{\rm net}$ are used for the net quantities.

From § 3 (see eq. (12)), it follows at once that

$$\sum_i \frac{d\vec{L}_i}{dt} = \sum_i \vec{\tau}_{i,\rm net} \ , \tag{16}$$

and thus that

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{\rm net} \ . \tag{17}$$

Now in the lecture *NEWTONIAN PHYSICS I*, we derived Newton's 2nd law for a system of particles

$$\vec{F}_{\rm net} = m\vec{a} \ , \tag{18}$$

where m is the total mass of the system, $\vec{a}$ is the center-of-mass acceleration, and $\vec{F}_{\rm net}$ is the net external force on the system—it is also the net force on the system since the internal forces sum to zero which justifies dropping any subscript indication of external force.

Is there an analogous formula to the 2nd law for rotational dynamics?

There is.

Yes, it's equation (17), of course.

But to make the analogy complete, we must prove that the internal torques sum to zero so that $\vec{\tau}_{\rm net}$ is also the net external torque as well as being the net torque on the system.

We will do the proof right now.

Decompose the net torque on particle i into external and internal terms:

$$\vec{\tau}_{i,\rm net} = \vec{\tau}_{i,\rm ext} + \vec{\tau}_{i,\rm int} \ . \tag{19}$$

The internal part can be taken as due to inter-particle forces. Thus,

$$\vec{\tau}_{i,\rm int} = \vec{r}_i \times \vec{F}_{i,\rm int} = \sum_{j,j\neq i} \vec{r}_i \times \vec{F}_{ji} \ , \tag{20}$$

where $\vec{F}_{ji}$ is the net force of particle j on particle i.

Now in the sum

$$\sum_i \vec{\tau}_{i,\rm int} = \sum_{ij,i\neq j} \vec{r}_i \times \vec{F}_{ji} \ , \tag{21}$$

we must have the terms $\vec{r}_k \times \vec{F}_{\ell k}$ and $\vec{r}_\ell \times \vec{F}_{k\ell}$.

By Newton's 3rd law, $\vec{F}_{k\ell} = -\vec{F}_{\ell k}$.

Thus, the sum $\sum_i \vec{\tau}_{i,\rm int}$ can be written as a sum of pairs of terms of the form

$$\vec{r}_k \times \vec{F}_{\ell k} + \vec{r}_\ell \times \vec{F}_{k\ell} = (\vec{r}_k - \vec{r}_\ell) \times \vec{F}_{\ell k} \ , \tag{22}$$

where $\vec{r}_k - \vec{r}_\ell$ is the displacement vector of particle k relative to particle ℓ and where we have used the distributive property of the cross product—which we just assumed and have not proven. If we now assume that the inter-particle forces of action and reaction point along the line joining the particles, then

$$(\vec{r}_k - \vec{r}_\ell) \times \vec{F}_{\ell k} = 0 \tag{23}$$

since the cross product of aligned vectors is always zero.

Such forces in physics jargon are called **CENTRAL FORCES** (e.g., Goldstein et al. 2002, p. 7).

The assumption of that the inter-particle forces of action and reaction point along the line joining the particles is called the strong version of Newton's 3rd law (e.g., Goldstein et al. 2002, p. 7). Actually, both Newton's 3rd law in its ordinary version (the weak version) and the strong version can be violated in classical physics—which is something intro physics books rarely/never mention—but in these cases some generalization can usually be found so that one can proceed in a similar fashion to having those laws hold (e.g., Goldstein et al. 2002, p. 7–8). Actually, the only violations of the 3rd law (weak version) that yours truly knows of are for the magnetic force on small systems that can be regarded as point-like or are particles (e.g., electrons treated classically). We won't worry about the fine points of the 3rd law at all—we just assume equation (23) holds.

Fig. 3.— Schematic diagram of internal torques canceling out pairwise.

Assuming equation (23) implies that

$$\sum_i \vec{\tau}_{i,\rm int} = 0 \ . \tag{24}$$

We say that the terms in the sum cancel pairwise. We now find that

$$\tau_{\rm net} = \sum_i \tau_i = \sum_i (\vec{\tau}_{i,\rm ext} + \vec{\tau}_{i,\rm int}) = \sum_i \vec{\tau}_{i,\rm ext} \ . \tag{25}$$

We see that $\tau_{\rm net}$ is the sum of the external torques. Thus, the net torque is the same as the net external torque.

This means that

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{\rm net} \tag{26}$$

is exactly analogous to

$$\frac{d\vec{p}}{dt} = \vec{F}_{\rm net} \ , \tag{27}$$

in that $\vec{\tau}_{\rm net}$ is the net external torque just as $\vec{F}_{\rm net}$ is the net external force.

One can drop the subscript "net" from both equations (26) and (27) if one knows what one means. But "net" is one subscript that these lectures seldom drop.

Equation (26) can be changed into a form exactly analogous to

$$\vec{F}_{\rm net} = m\vec{a} \tag{28}$$

in the special case of rigid body rotation about a fixed axis. In that case, the angular acceleration α of the rigid body (which is common to all particles making it up) is the analog to $\vec{a}$ and rotational inertia I is the analog of mass. We treat that special case in § 6.

Equation (26) shows that one can ignore all internal torques as long as only the total angular momentum is of interest. The internal torques may do lots of internal things to the system, but that is another story.

The formula equation (26) is the rotational Newton's 2nd law for a system of particles—which we usually just call the rotational Newton's 2nd law or the rotational 2nd law.

Note that if $\vec{\tau}_{\text{net}} = 0$, then $\vec{L}$ is constant: i.e., total angular momentum is conserved.

This is exactly analogous to the case where the momentum a system is conserved if the net force $\vec{F}_{\text{net}} = 0$.

Now the angular momenta of individual parts of the system can change in general.

For complex systems, all kinds of complex changes can occur to the system with $\vec{\tau}_{\text{net}} = 0$.

We'll consider some special cases of conservation of angular momentum in later sections.

But now for a special reading topic.

It's rather important for clear thinking, but can be omitted from verbal lecturing—and left to private delectation.

4.1. Equilibrium: Reading Only

The center-of-mass or translational acceleration of a system is zero if the net external force $\vec{F}_{\text{net}} = 0$. The system is then in translational equilibrium. But recall parts of the system can be undergoing accelerations (i.e., having changes in their momenta).

The angular momentum of a system is constant if the net external torque $\vec{\tau}_{\text{net}} = 0$. The system is then in overall rotational equilibrium although the angular momenta of parts of the system could be changing.

We can describe the system as being in general overall equilibrium if $\vec{F}_{\text{net}} = 0$ and $\vec{\tau}_{\text{net}} = 0$. But, of course, internal changes in momentum and angular momentum will be occurring in general.

General overall equilibrium for static systems with rigid bodies is usually just called equilibrium without qualification. Context must tell what is meant.

There is an important fact about general overall equilibrium is that it is origin and inertial reference frame independent. This implies that if one has a general overall equilibrium, the net torque is zero and total angular momentum is constant for all origins and in all inertial reference frames But how can this be since torque and angular momentum are actually origin dependent quantities?

It can be.

The proof is as follows.

Let $\vec{r}_i$ be the locations of particles i in one reference frame and $\vec{r}'_i = \vec{r}_i - \Delta\vec{r}$ be the locations in a general primed reference frame whose origin is at displacement $\Delta\vec{r}$ from first frame's origin. Note $\Delta\vec{r}$ can be changing at a constant rate in time: that is the two reference frames can be distinct inertial frames although they do not have to be. Say angular momentum is constant in the first frame, then $\vec{\tau}_{\rm net} = 0$ in this frame. Now the torque in the second frame $\vec{\tau}'_{\rm net}$ is

$$\vec{\tau}'_{\rm net} = \sum_i \vec{r}'_i \times \vec{F}_i = \sum_i (\vec{r}_i - \Delta\vec{r}) \times \vec{F}_i = \sum_i \vec{r}_i \times \vec{F}_i - \Delta\vec{r} \times \sum_i \vec{F}_i = \vec{\tau}_{\rm net} - \Delta\vec{r} \times \vec{F}_{\rm net} = 0 \ , \quad (29)$$

where we have used the distributive property of the cross product (which we just assumed), and the facts that $\vec{F}_{\rm net} = 0$ and $\vec{\tau}_{\rm net} = 0$ in general overall equilibrium and that forces are invariant under frame transformation.

Since $\vec{\tau}'_{\rm net} = 0$, there is no net torque in the primed frame and total angular momentum is constant in the primed frame. The net force is in the primed frame is zero too since forces are independent of reference frame in classical physics. So the system is in general overall equilibrium in the primed frame too.

Since the primed frame is a general inertial frame, if we have general overall equilibrium in one inertial reference frame, we have them in all inertial reference frames. Often the key point though is that in one inertial frame general overall equilibrium is origin independent. We have completed the proof.

The fact that the net torque is zero for cases of general overall equilibrium for all origins in a single inertial frame is often very useful in calculating the forces for systems in general overall equilibrium particularly when the systems are static and consist of rigid bodies. Some of the forces are unknown and can only be calculated from general overall equilibrium. The reason for the usefulness is that one chooses the origin for maximum convenience in calculating the forces.

5. ROTATIONAL DYNAMICS AND AXES

Angular momentum and torque are origin dependent as their definitions in § 3 show. But because they are vectors, angular momentum and torque are independent of the choice of orientation of the axes for the coordinate system. On the other hand, their components are dependent on the orientation of axes.

Just as with other vectors, even though the components of angular momentum and torque are not unique, treating angular momentum and torque using components is often the most efficient way of treating them. This is particularly true for rigid bodies rotating around a fixed axis which is conventionally the z axis. In this case, one often only needs to deal with the z components of angular momentum and torque.

We now proceed to consider angular momentum and torque components.

Consider a set of Cartesian axes and let us consider the z axis first. Let the cylindrical coordinates radius from the z axis be $\vec{R}$. We retain $\vec{r}$ for the radius vector from the origin.

For a general **PARTICLE**, we factorize $\vec{L}$ thusly

$$\vec{L} = \vec{r} \times \vec{p} = (\vec{R} + \vec{z}\,) \times (\vec{p}_{xy} + \vec{p}_z) \ , \tag{30}$$

where $\vec{z}$ is the z-component vector of $\vec{r}$, $\vec{p}_{xy}$ is the xy-component vector of $\vec{p}$, and $\vec{p}_z$ is the z-component vector of $\vec{p}$.

Now using the distributive property of the cross product, we find

$$\begin{aligned} \vec{L} &= \vec{R} \times \vec{p}_{xy} + \vec{R} \times \vec{p}_z + \vec{z} \times \vec{p}_{xy} + \vec{z} \times \vec{p}_z \\ &= \vec{R} \times \vec{p}_{xy} + \vec{R} \times \vec{p}_z + \vec{z} \times \vec{p}_{xy} \ , \end{aligned} \tag{31}$$

where we have used the fact that $\vec{z} \times \vec{p}_z = 0$ since $\vec{z}$ and $\vec{p}_z$ are aligned. A physical interpretation of $\vec{z} \times \vec{p}_z = 0$ is just that motion in the z direction right along the z axis is no sense rotation about the origin or the z axis.

We now note that $\vec{R} \times \vec{p}_z$ and $\vec{z} \times \vec{p}_{xy}$ are both perpendicular to the z axis since in both cases one of the factor vectors is aligned with the z axis. The relevant physical interpretation of $\vec{R} \times \vec{p}_z$ is that motion in the z direction at $\vec{R}$ is in no sense rotation around the z axis The relevant physical interpretation of $\vec{z} \times \vec{p}_{xy}$ is that motion in the x-y direction right on the z axis is in no sense rotation around the z axis.

On the other hand, both factor vectors of $\vec{R} \times \vec{p}_{xy}$ are in the xy plane, and thus $\vec{R} \times \vec{p}_{xy}$ points in the z direction, positively or negatively. The relevant physical interpretation of $\vec{R} \times \vec{p}_{xy}$ is that motion in the x-y plane is rotation around the z axis as long as $\vec{R}$ is not zero and $\vec{p}_{xy}$ is not aligned with $\vec{R}$: if it were aligned, one would just have radial motion.

The conclusion is that the z-component vector of $\vec{L}$ is given by

$$\vec{L}_z = \vec{R} \times \vec{p}_{xy} \ , \tag{32}$$

where the factor vectors are both in the xy plane. The z component of $\vec{L}$ itself or, z angular momentum as we'll often call it for brevity, is

$$L_z = (\vec{R} \times \vec{p}_{xy}) \cdot \hat{z} \ . \tag{33}$$

Note absolutely positively, only x-y momentum contributes to z angular momentum. The z momentum does **NOT** contribute to the z angular momentum. The relevant interpretation of z momentum is that in no sense implies rotation about the z axis.

Exactly analogously to the derivation of the $\vec{L}_z$ formula, one finds that the z-component vector of $\vec{\tau}$ is given by

$$\vec{\tau}_z = \vec{R} \times \vec{F}_{xy} \ , \tag{34}$$

where the factor vectors are both in the xy plane and $\vec{F}_{xy}$ is the component vector of $\vec{F}$ in the xy plane. The z component of $\vec{\tau}_z$ itself or, z angular momentum as we'll often call it for brevity, is

$$\tau_z = (\vec{R} \times \vec{F}_{xy}) \cdot \hat{z} \ . \tag{35}$$

Note absolutely positively, only x-y force contributes to z torque. The z force does **NOT** contribute to the z torque. This just the way the rotational dynamics formalism works out.

One can do the same analysis for the $\vec{L}_x$ and $\vec{L}_y$ and $\vec{\tau}_x$ and $\vec{\tau}_y$ component vectors with exactly analogous results, but we don't need those formulae for our developments.

Now for a subtle point. Angular momentum $\vec{L}$ and torque $\vec{\tau}$ are origin dependent quantities, but independent of the choice of the orientation of the axes.

But the z components of angular momentum and torque have no **EXPLICIT** origin dependence. They depend on the choice of the z axis. If you choose a z axis, then L_z and τ_z are determined without specifying an origin. Now if you want to determine the other components of angular momentum and torque an origin will be specified since determining the other components demands specifying the x and y axes which determines an origin since that is where the axes intersect. So you do need to specify an origin for determining vectors angular momentum and torque, but you don't need to do that if you only want to determine the components along some axis which is usually specified as the z axis.

Frequently, especially when dealing with rigid body rotation about a fixed axis (which is conventionally the z axis), the components of angular momentum and torque along other the axes are unneeded.

For brevity, one can say angular momentum and torque for an axis without specifying an origin provided that it is understood that you mean the components of angular momentum and torque along that axis (which is conventionally designated as the z axis) and the other components of angular momentum and torque are not being used.

Actually, one frequently uses the expressions 'angular momentum about an axis" and "torque about an axis". Both expressions mean that the vector is aligned with the axis. The expressions make sense to our ordinary sense of things, because angular momentum is a measure of the rotational motion about an axis and torque is a measure of a force that can increase the angular momentum about an axis.

Fig. 4.— Schematic diagram of the factors for the z-components of angular momentum and torque.

At this point, it is convenient to introduce angles γ_L and γ_τ: γ_L is the angle from $\vec{R}$ to $\vec{p}_{xy}$ and γ_τ is the angle from $\vec{R}$ to $\vec{F}_{xy}$. Counterclockwise is the positive direction for both γ_L and γ_τ. With these angle definition, we can write the z component angular momentum and torque for a particle as, respectively,

$$L_z = Rp_{xy} \sin \gamma_L \tag{36}$$

and

$$\tau_z = RF_{xy} \sin \gamma_\tau \ . \tag{37}$$

The setup for the formulae equations (36) and (37) is illustrated in Figure 4.

Note that R, p_{xy} and F_{xy} are magnitudes, but L_z and τ_z are vector components. The signs of L_z and τ_z are determined by the signs of the angles γ_L and γ_τ.

For a system of particles, we find the total z component angular momentum to be

$$L_z = \sum_i L_{i,z} = \sum_i R_i p_{i,xy} \sin \gamma_{i,L} \ , \tag{38}$$

where $L_{i,z}$ is the z component angular momentum of particle i, R_i is the cylindrical radius coordinate to particle i, $p_{i,xy}$ is the magnitude of xy component of particle i's momentum, and $\gamma_{i,L}$ is the angle between $\vec{R}_i$ and $\vec{p}_{i,xy}$ measured positive in the counterclockwise direction.

Likewise for a system of particles, we find the net z component torque to be

$$\tau_{z,\mathrm{net}} = \sum_i \tau_{i,z} = \sum_i R_i F_{i,xy} \sin \gamma_{i,\tau} \ , \tag{39}$$

where $\tau_{i,z}$ is the z component of particle i's net external torque, $F_{i,xy}$ is the magnitude of xy component of the net external force on particle i and $\gamma_{i,\tau}$ is the angle between $\vec{R}_i$ and $\vec{F}_{i,xy}$ measured positive in the counterclockwise direction. Recall that the internal forces contribute nothing to the net torque with our assumptions (see § 4), and so we do not have to include them in the summation for net z component torque. The quantity $\tau_{z,\mathrm{net}}$ is both the net z component torque and the net external z component torque.

Now in general for a system of particles, we have equation (26) from § 4

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{\rm net} \tag{40}$$

which leads to the z component form

$$\frac{dL_z}{dt} = \tau_{z,\rm net} \ . \tag{41}$$

With the formulae developed above, the rotational Newton's 2nd law z component for a system of particles is

$$\frac{dL_z}{dt} = \tau_{z,\rm net} = \sum_i \tau_{i,z} = \sum_i R_i F_{i,xy} \sin\gamma_{i,\tau} \ , \tag{42}$$

Note it $\tau_{z,\rm net} = 0$, then

$$L_z = \text{a constant} \ : \tag{43}$$

i.e., there is conservation of the z angular momentum. Of course, we already knew $\tau_{z,\rm net} = 0$ implied conservation of z angular momentum from our results in § 4.

Using the axis forms for rotational dynamics that we have developed in this section is often convenient particularly when all the complete rotations occurs around a single fixed axis. A special case of such a system is rigid-body rotation around a single fixed axis, conventionally chosen to be the z axis just as we have done in this section. In this case at our level, one usually only needs to deal with the z components of angular momentum and torque. The other components are not zero in general (see § 5.1 below and Appendix B), but they usually don't need to be considered to describe the motion at our level.

5.1. Rigid Body Rotation and the Non-z Components: Reading Only

As we just said above, for rigid body rotation about fixed axis (the z axis), one doesn't usually need to deal with the non-z components at our level.

This point needs a bit of investigation.

The other components are not necessarily zero. We give a proof of this in the Appendix B, but that proof is a bit involved.

Here's another, simpler proof.

Consider a particle in uniform circular motion around the z axis, but on path above the xy plane. Recall there is a centripetal force since there is circular motion. Just by qualitative right-hand evaluation using displacement vectors from the origin and the momentum and force vectors, one knows that there are non-z components of angular momentum and torque.

Now what if our uniform circular path were in the xy plane. Now by qualitative right-hand evaluation, one finds that only the z component of angular momentum is non-zero and the torque is zero.

So you see the non-z components of angular momentum and torque vanish with particular choice of reference frame. This suggests one really doesn't need them to describe the motion for a point particle executing a rigid-body rotation about the z axis.

What of a finite rigid body rotating about the z axis?

Well no change of reference frame can make all the non-z components of angular momentum and torque vanish for every particle.

But if one only wants to understand the overall behavior of the rigid body, one usually doesn't need those components.

The overall behavior being pretty fully described by the rigid-body angular displacement, angular velocity, and angular acceleration about the z axis and by the z components of total angular momentum and net torque.

One also doesn't need to know the internal forces and torques if one is satisfied by saying the body is rigid.

But what if one is **NOT** satisfied that the body is rigid or **NOT** satisfied that the fixed axis it is rotating on is unbreakable?

One may want to know what the internal forces and torques are in order to know under what conditions the body distorts or falls apart (i.e., ceases and desists being a rigid body). One may also want to know about the non-z components of the angular momenta and torques both internal and external.

So for detailed knowledge of the structure of the body and the fixed axis, all the components of the internal and external forces and torques and all components the angular momenta of body parts and the whole body may be of interest—but not to us in the rest of this lecture—partially excepting for understanding the fun demos in § 5.2.

5.2. Conservation of Angular Momentum

Recall that if $\vec{\tau}_{\rm net} = 0$, then the total angular momentum $\vec{L}$ of a body is constant: i.e., is conserved.

For complex bodies, the angular momenta of body parts change in general even when the total angular momentum is constant.

Lets consider just the z component of angular momentum for simplicity.

Say $\tau_{z,\rm net}$ (the net z torque) is zero.

This means L_z is conserved.

But the $L_{i,z}$ of components can change.

5.2.1. Bicycle Wheel Demo

Just for illustration consider a standard demonstration setup.

I need a volunteer—someone with enormous strength.

A person sits on a rotating stool whose axle is frictionless.

To understand the demo, let's assume an ideal system where there is no air resistance and the axle of the rotating stool is frictionless. Gravity acts down recall. None of these forces has any component in the x-y plane, and so cannot exert any z torque on the system of stool from the axle up.

If the person stays out of contact with outside objects, the total z component of the angular momentum of the stool system is conserved since all external z component torques are zero.

Other components can change since gravity and the stool axle can exert torques in x-y directions.

Say the person is set spinning.

If stool system stays rigid, it spins perpetually at a constant rate ideally. It is a rigid rotator.

Now say the person is holding a spinning bicycle wheel—the person holds an axle of the wheel and the axle is frictionless for the wheel, but not the person's hands.

The stool system is now rather complex.

The stool system is rotating about the stool axle and the wheel is rotating about the wheel axle at a different rate in general.

Say the person rotates the wheel axle.

The person needs some effort to rotate the wheel axle—the person is exerting internal torques inside the stool system.

External x-y torques do act during the wheel axle rotation since x-y angular momentum comes into and goes out of existence during the wheel axle rotation.

But there can be no change in the total z angular momentum.

So any change in the wheel z angular momentum—say by flipping it right over—must change the stool-person z angular momentum.

Say the wheel and stool-person—which sounds awful—are both rotating counterclockwise as seen looking down from the positive z axis which points up. The right-hand rule for rotation direction (see the lecture *ROTATIONAL KINEMATICS*) tells us this direction is to be regarded as counterclockwise: put the thumb of right hand along the positive direction of the z axis and the fingers curl in the conventional counterclockwise direction.

If the person flips the wheel over, the wheel z angular momentum contribution has changed from positive to negative.

The stool-person must rotation speed up in the counterclockwise direction, to conserve the total z angular momentum of the stool system.

Note the person can't change the magnitude of the wheel angular momentum about its own axle.

The frictionless axle of the wheel doesn't permit that.

The person can just change the direction of the wheel's angular momentum about its own axle.

But since angular momentum is a vector, changing the wheel's orientation is changing its angular momentum viewing the wheel as its own system and its angular momentum contribution to the stool system.

As mentioned above, x-y angular momentum can be created inside the system. To see this cleanly, say the person holds the wheel with the axle horizontal with everything initially motionless. The person can start the wheel spinning with their hand and thus create x-y angular momentum. The stool does **NOT** start spinning note. If the person now rotates the wheel axle and gives it z angular momentum, the stool will start spinning in just such a way as to cancel that z angular momentum: z angular momentum **CANNOT** be created inside the system. External x-y torques through the stool axle act to make the creation of x-y angular momentum—the person creates the x-y angular momentum making use of those external torques. We know it's the stool axle that gives the external x-y torques since the person can do the same thing in a gravity-free environment. Gravity can create x-y torques too. Say the person balances a ruler on finger and then moves the ruler center of mass off of the finger. The gravitational torque then causes x-y angular momentum of the ruler and the whole stool system. The ruler falls down.

You should note pretty darn complex internal interactions happen in this demo: forces and reaction forces, torques and reaction torques. Nevertheless, we can at least partially understand the demo using conservation of angular momentum.

In Appendix C, we give an mathematical analysis of the bicycle wheel demo. This analysis makes use of developments in § 6 below.

5.2.2. Rotating Person and Weights Demo

I need another volunteer now—someone who does not know the meaning of the word fear.

A person sits on a rotating stool.

To understand the demo, we assume an ideal system: the stool axle is frictionless and there is no air resistance.

If the person stays out of contact with outside objects, the total stool z angular momentum is conserved since no external z torques can act on the person-stool system.

The x-y angular momentum can change since gravity and the stool axle can exert torques in directions other than the z direction.

We give the person two weights to hold in their hands with their arms stretched out.

Say the person is set spinning.

If person stays rigid, the stool system spins perpetually at a constant rate ideally. The system is a rigid rotator.

Now the person pulls their hands in close to their body—after he/she is warned to expect dizziness.

What happens?

The system angular velocity increases.

There is no outside z torque on the stool-person-weight system.

Thus, the system z angular momentum is conserved.

In order for this to be so, what is called the rotational inertia about the z axis must decrease.

Rotational inertia is introduced below in § 6 and is given the symbol I.

We will prove that

$$L_z = I\omega \tag{44}$$

for a rigid rotator rotating about a fixed z axis, where ω is the angular velocity of the rigid rotator.

The stool system is a rigid rotator in both the stretched-out case and the held-in case.

Given L_z constant:

Stretched-out case: I is high, ω low.

Held-in case: I is low, ω high.

The person changes the stool system rotational inertia by changes the mass distribution of the the stool system.

Figure skaters change their rotational inertia during spins all the time. They go into a spin with hands spread out and then pull their hands in to increase their angular velocity—it looks really cool—even though it's just intro physics.

So after class, anyone can come down and try out being Kristi Yamaguchi.

6. RIGID-BODY ROTATION AROUND A SINGLE FIXED AXIS

Rigid-body rotation about a single fixed axis is a very special case of rotational motion, but it is very important in nature (e.g., rotating planets to a high degree of approximation) and in technology (e.g., wheels and rotators of all kinds). Rigid-body rotation is also easy to analyze which makes it heuristically useful in an intro physics course.

For rigid-body rotation around a single fixed axis at our level, one usually only needs to concern oneself with angular momentum and torque components along that single fixed axis. The angular momentum and torque components along orthogonal axes are not in general zero (see § 5.1 and Appendix B), but they are not usually needed for an adequate account of the motion at our level.

Fig. 5.— Schematic diagram of a rigid body in rotation about a fixed axis in the z direction.

We choose the axis of rotation to be the z axis: this is the conventional choice. See Figure 5 for a schematic illustration of rigid-body rotation.

We will make use of the formalism developed in § 5.

Consider a rigid body made up of particles i. For a rigid body, the R_i are fixed and magnitude of the momentum component in the x-y plane is

$$p_{i,xy} = m_i v_i = m_i R_i |\omega| \ , \tag{45}$$

where m_i is particle i's mass, v_i is particle i's speed which is all tangential for rigid-body rotation, and ω is the common angular velocity of all particles. For rigid-body rotation around the z axis, there is no z-component of the momentum.

Now we consider ω positive for counterclockwise rotation looking down from the positive z axis and negative otherwise. The right-hand rule for rotation direction (see the lecture *ROTATIONAL KINEMATICS*) tells us this direction is to be regarded as counterclockwise: put the thumb of right hand along the positive direction of the z axis and the fingers curl in the conventional counterclockwise direction.

Now using equation (36) (§ 5), the z-component of angular momentum for particle i is

$$L_{i,z} = R_i p_{i,xy} \sin\gamma_{i,L} = m_i R_i^2 \omega \ , \tag{46}$$

where $\sin\gamma_{L,i}$ can only be 1 for positive rotation and -1 for negative rotation: the sign of $\sin\gamma_{i,L}$ is used to eliminate the absolute value signs that appear around ω in equation (45).

The total z component angular momentum for the rigid body is

$$L_z = \sum_i m_i R_i^2 \omega = \left(\sum_i m_i R_i^2 \right) \omega \ . \tag{47}$$

We note again that ω is common to all particles, and thus we can take it out of the summation.

We now define the quantity **ROTATIONAL INERTIA** by

$$I = \sum_i m_i R_i^2 \ , \tag{48}$$

where I is the conventional symbol for rotational inertia. In older physics jargon, rotational inertia is called the moment of inertia.

The rotational inertia is the rotational analog of mass: just like z angular momentum and z torque, rotational inertia is an axis-dependent quantity and a position-weighted quantity.

As I does **NOT** need to be constant. It can vary if R_i vary and if the system changes mass.

But for a rigid-body, I is a constant and we have

$$L_z = I\omega \ . \tag{49}$$

We now note that

$$\frac{dL_z}{dt} = I\frac{d\omega}{dt} = I\alpha \ , \tag{50}$$

where $\alpha = d\omega/dt$ is the angular acceleration that we introduced in the lecture *ROTATIONAL KINEMATICS*. Using this last result and equation (42) (from § 5)

$$\frac{dL_z}{dt} = \tau_{z,\text{net}} = \sum_i \tau_{i,z} = \sum_i R_i F_{i,xy} \sin\gamma_{i,\tau} \ , \tag{51}$$

we now obtain

$$\tau_{z,\text{net}} = I\alpha \tag{52}$$

which is the special case rotational 2nd law for rigid-body rotation around a single fixed axis. Remember $\tau_{z,\text{net}}$ is the net external z torque (and also the net z torque too). Although a very special case, $\tau_{z,\text{net}} = I\alpha$ is very useful for rigid-body rotation. One can drop the subscript z if one knows what one means.

Note that we have

$$\tau_{z,\text{net}} = I\alpha = I\frac{d\omega}{dt} = \frac{dL_z}{dt} \tag{53}$$

which implies that if $\tau_{z,\text{net}} = 0$, then ω is constant for a rigid rotator as well as L_z being constant.

One can have a system changing between different rigid-rotator states as in § 5.2.2 where we had a person with weights on a spinning stool.

Now $L_z = I\omega$ for all the rigid-rotator states and L_z is constant at all times even during non-rigid rotator phases if $\tau_{z,\text{net}} = 0$ as we know from §§ 4 and 5.

Thus, one gets the situation of § 5.2.2 of the rotating person with weights.

Given L_z constant both in the rigid rotator states and in transitions between them, then for the rigid rotator states:

For I high, one has ω low.

For I low, one has ω high.

7. THE GRAVITATIONAL TORQUE ON A SYSTEM OF PARTICLES

Consider a system of particles and an arbitrary origin.

What is the net gravitational torque on the system?

We are considering only the ideal near Earth's surface case, where the gravitational field is constant with value $-g\hat{y}$, where g is the magnitude of the gravitational field or the acceleration due to gravity (with fiducial value $9.8\,\text{m/s}^2$) and $\hat{y}$ is a unit vector in the upward direction.

The torque about the origin is

$$\vec{\tau} = \sum_i \vec{r}_i \times (-m_i g \hat{y}) = \left(\sum_i m_i \vec{r}_i \right) \times (-g\hat{y}) = \vec{r} \times (-mg\hat{y}) \ , \tag{54}$$

where $\vec{r}_i$ is the 3-dimensional position vector of particle i from the origin, m_i is the mass of the particle i, m is the total mass of the system, and $\vec{r}$ is the 3-dimensional center-of-mass position vector from the origin.

To put it clearly, the gravitational torque formula is

$$\vec{\tau} = \vec{r} \times (-mg\hat{y}) \ . \tag{55}$$

Because of the peculiar nature of the gravitational field (i.e., it is constant in space and homogeneously linearly dependent on mass), it is as if all the mass were concentrated at the center of mass for the calculation of gravitational torque.

A consequence of equation (55) is that gravity can exert no torque about the center of mass and no torque about a fixed axis that runs through the center of mass (i.e., no torque component along an axis through the center of mass). This is because $\vec{r} = 0$ if the origin is the center of mass.

To help understand this result, imagine a symmetric wheel on axle at its center. The center is the center of mass by symmetry. The wheel will not rotate under gravity. There is no gravitational torque about the axis that the axle centers on. In this case, it is easy to see that all the forces/torques trying to rotate the wheel cancel by symmetry. Our formula for the gravitational torque equation (55) shows that an object doesn't have to be symmetrical. The gravitational torque about the center of mass of any object is zero.

Now I know what you are thinking.

There is another common force that is constant in space and is homogeneously linearly dependent on mass: the rectilinear inertial force. If you are in a non-inertial frame with acceleration $\vec{a}_{\rm in}$, then the inertial force in this frame is

$$\vec{F}_{\rm in} = -m\vec{a}_{\rm in} = -ma_{\rm in}\hat{a}_{\rm in} \ , \tag{56}$$

where $\hat{a}_{\rm in}$ is a unit vector that points in the direction of acceleration $\vec{a}_{\rm in}$.

Recall we consider inertial forces in lecture *NEWTONIAN PHYSICS II*. Newton's laws of motion apply relative to non-inertial frames if one introduces inertial forces.

What is the net inertial force torque on a system in the non-inertial frame?

The torque about a general origin fixed in the non-inertial frame is

$$\vec{\tau} = \sum_i \vec{r}_i \times (-m_i a_{\rm in}\hat{a}_{\rm in}) = \left(\sum_i m_i\vec{r}_i\right) \times (-a_{\rm in}\hat{a}_{\rm in}) = \vec{r} \times (-ma_{\rm in}\hat{a}_{\rm in}) \ . \tag{57}$$

Not surprisingly, we find that net inertial force torque on the system can be calculated as if all the mass were concentrated as the center of mass just as for gravity.

A consequence of equation (57) is that an inertial force for a non-inertial frame in rectilinear motion can exert no torque about the center of mass and no torque about a fixed axis that runs through the center of mass. This is because $\vec{r} = 0$ in equation (57) if the origin is the center of mass.

This zero-torque consequence may be seem a bit arcane, but actually its quite important in understanding rollers in § 12. We want to consider them when their centers of mass are accelerating. Thus, their rotation about a fixed axis passing through their centers of mass is rotation in a non-inertial frame with an inertial force of the type given by equation (56). If one wants to apply Newton's laws of motion in non-inertial frames, then in general one has use inertial forces. But for understanding the rotation of the roller in its center-of-mass non-inertial frame, the inertial force does nothing since it exerts no torque about the axis of rotation.

The zero-torque consequence of equation (57) is fine point that yours truly only realized in the 2009nov29 revision of this lecture. No intro textbook points this point out as far as yours truly knows.

7.1. The Physical Pendulum: Reading Only?

A physical pendulum is any rigid body that is allowed to rotate freely about some fixed pivot point.

To analyze the physical pendulum, we put an origin on pivot point and the z axis in the **HORIZONTAL** direction. The x and y axes have their usual right-hand coordinate system orientations relative to the z axis. The x is in the horizontal direction too and the y axis is the vertical direction. The positive y direction is upward and the negative y direction is downward. We put the orient the z axis so that the center of mass of the pendulum is in the x-y plane. The arrangement is such that the gravitational torque will be along the z axis as one can see from equation ?????? Gravity will not act to displace the origin from from the x-y plane as one can see from equation (55) since gravity only exerts as z torque. So as long as we don't displace the origin from the x-y plane, the origin will stay in the x-y plane and one will only get rotations about the z axis. For the analysis of the physical pendulum we only impose angular displacements inside the x-y plane.

We consider the ideal case where pivot point is frictionless. The pivot point holds the body up, but since it is frictionless it can only exert radial forces relative to the origin. Thus, it can exert no torques on the pendulum about the origin.

We could, of course, change the pendulum system by replacing the free pivot point by a frictionless, rigid axis that constrains the rotation to be about the axis direction. The analysis is essentially the same in the following—which should be clear enough.

The cylindrical azimuthal angular coordinate θ is measured counterclockwise from the downward vertical (i.e., the negative y direction) as viewed from the positive z axis. This zero-point for θ is convenient in for a pendulum.

We use θ to locate the center of mass in the angular direction. Let the displacement of the center of mass from the origin be $\vec{R}$. Now the gravitational z component torque (which is the only non-zero component of the gravitational torque since the center of mass is at the x and y origins) is

$$\tau_z = (\vec{R}\times\vec{F}_{xy})\cdot\hat{z} = [\vec{R}\times(-mg\hat{y})]\cdot\hat{z} = mg[R\times(-\hat{y})]\cdot\hat{z} = mg[R\sin\theta(-\hat{z})]\cdot\hat{z} = -Rmg\sin\theta \ , \tag{58}$$

where we've used equation (34) from § 5 and used a dot product to isolate the z component of the torque. The gravitational torque is negative for $\theta > 0$, positive for $\theta < 0$, and zero for $\theta = 0$ and $\theta = \pm\pi$.

We now assume that the gravitational torque is the only external torque.

For our pendulum, equation (52) $\tau_{z,\rm net} = I\alpha$ specializes to

$$-Rmg\sin\theta = I\alpha \qquad \text{or} \qquad I\frac{d^2\theta}{dt^2} = -Rmg\sin\theta \ . \tag{59}$$

Equation (59) is the equation of motion for the physical pendulum.

Fig. 6.— Schematic diagram of a physical pendulum.

There are three simple static equilibrium solutions for equation (59): i.e., three simple static equilibrium solutions for the motion of the pendulum. There are less simple non-static ones too: see below.

In the first solution, the pendulum starts at rest with $\theta = 0$ (i.e., starts with the center of mass directly below the pivot point). In this case, one has stable static equilibrium. The equilibrium is static since it starts at rest and the torque is zero. The equilibrium is stable since any perturbation from $\theta = 0$ causes a restoring torque that tries to return the pendulum to the $\theta = 0$ position. Frictional torques (which we've not included in eq. (59)) will damp out any oscillation and restore static equilibrium.

The first solution has a special use. Take any rigid object and hang it from two different free pivot points—at different times, of course—and allow the object to come to rest. In stable static equilibrium, the center of mass is directly below the pivot point. Thus, lines from the free pivot points downward will intersect at the center of mass. Using the free hanging procedure allows one to empirically identify the center of mass of the object. Finding the center of mass by this empirical procedure is often a lot easier than trying to calculate the center of mass from the specifications of the object which one often doesn't know anyway.

In the second solution, the pendulum starts at rest with $\theta = \pi$ (i.e., starts with the center of mass directly above the z axis). In this case, one has unstable static equilibrium. The equilibrium is static since it starts at rest and the torque is zero. The equilibrium is unstable since any perturbation from $\theta = \pi$ causes a torque that tries to move the pendulum farther from the equilibrium point.

In the third solution, the center of mass is at the origin. In this case the torque is zero for all θ. This situation is neutral equilibrium. If the pendulum is at rest in any orientation, it will just stay at rest. If the pendulum has non-zero angular momentum about z axis, the zero torque condition implies the angular momentum will be conserved. (Resistive torques at the z axis would cause a dissipation of energy to waste heat.) Since the pendulum is a rigid body, this means that the pendulum will rotate with constant angular velocity ω about the z axis.

The third solution brings out a significant point. The gravitational torque about the center of mass is always zero. This result is often useful in the analysis of rollers (balls, cylinder, wheels, etc.) since their centers of mass are usually on their axes of rotation.

Non-static solutions for the pendulum are obtained if one has a initial angular momentum about the z axis. For a large enough angular momentum, one would get a rotation with an non-constant angular velocity.

For a small enough angular momentum, one would get an oscillation.

If we make the small angle approximation $\sin\theta \approx \theta$ (where θ is in radians), then equation (59) becomes

$$I\frac{d^2\theta}{dt^2} = -Rmg\theta \tag{60}$$

which is just the simple harmonic oscillator differential equation discussed in the lecture *NEWTONIAN PHYSICS II*.

The oscillation (which is simple harmonic motion) recall is sinusoidal and the angular frequency, frequency, and period of the oscillation are independent of the amplitude of the oscillation. From solution in the lecture *NEWTONIAN PHYSICS II*, *mutatis mutandis*, we know that

$$\omega = 2\pi f = \frac{2\pi}{P} = \sqrt{\frac{Rmg}{I}}\ , \qquad f = \frac{1}{P} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{Rmg}{I}}\ , \qquad P = \frac{1}{f} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{Rmg}}\ . \tag{61}$$

These results are for the physical pendulum with the small angle approximation.

In the limit, that the body shrinks to a point particle the physical pendulum reduces to the simple pendulum and I goes to mr^2. In this case we get

$$\omega = 2\pi f = \frac{2\pi}{P} = \sqrt{\frac{g}{R}} \ , \qquad f = \frac{1}{P} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{R}} \ , \qquad P = \frac{1}{f} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R}{g}} \tag{62}$$

which are just the results for the simple pendulum with small angle approximation that we obtained in the lecture *NEWTONIAN PHYSICS II*.

8. ROTATIONAL INERTIA FORMULAE

In general the rotational inertia formulae for objects will be complex.

But for rigid objects of high symmetry and usually uniform density, simple standard formulae can be found. These formulae are summarized in Table 2 and Figure 7.

Recall that in general rotational inertia (or moment of inertia) is defined by

$$I = \sum_i m_i R_i^2 \tag{63}$$

(§ 6, eq. (48)) which for a continuous mass distribution goes over to the integral form

$$I = \int R^2 \rho \, dV \ , \tag{64}$$

where r is the cylindrical coordinates radius measured from the axis of rotation, ρ is the density, dV is differential volume, and the integral is over the whole volume of the object.

Note that rotational inertia tends to increase if mass is moved to larger radii and to decrease if it is moved to smaller radii. If the radii all go to zero (which is an ideal limit), rotational inertia vanishes.

The MKS unit of rotational inertia is given by

$$\text{unit}[I] = \text{kg}\,\text{m}^2 \ . \tag{65}$$

There is no special name or symbol for the unit of rotational inertia: the standard unit is just the $\mathrm{kg\,m^2}$. In macroscopic physics (except in astronomy), one almost always uses MKS units for rotational inertia calculations, and so no other unit of rotational inertia is in wide use (except in astronomy). For reporting final rotational values, one should write down the $\mathrm{kg\,m^2}$ unit explicitly.

We will derive some example rotational inertia formulae for rigid objects of high symmetry. Except for the thin ring and thin cylindrical shell, the objects have uniform density.

Fig. 7.— Schematic diagram of standard rigid bodies of uniform density for which standard rotational inertia formulae are given.

Table 2. Rotational Inertia Formulae for Constant-Density Objects of Standard Shape

Object	Formula
Thin ring or thin cylindrical shell	$I = MR^2$
Disk or Cylinder with a Hollow	$I = (1/2)M(R_2^2 + R_1^2)$
Solid disk or cylinder	$I = (1/2)MR^2$
Solid sphere	$I = (2/5)Mr^2$
Thin spherical shell	$I = (2/3)Mr^2$
Thin rod with axis through center of mass	$I = (1/12)ML^2$
Thin rod with axis through an end	$I = (1/3)ML^2$

Note. — The symbols are I for rotational inertia, M for mass, R for cylindrical coordinates radius R_1 for inner cylindrical coordinates radius, R_2 for outer cylindrical coordinates radius, and L for length. The axis of rotation is through the symmetry of axis of the object except for the last two cases. In both the last two cases the axis is perpendicular to the rod.

8.1. Thin Ring and Thin Cylindrical Shell

A thin ring in the present context is one where the ring has no thickness.

Obviously, it is an limiting form for actual finite-thickness rings.

We want the rotational inertia for the symmetry axis.

In this case, for a ring of radius R

$$I = \int R'^2 \rho \, dV' = R^2 \int \rho \, dV' = MR^2 \ , \tag{66}$$

where M is the mass of the ring.

For a thin ring, we take the limit of the density (volume density) going to infinity and volume going to zero in such a way that mass is fixed to M.

Actually, the above derivation did not assume uniform density, and so the rotational inertia result is valid for a thin ring with non-uniform linear density too. Linear density is the density per unit length.

So for the thin ring, the rotational inertia formula is

$$I = MR^2 \ . \tag{67}$$

Since a thin cylindrical shell is just a stack of thin rings, equation (67) applies to thin cylindrical shells too. The radius R is the same for all the rings in the stack, and so one just adds the masses of the rings to get the total mass which is labeled M too.

The thin cylindrical shell rotational inertia result does not require uniform density.

Say we have a ring of radius $R = 1\,\mathrm{m}$ and mass $M = 1\,\mathrm{kg}$, then

$$I = MR^2 = 1\,\mathrm{kg\,m^2} \ . \tag{68}$$

If we doubled the radius, $I = 4\,\mathrm{kg\,m^2}$.

8.2. Thin Disk and Cylinder

A thin disk means it has no thickness.

We want the rotational inertia for the symmetry axis.

We'll consider the case with a central hollow first.

The area density (mass per unit area) is constant and given by

$$\sigma = \frac{M}{\pi(R_2^2 - R_1^2)} \ , \tag{69}$$

where σ is the usual symbol for area density, M is the disk mass, R_1 is the inner disk radius, and R_2 is the outer disk radius.

To convert from a volume integral to an area integral, we note that

$$dm = \rho \, dV = \rho \, dz \, dA = \frac{\sigma}{dz} \, dz \, dA = \sigma \, dA \tag{70}$$

where we have set $\rho = \sigma/dz$ and where we let the disk have differential thickness dz which is then sent to zero without error after it has been eliminated from the expression for dm.

Converting from a volume integral to an area integral, we find the rotational inertia thusly

$$\begin{aligned} I &= \int R^2 \rho \, dV = \int R^2 \sigma \, dA \\ &= \frac{M}{\pi(R_2^2 - R_1^2)} \int_{R_1}^{R_2} R^2 (2\pi R) \, dr \ . \end{aligned} \tag{71}$$

Evaluate the integral and simplify as much as possible remembering the difference of squares result

$$a^2 - b^2 = (a+b)(a-b) \ . \tag{72}$$

You have 1 minute working individually or in groups. Go.

Behold:

$$\begin{aligned} I &= \frac{M}{\pi(R_2^2 - R_1^2)} \int_{R_1}^{R_2} R^2 (2\pi R)\, dr \\ &= \frac{M}{(R_2^2 - R_1^2)} \frac{R^4}{2}\bigg|_{R_1}^{R_2} \\ &= \frac{M}{2(R_2^2 - R_1^2)} (R_2^4 - R_1^4) \\ &= \frac{1}{2} M (R_2^2 + R_1^2) \ , \end{aligned} \tag{73}$$

where we have used the difference of squares result

$$R_2^4 - R_1^4 = (R_2^2 - R_1^2)(R_2^2 + R_1^2) \ . \tag{74}$$

Thus, the disk formula is

$$I = \frac{1}{2} M (R_2^2 + R_1^2) \ . \tag{75}$$

Since a cylinder is just a stack of disks, equation (75) applies to uniform cylinders with hollows too. The radii R is the same for all the disks in the stack, and so one just adds the masses of the disks to get the total mass which is labeled M too.

If the disk or cylinder has no central hollow, $R_1 \to 0$ and one has

$$I = \frac{1}{2} M R^2 \ , \tag{76}$$

where R without a subscript is the radius.

8.2.1. Another Approach: Reading Only

The formula for a disk with a hollow can be found by another approach using the formula for solid disk.

Say we have solid disk 1 with mass M_1 and radius R_1 and solid disk 2 with mass M_2 and radius R_2 (which is greater than R_1). Both disks have the same constant area density σ.

Now say we have a disk with a hollow of the same σ with inner radius R_1 and outer radius R_2. From the original discrete-particle formula for rotational inertia

$$I = \sum_i m_i R_i^2 \ , \tag{77}$$

it follows that the rotational inertia of the disk with a hollow must be

$$\begin{aligned} I &= \left(\sum_i m_i R_i^2\right)_{\text{hollow}} = \left(\sum_i m_i R_i^2\right)_2 - \left(\sum_i m_i R_i^2\right)_1 \\ &= I_2 - I_1 = \frac{1}{2}M_2 R_2^2 - \frac{1}{2}M_1 R_1^2 \\ &= \frac{1}{2}\sigma\pi\left(R_2^4 - R_1^4\right) \\ &= \frac{1}{2}\sigma\pi\left(R_2^2 - R_1^2\right)\left(R_2^2 + R_1^2\right) \\ &= \frac{1}{2}M\left(R_2^2 + R_1^2\right) \ . \end{aligned} \tag{78}$$

In the first line of the calculation, we added the smaller disk terms to the summation and subtracted them off by using the rotational inertias for the two unhollowed disks. In the fourth line, we have used the difference of squares. The quantity $M = \sigma\pi\left(R_2^2 - R_1^2\right)$ is the mass of the disk with a hollow.

The final result is for the disk with a hollow is just what we got before. The result also applies to a cylinder with a hollow as well, of course.

8.3. Sphere: Reading Only?

Consider a sphere of uniform density without a hollow.

We want the rotational inertia for the symmetry axis which in this case is any axis through the center.

The sphere density is

$$\rho = \frac{M}{(4\pi/3)r^3} \ , \tag{79}$$

where M is the sphere mass and r is the sphere radius.

The sphere is just a stack of disks of varying radius. The radius of one of the disks is $\sqrt{r^2 - z^2}$ where z is the coordinate of the axis of rotation with the origin at the sphere center. The differential rotational inertia for one of the disks is

$$dI = \frac{1}{2}(r^2 - z^2)\rho\, dV \ , \tag{80}$$

where $\rho\, dV$ is the disk mass and the differential volume is

$$dV = \pi(r^2 - z^2)\, dz \ . \tag{81}$$

Thus, for the total rotational inertia we have

$$\begin{aligned} I &= \int R^2 \rho\, dV = \rho \int_{-r}^{r} \frac{\pi}{2}(r^2 - z^2)^2\, dz \\ &= \frac{M}{(4\pi/3)r^3} \int_{-r}^{r} \frac{\pi}{2}(r^2 - z^2)^2\, dz \\ &= \frac{M}{(8/3)r^3} \int_{-r}^{r} (r^2 - z^2)^2\, dz \\ &= \frac{M}{(4/3)r^3} \int_{0}^{r} (r^4 - 2r^2 z^2 + z^4)\, dz \\ &= \frac{M}{(4/3)r^3} \left(r^4 z - \frac{2}{3} r^2 z^3 + \frac{1}{5} z^5 \right) \Bigg|_0^r \\ &= \frac{M}{(4/3)r^3} \left(r^5 - \frac{2}{3} r^5 + \frac{1}{5} r^5 \right) \\ &= \frac{M}{(4/3)r^3} \left(\frac{8}{15} r^5 \right) \\ &= \frac{2}{5} M r^2 \ . \end{aligned} \tag{82}$$

So the rotational inertial formula for the uniform density solid sphere is

$$I = \frac{2}{5} M r^2 \ . \tag{83}$$

8.3.1. Sphere with a Spherical Hollow

What of a sphere with a spherical hollow entirely contained in the sphere. We evaluate the rotational inertia about the symmetry axis of the system. The hollow need not be in the centered on the center of the sphere.

The density of the sphere is

$$\rho = \frac{M}{(4\pi/3)(r_2^3 - r_1^3)} \ , \tag{84}$$

where M is the sphere's mass, r_2 is the sphere radius, and r_1 is the hollow radius.

Now consider two imaginary uniform-density solid spheres: one of radius r_2 and mass $(4\pi/3)\rho r_2^3$ and one of radius r_1 and mass $(4\pi/3)\rho r_1^3$.

Substituting for density, the imaginary sphere masses are found to be

$$M\left(\frac{r_2^3}{r_2^3 - r_1^3}\right) \qquad \text{and} \qquad M\left(\frac{r_1^3}{r_2^3 - r_1^3}\right) \ . \tag{85}$$

If one subtracts the rotational inertial of the smaller imaginary sphere from that of the larger imaginary sphere, one gets the rotational inertia of the actual sphere. This rotational inertia is

$$I = \frac{2}{5}M\left(\frac{r_2^5 - r_1^5}{r_2^3 - r_1^3}\right) \tag{86}$$

which seems to have no obvious simplifications.

What about an infinitely thin spherical shell? The hollow must be central in this case.

Let $r = r_1$ and $r_2 = r + \Delta r$, where $\Delta r = r_2 - r_1$.

Using the binomial theorem, we find that

$$(r+\Delta r)^5 = r^5 + 5r^4\Delta r + \ldots \qquad \text{and} \qquad (r+\Delta r)^3 = r^3 + 3r^2\Delta r + \ldots \ , \tag{87}$$

where the ellipses replace the higher order terms in Δr: i.e., terms of Δr^2 up to Δr^5 for the first equation and up to Δr^3 for the second. Now we find that for the sphere with a hollow that

$$I = \frac{2}{5} M \left(\frac{5r^4 \Delta r + \ldots}{3r^2 \Delta r + \ldots} \right) . \tag{88}$$

In the limit that $\Delta r \to 0$, one finds that

$$I = \frac{2}{3} M r^2 . \tag{89}$$

This last result applies approximately to thin spherical shells. It is the ideal limiting case when the shell has zero thickness.

What is the qualitative explanation for the different coefficients? The uniform density solid sphere has 2/5 and the thin spherical shell has 2/3. Well rotational for a fixed mass tends to increase as mass is moved farther from the axis as the general formula for rotational inertia equation (63) shows. A thin spherical shell has more mass at large radii than a uniform density solid sphere of the same mass. Thus, the thin spherical shell should have a larger rotational inertia for the same mass. And that is the qualitative explanation for the different coefficients.

8.4. Thin Rod

A thin rod in this context is one of infinite thinness.

In the case of a infinitely thin linear mass distribution, the integral limit of the discrete rotational inertia formula

$$I = \sum_i m_i R_i^2 \tag{90}$$

is

$$I = \int x^2 \lambda \, dx , \tag{91}$$

where x is the coordinate along the mass distribution and λ is the mass per unit length or linear density.

What is our thin rod's rotational inertia for an axis through its center of mass and perpendicular the rod?

Say the rod has length L and mass M. In this case $\lambda = M/L$. Now

$$I = \int x^2 \lambda dx = \frac{M}{L} \int_{-L/2}^{L/2} x^2 \, dx = 2\frac{M}{L} \int_0^{L/2} x^2 \, dx = 2\frac{M}{L} \frac{x^3}{3} \bigg|_0^{L/2} = \frac{1}{12} M L^2 \ . \tag{92}$$

9. ROTATIONAL INERTIA THEOREMS

There are several well known theorems that can be used to help evaluate rotational inertia in some cases. We proof them below.

9.1. The Parallel-Axis Theorem

Say $I_{\rm cm}$ is the rotational inertia of an object for an axis through its center of mass.

The object's rotational inertia for an axis parallel to the center-of-mass axis and a distance R away from the center-of-mass axis is

$$I = I_{\rm cm} + M R^2 \ , \tag{93}$$

where M is the object's mass.

The proof is simple.

Let x and y be the planar coordinates measured from an origin on the parallel axis. The location of the center-of-mass axis in the planar coordinate system is (X, Y). Note

$$R^2 = X^2 + Y^2 \ . \tag{94}$$

Let the area density of object be σ. Consider the mass per unit area in the x-y plane. We label this area density σ. The quantity σ is implicitly a function of x and y.

The rotational inertia about the parallel axis is

$$\begin{aligned}
I &= \int R'^2 \rho \, dV' = \int (x^2 + y^2) \sigma \, dA \\
&= \int \left[(x - X + X)^2 + (y - Y + Y)^2 \right] \sigma \, dA \\
&= \int \left[(x - X)^2 + (y - Y)^2 \right] \sigma \, dA + 2 \int \left[(x - X)X + (y - Y)Y \right] \sigma \, dA \\
&\qquad + \int \left(X^2 + Y^2 \right) \sigma \, dA \\
&= I_{\rm cm} + 0 + MR^2 \\
&= I_{\rm cm} + MR^2 \ , \qquad (95)
\end{aligned}$$

where dA is the differential area perpendicular to the z axis, the integral is over all area, $R = \sqrt{X^2 + Y^2}$, and

$$\int \left[(x - X)X + (y - Y)Y \right] \sigma \, dA = 0 \qquad (96)$$

since

$$\frac{\int (x - X) \sigma \, dA}{M} \qquad \text{and} \qquad \frac{\int (y - X) \sigma \, dA}{M} \qquad (97)$$

are the center-of-mass x and y locations in the coordinate system where the center of mass is at the origin.

Fig. 8.— Schematic diagram of a parallel-axis-theorem case.

The proof is complete.

Note that

$$I \geq I_{\rm cm} \ , \tag{98}$$

where the equality only holds for $R = 0$ and/or $M = 0$.

9.1.1. Parallel Axis Theorem Applied to a Thin Rod

A thin rod (i.e., a rod of zero thickness) of mass M and length L has the rotational inertia formula

$$I_{\rm cm} = \frac{1}{12} M L^2 \tag{99}$$

for rotation about an axis perpendicular to the rod and passing through its center of mass.

What is the rotational inertia for an axis perpendicular to the rod and passing through one end?

You have 30 seconds working individually. Go.

Behold:

$$I = I_{\rm cm} + MR^2 = \frac{1}{12} M L^2 + M \left(\frac{L}{2} \right)^2 = \frac{4}{12} M L^2 = \frac{1}{3} M L^2 \ , \tag{100}$$

where in this case $R = L/2$.

9.2. The Perpendicular-Axis Theorem: Reading Only

If there is a parallel-axis theorem, there ought to be a perpendicular-axis theorem.

There is—but it's not exactly analogous.

For a planar object of infinite thinness, the rotational inertia through any z axis perpendicular to the plane of the object is

$$I_z = \int R^2 \sigma \, dA \ , \tag{101}$$

where R is the cylindrical coordinates radius, σ is the area density, and the integral is over the whole area of the object.

We assume the x and y axes are in the plane of the object. For the special case of our planar object, the x coordinate is the cylindrical coordinate for the x axis and the y coordinate is the cylindrical coordinate for the y axis.

Now

$$R^2 = x^2 + y^2 \ , \tag{102}$$

and thus

$$I_z = \int R^2 \sigma \, dA = \int (x^2 + y^2) \sigma \, dA = \int x^2 \sigma \, dA + \int y^2 \sigma \, dA = I_x + I_y \ , \tag{103}$$

where I_x is the rotational inertia about the x axis and I_y is the rotational inertia about the y axis.

The proof is complete since the perpendicular-axis theorem is

$$I_z = I_x + I_y \ . \tag{104}$$

9.3. Scaling Rotational Inertia: Reading Only

Note that if we just scale up any object keeping its density distribution constant in scale-free coordinates, then I increases as the 5th power of the scale factor.

To see this, let

$$\vec{\zeta} = f \vec{\zeta}_0 \ , \tag{105}$$

where $\vec{\zeta}$ is the planar vector coordinate, $\vec{\zeta}_0$ is the scale-free planar vector coordinate, the 0 subscript denotes scale-free quantities, and f is the scale factor. We are already using r and R for other things, and so have conscripted ζ for nonce for the planar vector coordinate symbol.

We assume that the density distribution (not density note) is constant in the scale-free coordinates: thus,

$$\rho(\vec{\zeta}, f) = \rho(f\vec{\zeta}_0, f) = \rho(\vec{\zeta}_0, f = 1) \ . \tag{106}$$

Now

$$I = \int \zeta^2 \rho(\vec{\zeta}, f)\, dV = f^5 \int \zeta_0^2 \rho(\vec{\zeta}_0, f = 1)\, dV_0 = f^5 I_0 \ , \tag{107}$$

where the integration is over the whole object and I_0 is the unscaled rotational inertia.

The scaling result most obviously applies to uniform-density objects.

For example, if one had a uniform-density solid cylinder. The rotational inertia formula is

$$I = \frac{1}{2} M R^2 \ , \tag{108}$$

M is total mass and R is cylinder radius. If one scaled the cylinder up by f, the mass would increase by f^3 since it increases like volume and the radius would scale up like f. Thus, I would scale up by f^5 as our scaling result predicts.

10. ROTATIONAL WORK, THE ROTATIONAL WORK-KINETIC-ENERGY THEOREM, ROTATIONAL KINETIC ENERGY

In this section, we consider the work done by forces on a rigid body that rotates around a single fixed axis which as usual is the z axis by convention.

The fundamental differential expression for work done by a force $\vec{F}$ when a particle is displaced $d\vec{s}$ is

$$dW = \vec{F} \cdot d\vec{s} \ . \tag{109}$$

For a rigid-body rotation, we find the work done to be

$$dW = \sum_i \vec{F}_i \cdot d\vec{s}_i = \sum_i \vec{F}_i \cdot R_i \hat{\theta}_i \, d\theta \ , \tag{110}$$

where $\vec{F}_i$ is the force on particle i (but it is **NOT** necessarily the total force or total external force), R_i is the cylindrical coordinates radius for particle i, θ is the polar coordinate and is measured to some arbitrary point on the rigid body from the x axis with the positive direction being counterclockwise as viewed from the positive z axis, $d\theta$ is the differential polar coordinate which is common to all particles since the body is rigid, and $\hat{\theta}_i$ the polar coordinate unit vector for particle i.

We can now decompose $\vec{F}_i$ into xy and z components thusly

$$\vec{F}_i = \vec{F}_{i,xy} + \vec{F}_{i,z} \ . \tag{111}$$

Using the fact that $\vec{F}_{i,z}$ is perpendicular to $\hat{\theta}$ and recalling our definition of angle $\gamma_{i,\tau}$ in § 5, we find that

$$\vec{F}_i \cdot \hat{\theta} = \vec{F}_{i,xy} \cdot \hat{\theta} + 0 = F_{i,xy} \cos \xi_i = F_{i,xy} \cos\left(\gamma_{i,\tau} - \frac{\pi}{2}\right) = F_{i,xy} \sin \gamma_{i,\tau} \ , \tag{112}$$

where ξ_i is the angle of force $F_{i,xy}$ measured from $\hat{\theta}$ with the positive direction being counterclockwise as viewed from the positive z axis.

Note that $F_i \cdot \hat{\theta}$, $\vec{F}_{i,xy} \cdot \hat{\theta}$, $F_{i,xy} \cos \xi_i$, and $F_{i,xy} \sin \gamma_{i,\tau}$ are all the same thing: the component of force $\vec{F}_i$ along the $\hat{\theta}$ direction (i.e, the direction in which the particle i is rotating).

sing this last result, we obtain for a rigid body

$$dW = \sum_i R_i F_{i,xy} \sin \gamma_{i,\tau} \, d\theta = \sum_i \tau_{i,z} \, d\theta = \tau_z \, d\theta \tag{113}$$

or in the indefinite integral version

$$W = \int \sum_i r_i F_{i,xy} \sin\gamma_{i,\tau}\, d\theta = \int \sum_i \tau_{i,z}\, d\theta = \int \tau_z\, d\theta \ , \tag{114}$$

where τ_z is a general sum of z component torques and not necessarily the net z component torque (which we designate $\tau_{z,\mathrm{net}}$) although it could be that.

For a rigid body in rotation about a single fixed axis, equation (114) is general.

But note it does not include any non-rotational displacement work. Such work is simple to handle in simple cases and not simple to handle in cases that are not simple. We consider that non-rotational displacement work in § 11.

What is the power expended by a general sum of z component torques? Using equation (113), we find

$$P = \frac{dW}{dt} = \tau_z \frac{d\theta}{dt} = \tau_z \omega \ . \tag{115}$$

Below we consider two important special cases of equation (114): 1) where τ_z is the sum of internal z torques and 2) where τ_z is the sum of external z torques (which is also the sum of all torques assuming the stron version of Newton's 3rd law).

10.1. The Net Work by Internal Torques

In this subsection, we prove that net work done by internal torques rotating about a fixed axis is zero.

We showed in § 4 that the sum of internal torques of a system of particles is zero—assuming the strong version of Newton's 3rd law (e.g., Goldstein et al. 2002, p. 7). This was a vector result, and so applies to each component, and thus the sum of internal torque z

components is

$$\sum_i \tau_{i,z} = \tau_z = 0 \ . \tag{116}$$

Now we apply equation (113) (§ 10) to the net internal torque z components. We find that

$$dW = \sum_i R_i F_{i,xy} \sin\gamma_{i,\tau}\, d\theta = \int \sum_i \tau_{i,z}\, d\theta = \int \tau_z\, d\theta = 0 \ . \tag{117}$$

The last equation shows that the internal torques in a rigid body do **ZERO** net work for rotation about a fixed axis assuming the strong version of Newton's 3rd law.

Actually the axis only has to be fixed for a differentially short time: i.e., for an instant. Here's an argument that makes sense if read slowly and carefully enough. Now any motion about a fixed point relative to a rigid body can be viewed as a rotation about a fixed axis with some alignment. Just imagine moving a particle in the rigid body from point A to point B in space holding the distance to the fixed point constant. That motion constitutes a rotation about some axis for that particle. But every other particle must follow a curved path that is parallel to the particle path since the body is rigid—this intuitively clear though proving it definitely takes more words than I care to use now. Thus, all the particles making up the object will rotate about some fixed axis for any motion about a point fixed relative to the body. So in fact for any motion of a rigid body about a fixed point, the net work done by the internal torques is zero.

If the body were not rigid, each part of the body would have its own individual differential $d\theta_i$ for rotation about some axis in general and the sum

$$dW = \sum_i R_i F_{i,xy} \sin\gamma_{i,\tau}\, d\theta_i = \sum_i \tau_{i,z}\, d\theta_i \neq 0 \tag{118}$$

in general.

There is a second proof that the internal torques do no net work on a rigid body rotating about any point fixed relative to the body. This second proof is instructive and more rigorous than the proof above. We give it in Appendix D.

10.2. The Rotational Work-Kinetic-Energy Theorem and Rotational Kinetic Energy

We now specialize the rigid-body work equation (114) to the case the where sum is over all external torques: thus

$$W = \int \sum_i R_i F_{i,xy} \sin\gamma_{i,\tau}\, d\theta = \int \sum_i \tau_{i,z}\, d\theta = \int \tau_{z,\rm net}\, d\theta \ , \tag{119}$$

where general z component torque τ_z is specialized to the the net z component torque $\tau_{z,\rm net}$.

Assuming the strong version of Newton's 3rd law, the internal forces of a rigid body do **ZERO** net work for rotation about a fixed axis (§ 10.1), Thus equation (119) gives the net work done on the rigid body for rotation about a fixed axis by all torques.

We now recall the rigid-body rotational 2nd law equation (52), $\tau_{z,\rm net} = I\alpha$, where I is the rotational inertia and α is the angular acceleration. Using this result in equation (119), we obtain

$$W = \int I\alpha\, d\theta = I \int \frac{d\omega}{dt}\, d\theta = I \int \frac{d\omega}{dt}\frac{d\theta}{dt}\, dt = I \int \frac{d\omega}{dt}\omega\, dt \ . \tag{120}$$

This last integral can be done generally. Do it. You have 30 seconds working individually. Go.

Behold:

$$W = I \int \frac{1}{2}\frac{d\omega^2}{dt}\, dt = \Delta\left(\frac{1}{2}I\omega^2\right) \ , \tag{121}$$

where we have made a change in integration variable from θ to t and where the Δ means change in $(1/2)I\omega^2$ for some unspecified time interval Δt.

The quantity

$$\frac{1}{2}I\omega^2 = \sum_i \frac{1}{2}m_i R_i^2 \omega^2 = \sum_i \frac{1}{2}m_i v_i^2 \ , \tag{122}$$

where the last expression is the sum of the particle kinetic energies that make up the rigid body: m_i are the particle masses and v_i are the particle velocities. Recall from rotational kinematics that tangential velocity is given by

$$v = \omega R . \tag{123}$$

Since the quantity in equation (122) is the sum of the particle kinetic energies making up the rigid body, we designate it as the rotational kinetic energy of the rigid body rotating about a single fixed axis: thus

$$KE = \frac{1}{2} I \omega^2 . \tag{124}$$

Our final result for this subsection obtained from equation (121) is

$$\Delta KE = W \tag{125}$$

which is the work-kinetic-energy theorem for rigid body rotation about a single fixed axis. Recall W is the net work on the rigid body rotating about a fixed axis by external torques. We do not allow the axis to move.

But motion is relative. Not moving in one frame is moving in another.

So, in fact, our results are general for inertial frames as long we keep the direction of the axis of rotation fixed relative to the inertial frames. If the axis did change direction, we'd have to deal with more than the z components of angular momentum and torque. This is possible, but it's beyond our scope. Non-inertial frames can be considered too if one introduces non-inertial forces. We consider a moving axis that defines an non-inertial frame in § 12.

We do allow that in § 11.

10.3. Rotational Potential Energy and the Rotational Work-Energy Theorem: Reading Only

Now say we take our general equation for work done by torques on a rigid body about a single fixed axis (i.e., eq. (114)),

$$W = \int \sum_i \tau_{i,z}\, d\theta \ , \tag{126}$$

and apply it to torques that sum to the conservative torque $\tau_{z,\text{con}}$. The quantity $\tau_{z,\text{con}}$ is a conservative torque because we specify that the work done by $\tau_{z,\text{con}}$ going from θ_a to θ_b is just a function of θ_a and θ_b: i.e.,

$$W(\theta_a, \theta_b) = \int_{\theta_a}^{\theta_b} \tau_{z,\text{con}}\, d\theta \ . \tag{127}$$

Now the work down going in reverse along any path must be

$$W(\theta_b, \theta_a) = -W(\theta_a, \theta_b) \tag{128}$$

since only the sign of the differential bits of path changes in doing the θ_b to θ_a integral. Thus, one can get back energy put into the system in a direct way and also the net work done for any closed path is zero. The property of path independence of the work done is the same as for work done by a conservative force. This is why we call we call the torque conservative. It is the analog, of course, of a conservative force.

Just as in the analogous translational motion case, the fact that $W(\theta_a, \theta_b)$ is path independent, allows us to define an energy of position: a potential energy. By analogy to the translational potential energy formula, we define a rotational potential energy by

$$\Delta PE = -W_{\text{con}} \tag{129}$$

where W_{con} is the work done by the conservative torque. If $W_{\text{con}} < 0$, there is an increase in PE; if $W_{\text{con}} > 0$, there is an decrease in PE. Equation (129) is identical in form to the potential formula for translational cases.

Now what if we decompose the net work done by external torques into work done by conservative torques $W_{\text{con}} = -\Delta PE$ and work done by non-conservative torques W_{non}? In this case, the work-kinetic-energy theorem for rigid body rotation about a fixed axis equation (125) becomes

$$W = W_{\text{non}} + W_{\text{con}} = W_{\text{non}} - \Delta PE = \Delta KE \tag{130}$$

We can now write

$$\Delta KE + \Delta PE = W_{\text{non}} \tag{131}$$

which is the rotational work-energy theorem for rigid body rotation about a single fixed axis. The mechanical energy, as for translational motion, is defined by

$$E = KE + PE \ , \tag{132}$$

and then the work-energy theorem becomes

$$\Delta E = W_{\text{non}} \ . \tag{133}$$

Our rotational work-energy theorem for rigid body rotation about a single fixed axis is exactly analogous in appearance to the original work-energy theorem for translational motion of a particle.

One has to interpret the symbols correctly in each case.

10.4. The Physical Pendulum Potential Energy: Reading Only

Recall our discussion of the physical pendulum in § 7.1. We use the same setup here. The net external torque on the pendulum was gravitational torque

$$\tau_z = -Rmg\sin\theta \ , \tag{134}$$

where R is the cylindrical coordinates radius of the center of mass measured from the fixed z axis of rotation, m is the pendulum's mass, g is the acceleration due to gravity, and θ is the angle of the center of mass measured from the positive x axis (which in this case points downward) with positive in the counterclockwise direction as viewed from the positive z axis as usual.

The work done by this torque going from θ_a to θ_b is

$$W(\theta_a,\theta_b) = \int_{\theta_a}^{\theta_b} \tau_z\,d\theta = \int_{\theta_a}^{\theta_b} (-Rmg\sin\theta)\,d\theta = Rmg\,(\cos\theta_b - \cos\theta_a) \ . \tag{135}$$

The work is path independent, and so gravitational torque is conservative. Since the gravitational force is a conservative force, this is not too surprising. Looking at the body from a rotational perspective using torques and rotation is a way of dealing collectively with the motions of all the particles making up the body. Each of those particles has a gravitational potential energy.

Conventionally for the physical pendulum, one defines zero PE as being at $\theta = 0$. Thus, the potential energy formula for the physical pendulum is

$$PE = -Rmg\,(\cos\theta - 1) = Rmg\,(1-\cos\theta) \ . \tag{136}$$

In the small angle approximation $\cos\theta \approx 1-(1/2)\theta^2$. Thus, the potential energy formula for physical pendulum for small angles is

$$PE = Rmg\frac{\theta^2}{2} \ . \tag{137}$$

11. COMBINED WORK-ENERGY THEOREMS AND CONSERVATION OF MECHANICAL ENERGY

Recall standard work-energy theorem for a system of particles is

$$\Delta E = \Delta KE + \Delta PE = W_{\rm non} \ , \tag{138}$$

where E is mechanical energy, KE is center-of-mass kinetic energy, PE is total potential energy of the system in external field forces, and $W_{\rm non}$ is the net work done by non-conservative forces on the center of mass.

The theorem is easiest to apply if the potential energy can be determined from a single representative point. For gravity near the Earth's surface, this representative point is the center of mass. For the linear force in one dimensions, the center of mass can be chosen to be this representative point. The net work done by non-conservative forces can also be called the net work done by external forces since the internal forces do no net work.

Note absolutely, positively the internal forces in general do things and do work, but they have zero net force, and thus do no work on the center of mass.

Also recall that there can be non-conservative forces that do no net work, but that still do things. They can be **WORKLESS CONSTRAINT FORCES** that guide the motions in some way.

The work-energy theorem can be generalized simply to include rigid-body rotation of the object in some special cases.

There are more general generalizations in advanced classical mechanics, but they are beyond our scope.

The two cases we will consider are the standard textbook cases of a roller rolling with the no-slip condition (§ 12) and Atwood's machine (§ 13). These are the standard intro textbook cases, but I'm not aware that any textbook ever proves the two cases. It seems to me that the textbooks pretend the generalizations of the work-energy theorem to these cases are obvious. We'll do the proofs.

But first let's consider a general case of rigid-body rotating around an axis with a fixed direction in space. But the rotator is also translating: i.e., it's center of mass is moving.

We have work-energy theorem equation (138) for the center-of-mass motion and we have rotational work-kinetic-energy theorem equation (121).

Nothing forbids us from simply adding these two equations to get a combined work-energy theorem

$$\Delta E = \Delta KE + \Delta KE_{\rm rot} + \Delta PE = W_{\rm non} + W_{\rm rot} \ , \tag{139}$$

where for clarity we have subscripted the rotational kinetic energy and work done torquing the rotator by "rot". The total mechanical energy E now includes center-of-mass kinetic energy, center-of-mass potential energy, and rotational kinetic energy, but not rotational potential energy if there is any. We don't want to consider rotational potential energy now.

By adding different kinds of energies to get equation (139), one is in danger of losing information about their individual values. In fact, equation (139) is actually of little use unless there are simple constraints that impose relationships between the various terms.

And there are such constraints in special cases. The most evident of those cases is the case of a roller rolling with the no-slip condition which we take up in § 12 just below.

The case of Atwood's machine is actually a bit different. In that case, one adds the (center-of-mass) work-energy theorem equations of two objects and then adds to this the rotational work-kinetic-energy theorem equation of the pulley wheel (which is the third object). Constraints that relate the terms in this version of the combined work-energy theorem make it useful.

It's probably generally true that combined work-energy theorems need constraints relating the term to be useful.

12. ROLLERS WITH THE NO-SLIP CONDITION

A roller is either a ball (i.e., a sphere) or a cylinder.

Let's consider a rigid roller.

We assume our roller has a symmetric density distribution, and so its center of mass is at the geometrical center of the roller and thus on the roller axis of rotation.

Let's consider a roller on a rigid surface. The surface can be curved, and so in space the motion is 2-dimensional. However, along the curved path of the surface the motion is 1-dimensional in the sense that only one coordinate is needed to specify the center-of-mass position of the roller. We can let that coordinate be s with s increasing to the right. The velocity is v and the acceleration is a. Vector signs are not needed since the sign of the components of the kinematic variables gives the directions.

In general, the motion can be complex even with the restrictions imposed so far. This is because the interaction of roller and surface can be complex.

There are two special ideal limiting cases though. The first case is where there is no interaction other than the normal force of the surface on the roller. This would be true if all frictional forces were turned off. In this case the translational and rotational motions would be decoupled (i.e., completely independent).

The second case is when the no-slip condition has been imposed. In this case, there is no relative motion between the roller and surface at the point of contact. Static friction can be invoked to maintain the no-slip condition, but as we'll see below one doesn't always need it. Because of the no-slip condition, the rotational axis must be perpendicular to the translational direction of motion.

The no-slip condition is generally what we want for vehicles and other rolling devices.

For the rest of the developments in this section, we only consider the case with the ideal no-slip condition imposed.

We assume a perfectly rigid roller and surface, and so the contact point is an ideal point.

We assume an ideal case with zero rolling friction. Since roller and surface are perfectly rigid, this is the consistent assumption. Rolling friction (which was briefly discussed in the lecture *NEWTONIAN PHYSICS I*) is an non-conservative force that opposes rolling motion. It is caused by the distortions of the roller and surface and imperfect elastic forces of restoration. In intro physics, we don't know how to deal with rolling friction.

The roller has radius r, mass m, and rotational inertia I. Since the roller could be a sphere or a cylinder, we just use r for the radius since it is the more generic symbol for radius.

Consider the roller rolling with the no-slip condition imposed.

If the center of mass of the roller moves at velocity v relative to the surface, then the roller rim at contact point must have velocity $-v$ relative to the center of mass in order to have zero velocity relative to the surface as required by the no-slip condition. This means that the angular velocity of the roller is

$$\omega = -\frac{v}{r} \ , \tag{140}$$

where as conventional we take counterclockwise rotation as positive, and so require a negative sign to satisfy all our conventions. If $v > 0$, then $\omega < 0$. If $v < 0$, then $\omega > 0$.

Equation (140) implies by integration that

$$\Delta\theta = -\frac{\Delta s}{r} \tag{141}$$

and by differentiation that

$$\alpha = -\frac{a}{r} \ , \tag{142}$$

where a is center-of-mass (or translational) acceleration.

Let's assume that only gravity, the normal force of the surface, and static friction act on the roller. The normal force and static friction can only act at the contact point. Because there is no relative motion, at the contact point, there can be no kinetic friction.

We assume that the roller never leaves the surface no matter how it wiggles. Minute jumps from the surface are instantly canceled by gravity. We assume large jumps don't occur (i.e., occasions when the normal force needs to become attractive to hold the roller to the surface—which of course, it can't do).

Let's consider the case where the roller is on the level. Gravity and the normal force are both in the vertical direction and cancel each other. The normal force must cancel gravity since the surface is rigid. What of the static friction force. It is entirely horizontal since it must be parallel to the surface. Let's make the hypothesis that it is zero.

Is that hypothesis consistent?

Yes.

There is no force to cause an acceleration of the center of mass since static friction is zero. Therefore v stays constant. There are no torques about the center of mass. Gravity exerts no torque about the center of mass (§ 7). The normal force is aligned with the radius vector from the center of mass, and so exerts no torque either. The zero static friction force exerts no torque about anything. Therefore ω is constant. As long as the motion is set up with equation (140) $\omega = -v/r$ holding (i.e., the no-slip condition), then v and ω should stay constant and the no-slip condition continue to hold. In other words, linear and angular momentum are conserved.

At first it may seem odd that the roller keeps rotating without static friction, but, in fact, this accords with common experience. If a roller rolls off a table and goes into free fall, it keeps rotating because one knows that if the roller rolls off an edge into free-fall, it keeps rotating.

Now if there **IS** center-of-mass acceleration, then the no-slip condition implies static friction will maintain $\omega = -v/r$. The static friction must torque the roller to give it angular acceleration. Note that in center-of-mass acceleration cases the non-inertial force due to the acceleration in the frame of the center of mass causes no torque about the center of mass as discussed in § 7.

In real roller cases, static friction is probably acting most of the time. For example, say $v > 0$ and there is rolling friction decreasing v. Then static friction must point in the positive direction to increase the negative ω (which means decrease ω's magnitude) in order to maintain the no-slip condition. Note rolling friction points opposite velocity and static friction points in the direction of velocity in this case. If an axle torque tries to decrease ω (i.e., increase it's magnitude), then again there must be a positive static friction force to increase v in a compensating way in order to maintain the no-slip condition. This is, of course, how cars accelerate forward. A car brakes by the axle torque trying to increase ω (i.e., decrease its magnitude). To maintain no-slip, the static friction force points opposite the velocity and acts to decrease it.

What if the force demanded from static friction to maintain the no-slip condition exceeds the upper limit on static friction which we discussed in the lecture *NEWTONIAN PHYSICS I*? The roller skids.

In our current development, we assert the no-slip condition always holds, and so the demand for static friction never exceeds the static friction upper limit.

Now say the roller is on a part of the path with an inclination angle to the horizontal θ. We have $\theta > 0$ for the path rising in the positive direction and $\theta < 0$ for the path falling in the positive direction.

Note that θ is also used for angular displacement. Both usages are conventional, and so context must decide which is meant—*caveat lector*.

Let's analyze the force and torque situation first.

Recall s increases to the right no matter what the inclination angle. The y' direction we take perpendicular to the surface at all points and outward from the surface, and thus the normal force is always aligned with the positive y' direction. Note that the y' direction depends on the local inclination angle. The positive z axis is given by a right-hand rule from the positive s and y' axes. It points out of the page actually.

Note we used y' for the direction perpendicular to the surface. We use y for the vertical height coordinate.

In the s direction, Newton's 2nd law $\vec{F}_{\rm net} = m\vec{a}$ gives

$$ma = -mg\sin\theta + F_{\rm st} \ , \tag{143}$$

where a is the center-of-mass acceleration again and $F_{\rm st}$ is the static frictional force than enforces the no-slip condition.

A little thought shows that $F_{\rm st}$ points uphill regardless of whether the roller is rolling

Fig. 9.— Schematic diagram of a roller on an incline.

uphill or downhill. To be definite, let's assume the roller is moving to the right (i.e., in the positive s direction). If the roller is rolling uphill, the gravitational force is decreasing the center-of-mass velocity. If there were no friction (static or kinetic), there would be no change in angular velocity. But the static friction enforces the no-slip condition and points uphill to reduce the angular speed (which actually increases the angular velocity) and maintain the relationship $\omega = -v/r$. If the roller is rolling downhill, the gravitational force is increasing the center-of-mass velocity. If there were no friction (static or kinetic), there would be no change in angular velocity. But the static friction enforces the no-slip condition and points uphill (which is in the negative s direction) to increase the angular speed (which actually decreases the angular velocity) and maintain the relationship $\omega = -v/r$. One can think of static friction as continuously tripping the roller.

Making the roller roll to the left just vice-versa everything—to verb vice versa.

In the y' direction, there is, of course, no motion and no acceleration. Thus, we know that the normal force

$$F_{\text{normal}} = mg\cos\theta \ . \tag{144}$$

We have no intrinsic formula for F_{st}, but we can solve for it using the no-slip condition and the rotational Newton's 2nd law $\tau_{z,\text{net}} = I\alpha$.

First note that the arguments given above for zero torque from gravity and the normal force still apply: those torques about the roller axis are zero. Second note that since the roller can be accelerating now, in the frame of the center of mass of the roller there will be an inertial force. But that inertial force will also exert zero torque about an axis through the center of mass as proven in § 7. Actually, the acceleration of the center-of-mass frame will vary with slope. But at each instant it can be regarded as a constant and at that instant the inertial force exerts zero torque about the center of mass.

With the gravitational, normal, and inertial forces disposed of, the rotational Newton's 2nd law applied to the roller about the axis gives

$$rF_{\rm st} = I\alpha \ , \tag{145}$$

where I is the roller rotational inertia recall and where only the friction force exerts a torque about the center of mass and the sine of the angle between radius to the point of application of the friction force on the rim and the friction force is 90°.

The torque is positive for $F_{\rm st}$ positive and negative for $F_{\rm st}$ negative. Recall, the positive torque vector points in the positive z direction and the negative torque vector in the negative z direction. A positive torque tries to accelerate the roller in the counterclockwise sense and the a negative torque in the negative sense.

Substituting with $\alpha = -a/r$ in equation (145), we get

$$rF_{\rm st} = -\frac{Ia}{r} \ . \tag{146}$$

From equations (143) and (146), we can solve for a and $F_{\rm st}$. First eliminate $F_{\rm st}$ to get

$$\begin{aligned} ma &= -mg\sin\theta - \frac{Ia}{r^2} \\ ma + \frac{Ia}{r^2} &= -mg\sin\theta \\ a &= -\frac{mg\sin\theta}{m + I/r^2} \\ a &= -\frac{g\sin\theta}{1 + I/(mr^2)} \ . \end{aligned} \tag{147}$$

Note that the quantity $I/(mr^2)$ is dimensionless. For example, for a solid uniform cylinder it is 1/2 and for a solid uniform sphere is 2/5 (see Table 2).

The formula equation (149) for acceleration is similar to the formula we would get for an object sliding without friction on the incline:

$$a = -g\sin\theta \ . \tag{148}$$

If the roller has vanishing rotational inertia because all mass is concentrated on its axis of rotation, then we, in fact, recover $a = -g\sin\theta$ from equation (149). But note that the cases are still physically different: one is a roller with the no-slip condition and the other is slider with no friction.

Having found the acceleration formula, we can now easily solve for the angular acceleration and using $\alpha = -v/r$ and the static friction force using $F_{\rm st} = -Ia/r^2 = I\alpha/r$. The complete set of solutions is

$$a = -\frac{g\sin\theta}{1 + I/(mr^2)} , \tag{149}$$

$$\alpha = \frac{1}{r}\left[\frac{g\sin\theta}{1 + I/(mr^2)}\right] , \tag{150}$$

$$F_{\rm st} = \frac{I}{r^2}\left[\frac{g\sin\theta}{1 + I/(mr^2)}\right] . \tag{151}$$

Note that the acceleration always points downhill. If $\theta > 0$, the acceleration points to the left which is downhill. If $\theta < 0$, the acceleration points to the right which is also downhill.

The negative sign in equation (149) is merely due to the conventions we have used. The equation without the negative sign is just as valid as long as it is interpreted correctly—which is that acceleration is always downhill and θ is the angle from the horizontal measured as a positive value in all cases.

Note also that a and α are dependent on θ which can vary with the curved surface and can be positive or negative. If θ is constant, then a and α are constant.

Note moreover that static friction force depends on θ and is zero for $\theta = 0$ which we already proved above. Also the formula for static friction shows that it always points uphill as we anticipated above.

If we take θ as constant, we have a constant acceleration case and we can use the constant-acceleration kinematic equations plus initial conditions to determine the whole motion of the roller.

If θ is not constant, then we cannot solve for the whole motion without knowing the slope in detail and using numerical means in general.

On the other hand, an energy approach to the system will give us partial information for any variation of θ with position—but only partial information.

Recall our combined work-energy theorem equation (139) from § 11.

$$\Delta E = \Delta KE + \Delta KE_{\rm rot} + \Delta PE = W_{\rm non} + W_{\rm rot} \ . \tag{152}$$

The no-slip condition allows us make this a useful formula. The no-slip condition is one of those constraints that make the combined work-energy theorem useful (see § 11).

First, for our development

$$KE + KE_{\rm rot} = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}\left(m + \frac{I}{r^2}\right)v^2 \ . \tag{153}$$

We have eliminated the ω variable relying on the no-slip condition result $\omega = -v/r$.

Second, for our development

$$\begin{aligned} W_{\rm non} + W_{\rm rot} &= \int_{s_0}^{s} F_{\rm st}\,ds' + \int_{\theta_0}^{\theta} \tau_{z,\rm net}\,d\theta' \\ &= \int_{s_0}^{s} F_{\rm st}\,ds' + \int_{\theta_0}^{\theta} rF_{\rm st}\,d\theta' \\ &= \int_{s_0}^{s} F_{\rm st}\,ds' + \int_{\theta_0}^{\theta} F_{\rm st}\,d(r\theta') \\ &= \int_{s_0}^{s} F_{\rm st}\,ds' - \int_{s_0}^{s} F_{\rm st}\,ds' \\ &= 0 \ , \end{aligned} \tag{154}$$

where s_0 and θ_0 are initial conditions. Note made change in variable of integration: i.e., θ' to s' using $r\,d\theta' = d(r\theta') = -ds'$ derived from equation (141). We see that there is no net work done by the non-conservative static friction force. The center-of-mass work and the rotational work cancel out. Note the no-slip condition result $\theta = -s/r$ was needed to get the cancellation.

Static friction turns out to be a **WORKLESS CONSTRAINT FORCE**. It imposes conditions on the motion of the roller, but does no net work.

The way to look at this cancellation is that the static friction force continuously transforms energy from center-of-mass kinetic energy to rotational energy for the roller going downhill and vice versa for the roller going uphill. Recall in the lecture *ENERGY* we said that forces transform energy. Here we have example of that which at first glance seems unusual. Of course, its really a very usual way for cases where energy is transformed from center-of-mass kinetic energy to rotational energy. Note that the static friction force does no work on the surface—which is the proverbial immovable object—and so no energy is transferred to the surface.

From another perspective, the ideal no-slip condition means no sliding with kinetic friction, and thus no loss of macroscopic energy to waste heat. We discuss the dissipation to waste heat caused by kinetic friction in lecture *ENERGY*.

Since the net work is zero in our combined work-energy theorem application, mechanical energy is conserved. Using $PE = mgy$ and equation (153), we can solve for $|v|$ for any height y from the combined work-energy theorem:

$$\begin{aligned}
0 &= \Delta KE + \Delta KE_{\rm rot} + \Delta PE \\
0 &= \frac{1}{2}\left(m + \frac{I}{r^2}\right)(v^2 - v_0) + mg\Delta y \\
v^2 - v_0^2 &= -\frac{2mg\Delta y}{m + I/r^2} \\
|v| &= \sqrt{v_0^2 - \frac{2g\Delta y}{1 + I/(mr^2)}} \ ,
\end{aligned} \tag{155}$$

where v_0 is the initial velocity and Δy is the change in height from the initial height.

We could invert equation (155) and solve for Δy as a function of v, but we'll forbear.

The above development shows how to analyze ideal rolling motion and shows how a combined work-energy theorem can be used to give partial information easily.

The partial information would be hard to obtain for varying θ (implying varying acceleration) using the 2nd law approach. One does need the 2nd law approach to get total information. But that approach can't be done analytically in general if θ is varying.

13. ATWOOD'S MACHINE

George Atwood (1745–1807) invented his eponymous machine in 1784. The device as technological machine has been known since prehistory. But Atwood's idea was to use it to study the laws of motion. With Atwood's machine one can observe nearly constant accelerations that are arbitrarily small.

Atwood's machine is a pretty simple machine.

In Atwood's machine, there is a pulley with rope over it and two masses attached to the ends of the rope and the ends of the rope hang straight down.

We analyzed Atwood's machine in the case of a massless pulley in the lecture *NEWTONIAN PHYSICS I*.

Here we will give the pulley mass. Now we need translational and rotational dynamics to analyze Atwood's machine.

Let mass on the left be mass 1 and the one on the right be mass 2.

The rope is an ideal massless rope that doesn't slip on the pulley and the pulley axle is frictionless.

We have three objects.

The two masses only translate and don't rotate. Thus, the two masses can be treated as point masses with all their mass concentrated at their centers of mass. Other than gravity and the tension force, there are no external forces on the masses.

The pulley can be treated as a rigid body that rotates about a fixed axis. It does not have translational motion. Gravity and the pulley axle's normal force cancel, and so give no net force on the pulley. They also give no torque. The pulley center of mass is at the center of axle (even though the axle is not in the pulley) and so there is no gravitational torque. The normal force is radial, and so gives no torque by the cross product in the torque definition.

We can write down the equations of motion for the three objects:

$$m_1 a_1 = -m_1 g + T_1 , \tag{156}$$

$$m_2 a_2 = m_2 g - T_2 , \tag{157}$$

$$I\alpha = RT_2 - RT_1 , \tag{158}$$

where we have set some conventions: up is positive for mass 1, down is positive for mass 2,

Fig. 10.— Schematic diagram of an Atwood machine.

clockwise is positive for the pulley (which is opposite the usual convention), I is the pulley rotational inertia, R is the pulley radius, T_1 is the tension force in the rope that attaches to mass 1, and T_2 is the tension force in the rope that attaches to mass 2.

The two tension forces act as external forces on the masses and on the pulley. Then tension forces can be said to act on the pulley where the rope contacts the pulley. The rope as it wraps around the pulley can be regarded as part of the pulley, and so this part of the rope's forces and torques can be regarded as internal and disregarded in our analysis.

Since the rope is taut and there is no slip on the pulley,

$$a_1 = a_2 = R\alpha \ , \tag{159}$$

where there is no minus with the last member of this equation because we choose clockwise rotation to be positive for out Atwood's machine discussion. We replace a_1 and a_2 by a.

Now we can write

$$\begin{aligned} m_1 a &= -m_1 g + T_1 \ , & (160) \\ m_2 a &= m_2 g - T_2 \ , & (161) \\ I\frac{a}{R^2} &= T_2 - T_1 \ , & (162) \end{aligned}$$

where these three equations are solvable for the three unknowns a, T_1, and T_2.

Just adding the equations and solving for a gives

$$\begin{aligned} \left(m_1 + m_2 + \frac{I}{R^2}\right) a &= (m_2 - m_1) g \\ a &= \frac{g(m_2 - m_1)}{m_1 + m_2 + I/R^2} \ . & (163) \end{aligned}$$

Note if I/R^2 goes to zero, we recover the Atwood's machine acceleration result of the lecture *NEWTONIAN PHYSICS I*, where we assumed a massless pulley wheel.

We really, really don't want to know what T_1 and T_2 are—but we could find them easily now if we did.

Since the acceleration is a constant, we can use the timeless equation of the constant-acceleration kinematic equations to find the speed for any position y along the incline:

$$|v| = \sqrt{v_0^2 + 2a\Delta y} = \sqrt{v_0^2 + \frac{2(m_2 - m_1)g\Delta y}{m_1 + m_2 + I/R^2}} \ , \tag{164}$$

where v_0 is the initial speed and Δy is change in height: $\Delta y > 0$ is for mass 1 moving up and mass 2 moving down and $\Delta y < 0$ is for the reverse case.

We can get the same result for speed using conservation of mechanical energy.

I don't think the application is obviously true the way textbooks do. We make need to make use of the combined work-energy theorem equation (139) which was discussed in § 11:

$$\Delta E = \Delta KE + \Delta KE_{\text{rot}} + \Delta PE = W_{\text{non}} + W_{\text{rot}} \ . \tag{165}$$

We apply the combined work-energy theorem to three objects: the two masses that translate, but don't rotate and the pulley wheel that rotates, but doesn't translate. We sum those applications to get

$$\Delta E = \Delta\left(\frac{1}{2}m_1 v^2 + \frac{1}{2}m_2 v^2 + \frac{1}{2}\frac{I}{R^2}v^2\right) + (m_1 - m_2)g\Delta y = W_{\text{non},1} + W_{\text{non},2} + W_{\text{pulley}} \ , \tag{166}$$

where $W_{\text{non},1}$ is the non-conservative work done on mass 1, $W_{\text{non},2}$ is the non-conservative work done on mass 2, and W_{pulley} is the rotational work done on the pulley. Note the fact that all the velocity quantities (mass translational velocities and pulley wheel tangential velocity) have the same value v vastly simplifies the sum of the applications of the combined work-energy theorem. Also there is no change under rotation of the pulley's potential energy. The sameness of the velocities and the lack of rotational potential energy constraints that make the combined work-energy theorem useful (see § 11).

Now, in fact,

$$W_{\rm non,1} + W_{\rm non,2} + W_{\rm pulley} = T_1\Delta y - T_2\Delta y + (T_2 - T_1)R\Delta\theta = 0 \ , \tag{167}$$

where we've used the fact that $R\Delta\theta = \Delta y$ by the no-slip condition. Note that with our conventions, the work is positive on mass 1 for $\Delta y > 0$ and negative on mass 2 for $\Delta y > 0$. Once again, constraints have led to a simplification of the combined work-energy theorem application.

The tension forces have turned out to be a **WORKLESS CONSTRAINT FORCE**. It imposes conditions on the motion of Atwood's machine, but do no net work.

Now using equations (166) and (167), we find

$$\begin{aligned} 0 &= \Delta\left(\frac{1}{2}m_1 v^2 + \frac{1}{2}m_2 v^2 + \frac{1}{2}\frac{I}{R^2}v^2\right) + (m_1 - m_2)g\Delta y \\ |v| &= \sqrt{v_0^2 + \frac{2(m_2 - m_1)g\Delta y}{m_1 + m_2 + I/R^2}} \end{aligned} \tag{168}$$

which is the same result as before the speed for change in height Δy.

The Atwood's machine results we have developed above can easily be generalized for the double-incline-pulley system which we analyzed in the lecture *NEWTONIAN PHYSICS I*. The analysis would be the same as there except now the pulley wheel would have a finite rotational inertia.

Actually multi-component systems like Atwood's machine can be treated in a systematic way using the general formalism of advanced classical mechanics (e.g., Goldstein et al. 2002, p. 27–28). But that's well beyond our scope.

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A. SOME CROSS PRODUCT FORMALISM

Recall from § 2, the formula for the cross product of general vectors $\vec{A}$ and $\vec{B}$ (eq. (1)) is

$$\vec{A} \times \vec{B} = AB \sin\theta \hat{n} \ , \tag{A1}$$

where A and B are, respectively, the magnitudes of $\vec{A}$ and $\vec{B}$, θ is the angle between $\vec{A}$ and $\vec{B}$, and $\hat{n}$ is a unit vector normal to the plane defined by $\vec{A}$ and $\vec{B}$ whose sense is determined by a right-hand rule.

The alternative component-form formula for the cross product can be developed by writing general vectors $\vec{A}$ and $\vec{B}$ in component form:

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z} \qquad \text{and} \qquad \vec{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z} \ . \tag{A2}$$

From the original cross product formula, we find that

$$\hat{x} \times \hat{x} = 0 \ , \qquad \hat{y} \times \hat{y} = 0 \ , \qquad \hat{z} \times \hat{z} = 0 \ , \tag{A3}$$

and

$$\hat{x}\times\hat{y} = \hat{z} \ , \qquad \hat{y}\times\hat{z} = \hat{x} \ , \qquad \hat{z}\times\hat{x} = \hat{y} \ , \qquad \hat{y}\times\hat{x} = -\hat{z} \ , \qquad \hat{z}\times\hat{y} = -\hat{x} \ , \qquad \hat{x}\times\hat{z} = -\hat{y} \ . \tag{A4}$$

Using these results, it follows—and after some algebra—that

$$\vec{A} \times \vec{B} = (A_yB_z - A_zB_y)\hat{x} + (A_zB_x - A_xB_z)\hat{y} + (A_xB_y - A_yB_x)\hat{z} \ . \tag{A5}$$

This result is the component form formula for the cross product. There is an easy rule for remembering the component form formula, but we won't need it in this course—and so we won't show it.

Actually the derivation of equation (A5) required us to assume the distributive property for the cross product. This property really needs a proof as mentioned in § 2. A proper rigorous mathematical way to establish to the cross product and its properties is to assume equation (A5) as a starting point, show that it is a pseudovector (e.g., Arfken 1970, p. 131) (which means to show that it transforms as a pseudovector), show that it yields the original formula (eq. (A1)), and then prove from equation (A5) that the distributive property holds. But to do all that is to go off on a tangerine beyond our scope.

Now for what all the foregoing is foregone for: the establishment of the product rule for the cross product. Differentiating with respect to independent variable t (which could be time), we obtain

$$\begin{aligned} \frac{d(\vec{A}\times\vec{B}\,)}{dt} &= \left(\frac{dA_y}{dt}B_z + A_y\frac{dB_z}{dt} - \frac{dA_z}{dt}B_y - A_z\frac{dB_y}{dt}\right)\hat{x} + \ldots \\ &= \left(\frac{dA_y}{dt}B_z - \frac{dA_z}{dt}B_y + A_y\frac{dB_z}{dt} - A_z\frac{dB_y}{dt}\right)\hat{x} + \ldots \\ &= \frac{d\vec{A}}{dt}\times\vec{B} + \vec{A}\times\frac{d\vec{B}}{dt}\ , \end{aligned} \tag{A6}$$

where we have left implicit the operations that are similar to the ones displayed. We can see the product rule for the cross product equation (A6) is just what you'd have guessed. But a proof is necessary.

B. ANGULAR MOMENTUM VECTOR DIRECTION AND RIGID-BODY ROTATION

An important result is that if you have a rigid body in rotation about an axis (which we label the z axis) and the origin for the angular momentum calculation is on the z axis, then the angular momentum about the x and y axes is **NOT** necessarily zero. In other words, the angular momentum vector is not always the z axis. The result is slightly surprising. In important special cases, the angular momentum vector does align with the z axis.

We gave a simple proof of this fact in § Rigid Body Rotation and the Non-z Components:

Reading Only.

Here we give a second, more involved proof.

The proof of the result is straightforward/tricky.

First, note that the velocity of any point mass i of the body is given by

$$\vec{v}_i = \vec{\omega} \times \vec{r}_i \ , \tag{B1}$$

where $\vec{\omega}$ is the vector angular velocity of the rigid-body rotation and $\vec{r}_i$ is the position of the point mass. The vector angular velocity has a magnitude equal to the magnitude of the angular velocity and points along the z axis: it points in the positive z direction for counterclockwise rotation and in the negative z direction for clockwise rotation. Since the body is in rigid-body rotation all point masses have the same $\vec{\omega}$. The validity of equation (B1) is easy to see. Note that

$$\vec{v}_i = \vec{\omega} \times \vec{r}_i = \omega r_i \sin\theta_i \hat{\phi} = \omega r_{i,xy} \hat{\phi} \ , \tag{B2}$$

where ω is the angular velocity (not the magnitude of the angular velocity which is what we conventionally mean by a vector quantity without its vector symbol), θ_i is the angle of $\vec{r}$ from the positive z axis, $\hat{\phi}$ is the spherical polar coordinate azimuthal unit vector at the position of the point mass, and $r_{i,xy}$ is the xy plane component of r_i. The last expression in equation (B2) clearly gives the point mass velocity using the rules we've established in earlier work.

Now the total angular momentum of the rigid body is given by

$$\begin{aligned} \vec{L} &= \sum_i \vec{r}_i \times \vec{p}_i \\ &= \sum_i \vec{r}_i \times m_i \vec{v}_i \\ &= \sum_i \vec{m}_i r_i \times (\vec{\omega} \times \vec{r}_i) \end{aligned}$$

$$= \sum_i [m_i(\vec{r}_i \cdot \vec{r}_i)\vec{\omega} - m_i(\vec{r}_i \cdot \vec{\omega})\vec{r}_i]$$

$$= \left[\sum_i m_i(\vec{r}_i \cdot \vec{r}_i)\right]\vec{\omega} - \left[\sum_i m_i(z_i x_i \hat{x} + z_i y_i \hat{y} + z_i^2 \hat{z})\right]|\omega| \ , \qquad \text{(B3)}$$

where we have used the cross product identity

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} \qquad \text{(B4)}$$

(e.g., Arfken 1970, p. 25) and the fact that $\vec{r}_i \cdot \vec{\omega} = r_i|\omega| \cos\theta_i = z_i|\omega|$.

The first term in last expression of equation (B3) has the direction of $\vec{\omega}$, and so is aligned with the z axis.

But the second term does **NOT**. This is because the sums

$$\sum_i m_i z_i x_i \qquad \text{and} \qquad \sum_i m_i z_i y_i \qquad \text{(B5)}$$

are not zero in general.

This means that in general, a rigid body rotating about the z axis (which actually general axis) does not have it's angular momentum vector aligned with the z axis. This is the result we set out to prove.

The result means that to start a rigid body rotating about the z axis from rest, one needs to exert not only a z (external) torque, but also x and y (external) torques too. Now if the rigid body is rotating about a physical fixed axis, then that fixed axis can supply those torques if nothing else does and thus can create non-zero x and y components of angular momentum. In fact, in many cases one can describe the motion of such a rotating rigid body only referring to the z torques and z component of angular momentum. We'll discuss this point further in § 5. But the fact that that is an adequate description in most cases does not mean that the x and y torques are zero and that x and y components of angular momentum are zero. Knowledge about the x and y torques and that x and y components of angular momentum may be needed if for example your physical fixed axis can break. Those data are probably needed to understand the breaking point of the physical fixed axis. See further discussion of the need to go beyond just knowing the z torque and the z component of angular momentum see § 5.1

Another case where our result is relevant is for a rigid body isolated floating in space. The body could be in orbit for interest and in an approximately inertial frame. Or it could be a microscopic particle floating in a medium. Microscopic particles really need quantum mechanics, but they can be approximately treating classically sometimes and the classical description is useful to know among other things since it gives insight into the quantum mechanical description.

Say the floating rigid body is at rest initially and we then exert only z torques on the body about the z axis. Only the z angular momentum can be non-zero and precisely for that reason the body can't in general be rotating around the z axis. It will be rigidly rotating, but in a perhaps complex manner. My knowledge of such systems has reached it's limit.

There are important special cases where rigid body rotation about the z axis does have zero x and y components of angular momentum or, in other words, have its angular momentum vector aligned with the z axis. These special cases arise from special symmetries.

Consider again the sums

$$\sum_i m_i z_i x_i \qquad \text{and} \qquad \sum_i m_i z_i y_i \ . \tag{B6}$$

First, if the body has mirror symmetry about the xy plane, then both sums are zero since for every mass point at (x_i, y_i, z_i), there is an equal mass point at $(x_i, y_i, -z_i)$ and the terms in the both sums cancel out pairwise.

Second, if the body has mirror symmetry about both the yz plane and xz plane both sums cancel out pairwise. The first sum cancels out pairwise since for every at (x_i, y_i, z_i), there is an equal mass point at $(-x_i, y_i, z_i)$. The second sum cancels out pairwise since for every at (x_i, y_i, z_i), there is an equal mass point at $(x_i, -y_i, z_i)$.

Third, if the body has mirror symmetry about any two planes perpendicular to each other and perpendicular to the z axis, the sums cancel out pairwise. The proof of this case is just to note that the orientation of the xy axes is arbitrary, and so we can always orient our axes with the symmetry planes and then this case is proven by the second case.

Fourth, if a body has rotational symmetry about the z axis then it is a special case of our second case and both sums cancel out pairwise.

These special cases are important since in technology, we can built bodies with the special symmetries and since microscopic particles often exhibit the special symmetries.

C. ANGULAR MOMENTUM FOR MULTIPLE OBJECTS

The total angular momentum for a system of particles about some origin is

$$\vec{L} = \sum_j \vec{r}_j \times \vec{p}_j \ , \tag{C1}$$

where j labels the particles.

Say the particles can be grouped in objects or subsystems. Let's label the objects by i.

We can now write

$$\vec{L} = \sum_i {\sum_j}' \vec{r}_{ij} \times \vec{p}_{ij} \ , \tag{C2}$$

where the primed sum is only over the particles belonging to object i and where $\vec{r}_{ij}$ is the position of particle j that belongs to object i (i.e., particle ij) and $\vec{p}_{ij}$ is the momentum of particle j that belongs to object i (i.e., particle ij).

We cannot decompose $\vec{r}_{ij}$ by

$$\vec{r}_{ij} = \vec{r}_i + \Delta\vec{r}_{ij} \ , \tag{C3}$$

where $\vec{r}_i$ is a representative position for object i (which could be chosen be the object center of mass for example) and $\Delta\vec{r}_{ij}$ is the position of particle ij relative to position $\vec{r}_i$.

We now have for total angular momentum

$$\begin{aligned}
\vec{L} &= \sum_i {\sum_j}' \vec{r}_{ij} \times \vec{p}_{ij} \\
&= \sum_i \vec{r}_i \times {\sum_j}' \vec{p}_{ij} + \sum_i {\sum_j}' \Delta\vec{r}_{ij}\vec{p}_{ij} \\
&= \sum_i \vec{r}_i \times \vec{p}_i + \sum_i {\sum_j}' \Delta\vec{r}_{ij}\vec{p}_{ij} \\
&= \sum_i L_i + \sum_i L_i' \ ,
\end{aligned} \tag{C4}$$

where p_i is the total momentum of object i, L_i is the angular momentum of object i treated as point particle at $\vec{r}_i$ and L_i' is the angular momentum of object i about the representative point $\vec{r}_i$.

Equation (C4) has lots of interesting consequences.

But let's just use it to analyze the bicycle wheel demo of § 5.2.1.

Our analysis is for the ideal system where no external z torque can be exerted on the system and where the wheel axle is frictionless.

Let the rotating person be object 1 and the bicycle wheel be object 2. Let the origin for the system and representative point in object 1 be the same point. This means $\vec{L}_1 = 0$.

The total angular momentum of the system is

$$\vec{L} = \vec{L}_1' + \vec{L}_2 + \vec{L}_2' \ . \tag{C5}$$

Only the z component of angular momentum is conserved, and so let's consider that only the z component angular momentum expression

$$L_z = L'_{z,1} + L_{z,2} + L'_{z,2} \ . \tag{C6}$$

Now the person and the wheel representative point can be treated as a single rigid rotator between wheel flips if we put the representative point at the center of the wheel axle which we do. Thus, using the results § 6, we can write

$$L'_{z,1} + L_{z,2} = I\omega \ , \tag{C7}$$

where I is the joint rotational inertia of the person and the wheel representative point and ω is their joint angular momentum.

The angular momentum of the wheel about the representative point when the wheel axle is aligned with the vertical is

$$L'_{z,2} = I_{\rm wh}\omega_{\rm wh} \ , \tag{C8}$$

where $I_{\rm wh}$ is the rotational inertia of the wheel and $\omega_{\rm wh}$ is the wheel angular velocity.

Thus for the total z angular momentum we have

$$L_z = I\omega + I_{\rm wh}\omega_{\rm wh} \ . \tag{C9}$$

Now in flipping the wheel over $\omega_{\rm wh}$ can't change at all since the wheel axle can exert no torque on the wheel. Remember wheel axle is frictionless.

Also L_z and $I_{\rm wh}$ cannot change during the flips: the first because it is conserved when no external z torques act and the second because the wheel is rigid.

But the I and ω can change during the flip over and $L'_{z,2}$ clearly changes since the direction of the wheel axle changes.

Without a more detailed analysis, we can't follow the changes going on during the flip. But we can write down the expression for total angular momentum after a complete flip. In an ideal flip, I will be the same after by symmetry and $L'_{z,2}$ will change from $I_{\rm wh}\omega_{\rm wh}$ to $-I_{\rm wh}\omega_{\rm wh}$.

So after the flip, we have

$$L_z = I\omega_{\rm after} - I_{\rm wh}\omega_{\rm wh} \ . \tag{C10}$$

So we can solve for $\omega_{\rm after}$ using conservation of the z component of angular momentum. We get

$$\omega_{\rm after} = \omega + 2\left(\frac{I_{\rm wh}}{I}\right)\omega_{\rm wh} \ . \tag{C11}$$

This result is thankfully confirmed by (e.g., Halliday et al. 2001, p. 262).

You should note pretty darn complex internal interactions happen in the bicycle wheel demo: forces and reaction forces, torques and reaction torques. Nonetheless we can at least partially understand the demo using conservation of angular momentum.

D. A SECOND PROOF THAT INTERNAL TORQUES DO NO NET WORK ON A RIGID BODY ROTATING ABOUT ANY POINT FIXED RELATIVE TO THE BODY

In § 10.1, we gave a proof that the internal torques do no net work on a rigid body rotating about any point fixed relative to the body.

There is a second proof.

The second proof is instructive since along the way we prove that the internal forces do no net work on rigid body for any motion of the body. It ia also more rigorous than the first proof.

Actually, it is quasi-obvious that the internal forces (which are the origin of the internal torques) of a rigid body do **ZERO** net work for general motions assuming the strong version of Newton's 3rd law.

The "quasi" is because it took me some time to think of the "obvious" proof.

Say $\vec{F}_{ji}$ is the internal force of particle j on particle i of a rigid body. The net work done on the system by the internal forces in a arbitrary differential overall motion is

$$dW = \sum_{ij,i\neq j} \vec{F}_{ji} \cdot d\vec{s}_i \ , \tag{D1}$$

where the $d\vec{s}_i$ are arbitrary displacements, except that they are consistent with the body being rigid.

Now consider the terms $\vec{F}_{k\ell} \cdot d\vec{s}_\ell$ and $\vec{F}_{\ell k} \cdot d\vec{s}_k$ in equation (D1). By Newton's 3rd law

$$\vec{F}_{k\ell} \cdot d\vec{s}_\ell + \vec{F}_{\ell k} \cdot d\vec{s}_k = \vec{F}_{k\ell} \cdot (d\vec{s}_\ell - d\vec{s}_k) = \vec{F}_{k\ell} \cdot d\vec{s}_{k\ell} \ , \tag{D2}$$

where we have defined the relative differential displacement vector $d\vec{s}_{k\ell} = d\vec{s}_\ell - d\vec{s}_k$. Since the body is rigid, $d\vec{s}_{k\ell}$ can have no component along the line joining particles k and ℓ. We now invoke the strong version of Newton's 3rd law (e.g., Goldstein et al. 2002, p. 7), which requires that $\vec{F}_{k\ell}$ be aligned with the line joining particles k and ℓ. Clearly, then

$$\vec{F}_{k\ell} \cdot d\vec{s}_\ell + \vec{F}_{\ell k} \cdot d\vec{s}_k = \vec{F}_{k\ell} \cdot (d\vec{s}_\ell - d\vec{s}_k) = \vec{F}_{k\ell} \cdot d\vec{s}_{k\ell} = 0 \ . \tag{D3}$$

Thus, in equation (D1) the terms cancel out pairwise.

And now we've proven that the internal forces of a rigid body do **ZERO** net work for general motion.

Remember that we assumed the strong version of Newton's 3rd law. What if the strong version does **NOT** apply to a rigid body? From the little I know of cases where the strong version does not apply (e.g., Goldstein et al. 2002, p. 8), I can't imagine a rigid body that was such a case. So we can assume the strong version of Newton's 3rd law does apply to a rigid bodies usually.

Now what about the torque proof. Let's first assume an arbitrary origin. The origin could be fixed in space or fixed relative to the rigid body or moving relative to space and the rigid body. In the middle case, it could be chosen to be the center of mass, but this is not necessary. Let's decompose the differential displacements thusly

$$d\vec{s}_i = r_i\hat{\phi}_i\,d\phi_i + \hat{r}_i\,dr_i \ , \tag{D4}$$

where the r_i is the radial position of particle i relative to the origin, $\hat{\phi}_i$ is unit vector that gives direction of the component of particle motion perpendicular to the radial direction, ϕ_i is the angular motion of the particle i, $\hat{r}_i$ is unit vector that points in the radial direction, and dr_i is the change in the radial position of particle i.

Now we expand the differential work done by the internal forces thusly

$$\begin{aligned}
dW &= \sum_{ij,i\neq j}\vec{F}_{ji}\cdot d\vec{s}_i \\
&= \sum_{ij,i\neq j}\vec{F}_{ji}\cdot r_i\hat{\phi}_i\,d\phi_i + \sum_{ij,i\neq j}\vec{F}_{ji}\cdot\hat{r}_i\,dr_i \\
&= \sum_{ij,i\neq j} r_iF_{ji}\cos\xi_{ji}\,d\phi_i + \sum_{ij,i\neq j}\vec{F}_{ji}\cdot\hat{r}_i\,dr_i \\
&= \sum_{ij,i\neq j} r_iF_{ji}\cos\left(\gamma_{ji,\tau}-\frac{\pi}{2}\right)d\phi_i + \sum_{ij,i\neq j}\vec{F}_{ji}\cdot\hat{r}_i\,dr_i \\
&= \sum_{ij,i\neq j} r_iF_{ji}\sin\gamma_{ji,\tau}\,d\phi_i + \sum_{ij,i\neq j}\vec{F}_{ji}\cdot\hat{r}_i\,dr_i \\
&= \sum_{ij,i\neq j}\tau_{ji}\,d\phi_i + \sum_{ij,i\neq j}\vec{F}_{ji}\cdot\hat{r}_i\,dr_i \\
&= W_\tau + W_R
\end{aligned} \tag{D5}$$

where ξ_{ji} is the angle measured from the force $\vec{F}_{ji}$ direction to the $\hat{\phi}_i$ direction, $\gamma_{ji,\tau}$ is the angle measured from radial direction to the force $\vec{F}_{ji}$ direction, τ_{ji} is the torque of the force F_{ji} about the origin, $\tau_{ji}\,d\phi_i$ is the work done by torque of the force F_{ji} about the origin W_τ

is the net torque work done by the internal forces and W_R is the net non-torque or net radial work done by the internal forces

Now, in fact, $dW = 0$ in general for a rigid body as we have proven above.

If the origin is fixed relative to the rigid body, then all $dr_i = 0$ and thus $dW_R = 0$. In this case, $dW = 0$ implies $dW_\tau = 0$. So we have proven the result we wished to prove: the internal torques do no net work on a rigid body rotating about any point fixed relative to the body.

What if the origin is not fixed relative to the rigid body? Then dW_R is not zero in general and thus dW_τ is not zero in general.

Actually, I think when wishes to analyze a rigid body's motion in terms of internal torques, one pretty well always chooses to do so for an origin that is fixed relative to the moving body. Thus, one usually doesn't have to worry about net work done by internal torques on a rigid body.

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