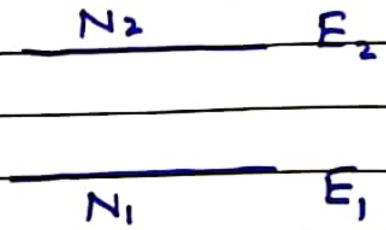
why population inversion is not possible in 2-level laser

$N_2 > N_1$



Absorption rate

$$\frac{dN_1}{dt} = N_1 B_{12} \rho(\omega)$$

$$\frac{dN_1}{dt} = N_{12} = N_1 \underbrace{B_{12} \rho(\omega)} \qquad W_{12} = B_{12} \rho(\omega)$$

$$N_{12} = N_1 W_{12}$$

stimulated emission rate

$$\frac{dN_2}{dt} = N_{21} = N_2 B_{21} \rho(\omega)$$

dynamic equlibrium

steady state condition $B_{12} \approx B_{21}$

$W_{12} \approx W_{21}$

$$\frac{dN_2}{dt} = N_{21} = N_2 W_{21}$$

$$W_{21} = B_{21} \rho(\omega)$$

$$N_{21} = W_{12} N_2$$

spontanious emission Rate

A_{21} = radiative transion coefficient

S_{21} = non radiative transition

$$T_{21} = A_{21} + S_{21}$$

$$\frac{dN_2}{dt} = N_{21}^{sp} = N_2 T_{21}$$

$$N_{21}^{sp.} = N_2 (A_{21} + S_{21})$$

Rate equation of ~~absorption~~ population inversion E_2

$$\frac{dN_2}{dt} = W_{12} N_1 - W_{12} N_2 - N_2 T_{21}$$

$$= W_{12}(N_1 - N_2) - N_2 T_{21}$$

Rate equation E_1

$$\frac{dN_1}{dt} = -W_{12} N_1 + W_{12} N_2 + N_2 T_{21}$$

$$= -W_{12}(N_1 - N_2) + N_2 T_{21}$$

total number of atoms = N

$$N = N_1 + N_2$$

$$\frac{d}{dt}(N_1+N_2) = 0 \qquad N_1 + N_2 = \text{constant}$$

steady state or Dyna equili

$$\frac{dN_1}{dt} = \frac{dN_2}{dt}$$

$$-W_{12}(N_1-N_2) + N_2T_{21} = W_{12}(N_1-N_2) - N_2T_{21}$$

$$\cancel{2}N_2T_{21} = \cancel{2}W_{12}(N_1-N_2)$$

divide with N_1 on both sides

$$\frac{N_2}{N_1}T_{21} = W_1\left(\frac{N_1-N_2}{N_1}\right)$$

$$\frac{N_2}{N_1} T_{21} = W_{12}\left(1 - \frac{N_2}{N_1}\right)$$

$$\frac{N_2}{N_1} T_{21} = W_{12} - W_{12}\frac{N_2}{N_1}$$

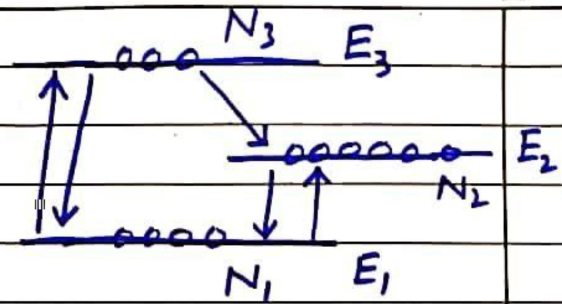
$$\frac{N_2}{N_1}(T_{21} + W_{12}) = W_{12}$$

$$\frac{N_2}{N_1} = \frac{W_{12}}{T_{21} + W_{12}}$$

$N_2 < N_1$ always

total number of atoms $= N$

$$N = N_1 + N_2 + N_3$$



Spontanous emission

Radiative spont emission coefficient $= A$

non radiative " " " $= S$

total spont " " $= T$

3 to 1 $\quad T_{31} = S_{31} + A_{31} \rightarrow$ neglect

3 to 2 $\quad T_{32} = S_{32} + A_{32}$

2 to 1 $\quad T_{21} = S_{21} + A_{21}$

Scanned with CamScanner

T_{31} is neglected because probability of spont transition from $3 \rightarrow 1$ is very very small

Absorption rate from $1 \rightarrow 3$

$= N_1 B_{13} \rho(\omega)$

denote

$= N_1 W_{13}$

$W_{13} = W_p$

$= N_1 W_p$

stimulated emission from $3 \rightarrow 1$

Rate

$= N_3 B_{31} \rho(\omega)$

as $B_{13} = B_{31}$

$= N_3 W_{31}$

so $W_{13} = W_{31}$

$= N_3 W_{13}$

Scanned with CamScanner

and we denote W_{13} with W_p so

$= N_3 W_p$

Rate of change of population of level 3 is

$$\frac{dN_3}{dt} = \underbrace{W_p N_1}_{\substack{\text{Absorption} \\ 1\to 3}} - \underbrace{W_p N_3}_{\substack{\text{stimulated} \\ \text{emission} \\ 3\to 1}} - \underbrace{T_{32} N_3}_{\substack{\text{spontanious} \\ \text{emission} \\ 3\to 2}}$$

$$\frac{dN_3}{dt} = W_p (N_1 - N_3) - T_{32} N_3$$

Absorption rate $1 \rightarrow 2$

$$= N_1 B_{12} \rho(\omega) \qquad W_{12} = B_{12} \rho(\omega)$$

$$= N_1 W_{12}$$

stimulated emission rate $2 \rightarrow 1$

$$= N_2 B_{21} \rho(\omega)$$

$$= N_2 B_{12} \rho(\omega) \qquad \text{as } B_{12} = B_{21}$$

$$\text{so } W_{12} = W_{21}$$

$$= N_2 W_{12}$$

Rate of change of population of level 2

$$\frac{dN_2}{dt} = \underbrace{W_{12} N_1}_{\substack{\text{Absorption} \\ 1\rightarrow 2}} - \underbrace{W_{12} N_2}_{\substack{\text{stimu emission} \\ 2\rightarrow 1}} - \underbrace{T_{21} N_2}_{\substack{\text{spon emission} \\ 2\rightarrow 1}} + \underbrace{N_3 T_{32}}_{\substack{\text{spon emissi} \\ 3\rightarrow 2}}$$

$$\frac{dN_2}{dt} = -W_{12}(N_2 - N_1) - T_{21} N_2 + T_{32} N_3$$

Rate of change of population of level 1

$$\frac{dN_1}{dt} = \underbrace{W_p N_3}_{\substack{\text{stimu emiss} \\ 3\to 1}} - \underbrace{W_p N_1}_{\substack{\text{absorption} \\ 1\to 3}} + \underbrace{W_{12} N_2}_{\substack{\text{stimu emiss} \\ 2\to 1}} - \underbrace{W_{12} N_1}_{\substack{\text{absorption} \\ 1\to 2}} + \underbrace{T_{21} N_2}_{\substack{\text{spon emiss} \\ 2\to 1}}$$

$$\frac{dN_1}{dt} = W_p(N_3 - N_1) + W_{12}(N_2 - N_1) + N_2 T_{21}$$

Steady state condition

$$\frac{dN_1}{dt} = \frac{dN_2}{dt} = \frac{dN_3}{dt} = 0$$

$$\frac{dN_3}{dt} = \frac{dN_1}{dt}$$

$$\frac{dN_2}{dt} = 0$$

$$-W_{12}(N_2 - N_1) - T_{21} N_2 + T_{32} N_3 = 0$$

$$W_{12}(N_2 - N_1) = T_{32} N_3 - T_{21} N_2 \quad \text{— (i)}$$

Now

$$\frac{dN_3}{dt} = \frac{dN_1}{dt}$$

$$W_p(N_1 - N_3) - T_{32} N_3 = W_p(N_3 - N_1) + W_{12}(N_2 - N_1) + N_2 T_{21}$$

$$-T_{32}N_3 + W_p(N_1 - N_3) + W_p(N_1 - N_3) = W_{12}(N_2 - N_1) + N_2 T_{21}$$

put value of $W_{12}(N_2 - N_1)$ from ①

$$\cancel{2} W_p(N_1 - N_3) = \cancel{2} T_{32} N_3 - \cancel{T_{21} N_2} + \cancel{N_2 T_{21}}$$

$$W_p N_1 - W_p N_3 = T_{32} N_3$$

$$W_p N_1 = N_3 (W_p + T_{32})$$

$$N_3 = \frac{W_p N_1}{W_p + T_{32}}$$

Three level lasing system

$$N_3 = \frac{N_1 W_p}{T_{32} + W_p}$$

rate of change of population for level E_2

$$\frac{dN_2}{dt} = W_{12}(N_1 - N_2) + N_3 T_{32} - N_2 T_{21}$$

at steady state condition

$$\frac{dN_2}{dt} = 0 = \frac{W_{12}(N_1 - N_2)}{1} + \frac{N_1 W_p T_{32}}{T_{32} + W_p} - \frac{N_2 T_{21}}{1}$$

$$0 = \frac{W_{12}(N_1 - N_2)(T_{32} + W_p) + N_1 W_p T_{32} - N_2 T_{21}(T_{32} + W_p)}{T_{32} + W_p}$$

$$0 = (W_{12} N_1 - W_{12} N_2)(T_{32} + W_p) + N_1 W_p T_{32} - N_2 T_{21}(T_{32} + W_p)$$

$$0 = W_{12} N_1 T_{32} + W_{12} N_1 W_p - W_{12} N_2 T_{32} - W_{12} N_2 W_p + N_1 W_p T_{32} - N_2 T_{21} T_{32} - N_2 T_{21} W_p$$

$$0 = N_1(W_{12} T_{32} + W_{12} W_p + W_p T_{32}) - N_2(W_{12} T_{32} + W_{12} W_p + T_{21} T_{32} + T_{21} W_p)$$

$$N_1(T_{32}W_{12} + W_{12}W_p + W_pT_{32}) = N_2(W_{12}T_{32} + W_{12}W_p + T_{21}T_{32} + T_{21}W_p)$$

$$\frac{N_2}{N_1} = \frac{T_{32}W_{12} + W_{12}W_p + W_pT_{32}}{W_{12}T_{32} + W_{12}W_p + T_{21}T_{32} + T_{21}W_p}$$

$$\frac{N_2}{N_1} = \frac{T_{32}W_{12} + W_{12}W_p + W_pT_{32}}{(W_{12}+T_{21})(T_{32}+W_p)}$$

$$\frac{N_2}{N_1} - 1 = \left(\qquad \right) - 1$$

$$\frac{N_2 - N_1}{N_1} = \frac{\cancel{T_{32}W_{12}} + \cancel{W_{12}W_p} + W_pT_{32} - \cancel{W_{12}T_{32}} - \cancel{W_{12}W_p} - T_{21}T_{32} - T_{21}W_p}{(W_{12}+T_{21})(T_{32}+W_p)}$$

$$\frac{N_2 - N_1}{N_1} = \frac{W_pT_{32} - T_{21}T_{32} - T_{21}W_p}{(W_{12}+T_{21})(T_{32}+W_p)}$$

$$\frac{N_2 - N_1}{N_1} = \frac{W_p(T_{32} - T_{21}) - T_{21}T_{32}}{(W_{12}+T_{21})(T_{32}+W_p)} \quad \text{--- (1)}$$

for positive $N_2 - N_1 = +ve$

if $T_{32} > T_{21}$ $\rightarrow$ $N_2 > N_1$

population inversion condition

minimum pumping power $N_2 = N_1$

$N_2 - N_1 = 0$

$$0 = \frac{0}{N_1} = \frac{W_p(T_{32} - T_{21}) - T_{32}T_{21}}{(W_{12} + T_{21})(T_{32} + W_p)}$$

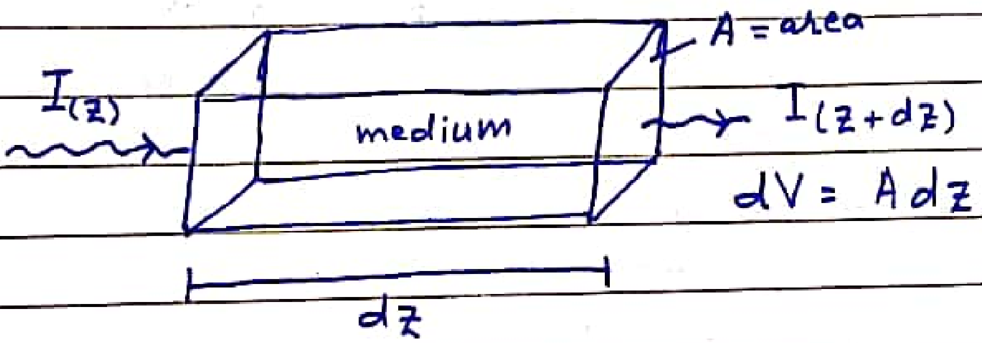
$$0 = W_p(T_{32} - T_{21}) - T_{32}T_{21}$$

$$T_{32}T_{21} = W_p(T_{32} - T_{21})$$

$$\frac{T_{32}T_{21}}{(T_{32} - T_{21})} = W_p$$

$$W_p = \frac{T_{32} T_{21}}{(T_{32} - T_{21})}$$

Gain coefficient of Laser



Energy absorption rate of the medium

$$= B\rho(\omega)\, dz A \cdot N_1$$

Energy rate emitted from the medium =

$$B\rho(\omega)\, A dz\, N_2$$

Net emitted Energy per unit time (rate)

$$= B\rho(\omega) A dz N_2 - B\rho(\omega) A dz N_1$$

$$= B\rho(\omega) A\,dz\,(N_2 - N_1)$$

$$= B\rho(\omega) A\,dz\,(\Delta N)$$

$$I_{(z+dz)} = \frac{\partial I}{\partial z} dz + I_{(z)}$$ Taylor expension

we know

$$I = \frac{E}{At}$$

$$IA = \frac{E}{t} = \text{Energy rate}$$

Net energy emitted rate.

$$= B\rho(\omega) A\,dz(\Delta N) = I\ A$$

$$B\rho(\omega)\cancel{A}\cancel{dz}\,\Delta N = \frac{\partial I}{\partial z}\cancel{dz}\,\cancel{A}$$

$$\frac{\partial I_{(z)}}{\partial z} = B\rho(\omega)\,\Delta N$$

$$\frac{dI}{dz} = B\rho(\omega)\,\Delta N$$

we know the relation b/w coefficients A & B

$$B = \frac{\pi^2 c^3}{\hbar \omega^3} A \qquad \because A = \frac{1}{\tau}$$

$$B = \frac{\pi^2 c^3}{\hbar \omega^3} \frac{1}{\tau}$$

$$\frac{dI}{dz} = \frac{\pi^2 c^3}{\hbar \omega^3 \tau} \rho(\omega) \Delta N \quad \text{—— (i)}$$

As

$$I = \frac{E}{At}$$

$$I = \frac{E}{A\frac{s}{v}} \qquad \because s = vt, \quad \frac{s}{v} = t$$

refrective index $= n$

$$n = \frac{c}{v}$$

$$v = \frac{c}{n}$$

$$I = \frac{E\,v}{Volu} \qquad \because Volume = A \times S$$

$$I = \rho(\omega) v \qquad \rho(\omega) = \frac{E}{Volu}$$

$$I = \rho(\omega) \frac{c}{n}$$

put $\rho(\omega)$ in eq (i)

$$\frac{In}{c} = \rho(\omega)$$

$$\frac{dI}{dz} = \frac{\pi^2 c^3}{\hbar \omega^3 \tau} \frac{I n}{c} \Delta N$$

$$= \frac{\pi^2 c^2 I_{(z)} n \Delta N}{\hbar \omega^3 \tau}$$

$$= \frac{\pi^2 c^2 n \Delta N}{\hbar \omega^3 \tau} I_{(z)}$$

$$\frac{dI}{dz} = G(\omega) \, I_{(z)} \qquad G(\omega) = \frac{\pi^2 c^2 n \Delta N}{\hbar \omega^3 \tau}$$

$$\int \frac{dI}{I} = G(\omega) \int dz$$

$$\ln I = G(\omega) z + C$$

$$\ln I = G(\omega) z + \ln(I_0) \qquad C = \ln(I_0)$$

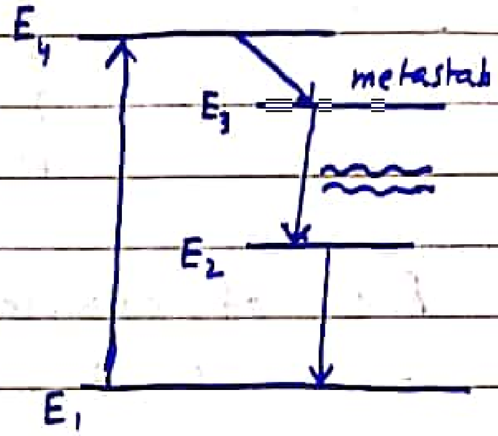
$$\ln I - \ln I_0 = G(\omega) z$$

$$\ln \left(I/I_0 \right) = G(\omega) z$$

$$I/I_0 = e^{G(\omega) z}$$

$$I_{(z)} = I_0 \, e^{G(\omega) z}$$

4 - level lasing system



1- rapidly transition from level E_2 to level E_1 (spontanious)

we are interested in the rate equations of level E_3 and level E_2

$E_3 - E_2$ = energy of laser

R_3 = rate at which atoms are pumped to level E_4, these atoms make quick ~~to~~ non radiative transition to level E_3

So

→ R_3 represents the rate at which the atoms reach to level E_3

Rate of change of population of level E_3 (on next page)

→ R_2 denote the rate at which atoms are pumped to level E_2

page 2

(i) — $$\frac{dN_3}{dt} = R_3 + W_{32}N_2 - W_{32}N_3 - T_{32}N_3$$

$2 \to 3$ absorption $3 \to 2$ stimulated emission

Rate of change of population of level E_2

(ii) — $$\frac{dN_2}{dt} = R_2 + W_{32}N_3 - W_{32}N_2 + T_{32}N_3 - T_{21}N_2$$

$3 \to 2$ stimulated emission $2 \to 3$ absorption

steady state condition

$$\frac{dN_2}{dt} = \frac{dN_3}{dt} = 0$$

$$0 = R_3 - \cancel{W_{32}(N_3 - N_2)} - \cancel{T_{32}}N_3$$

$$0 = R_2 + \cancel{W_{32}(N_3 - N_2)} + \cancel{T_{32}}N_3 - T_{21}N_2$$

$$0 = R_2 + R_3 - T_{21}N_2$$

$$R_2 + R_3 = T_{21} N_2$$

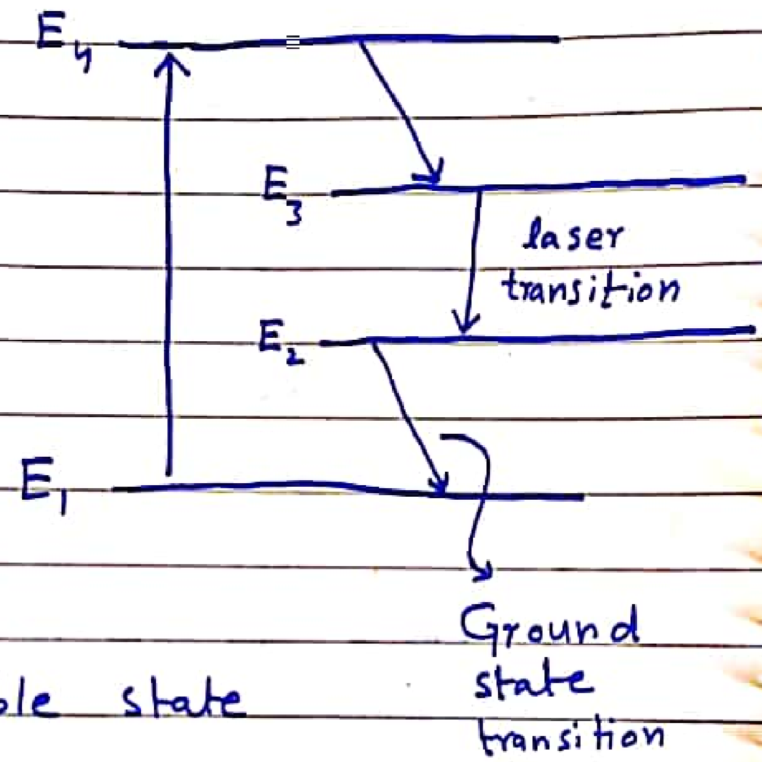
$$R_3 = T_{21} N_2 \qquad R_3 >> R_2$$

$$N_2 = R_3 / T_{21} \qquad R_2 \approx 0$$

$$\frac{dN_3}{dt} = R_3 - W_{32}(N_3 - N_2) - T_{32} N_3$$

$$0 = R_3 - W_{32}\left(N_3 - \frac{R_3}{T_{21}}\right) - T_{32} N_3$$

4- level lasing system



E_3 = metastable state

$E_3 \longrightarrow E_2$ laser transition

E_4 = pumping level

atoms are pumped from E_1 to E_4 they make quick non-radiative transition to level E_3

atoms do not accumulate at E_2 level otherwise sustained population inversion will not be possible

For this purpose it is important

i) level E_2 has very small life time

ii) Normal thermal processes do not populate this level $E_2 >> KT$
it means, this level has to be sufficiently above the ground level

we are interested only in level E_3 and E_2, whose transition gives laser

R_3 be the rate at which atoms are pumped to level E_4 and from there atom make quick transition to level E_3. So effectively R_3 represents the rate at which the atoms reach level E_3.

Now we can write

Rate of change of population of level E_3

$$\frac{dN_3}{dt} = R_3 - W_{32} N_3 + W_{32} N_2 - T_{32} N_3$$

$$\frac{dN_3}{dt} = R_3 - W_{32}N_3 + W_{32}N_2 - T_{32}N_3$$

$3 \to 2$ stimulated emission

$2 \to 3$ absorption

$3 \to 2$ spontanious emission

$$= R_3 - W_{32}(N_3 - N_2) - T_{32}N_3$$

Similarly we can write ~~eq~~ rate equation for level E_2

The rate of change of population of level E_2 is —2

$$\frac{dN_2}{dt} = + W_{32}N_3 - W_{32}N_2 + T_{32}N_3 - T_{21}N_2$$

$$\frac{dN_2}{dt} = W_{32}(N_3 - N_2) + T_{32}N_3 - T_{21}N_2$$

At stead state condition

$$\frac{dN_3}{dt} = \frac{dN_2}{dt} = 0$$

$$\frac{dN_3}{dt} = 0 = R_3 - W_{32}(\cancel{N_3 - N_2}) - \cancel{T_{32} N_3} \quad \text{— (i)}$$

also

$$\frac{dN_2}{dt} = 0 = W_{32}(\cancel{N_3 - N_2}) + \cancel{T_{32} N_3} - T_{21} N_2 \quad \text{— ii}$$

add eq (i) and (ii)

$$0 = R_3 - T_{21} N_2$$

$$R_3 = T_{21} N_2$$

$$R_3 = T_{21} N_2$$

$$N_2 = \frac{R_3}{T_{21}}$$

put the value of N_2 in equation $\frac{dN_3}{dt}$

we get

$$\frac{dN_3}{dt} = R_3 - W_{32}(N_3 - N_2) - T_{32} N_3$$

$$0 = R_3 - W_{32}\left(N_3 - \frac{R_3}{T_{21}}\right) - T_{32} N_3$$

$$R_3 = W_{32}\left(\frac{N_3 T_{21} - R_3}{T_{21}}\right) + T_{32} N_3$$

$$R_3 = (W_{32}T_{21}N_3 - W_{32}R_3 + T_{32}T_{21}N_3)/T_{21}$$

$$T_{21}R_3 = N_3(W_{32} + T_{32})T_{21} - W_{32}R_3$$

$$R_3(T_{21} + W_{32}) = N_3(W_{32} + T_{32})T_{21}$$

$$R_3\left(\frac{T_{21} + W_{32}}{T_{21}}\right) = N_3(W_{32} + T_{32})$$

$$R_3\left(\frac{T_{21} + W_{32}}{T_{21}}\right)\frac{1}{(W_{32} + T_{32})} = N_3$$

As we know that $N_2 = \frac{R_3}{T_{21}}$

$$N_3 - N_2 = \left[R_3\left(\frac{T_{21} + W_{32}}{T_{21}}\right)\frac{1}{(W_{32} + T_{32})}\right] - \frac{R_3}{T_{21}}$$

$$= R_3\left(\frac{1}{T_{32} + W_{32}}\right)\left[\frac{T_{21} + W_{32}}{T_{21}} - \frac{(T_{32} + W_{32})}{T_{21}}\right]$$

$$= \left[\frac{T_{21} + \cancel{W_{32}} - T_{32} - \cancel{W_{32}}}{T_{21}}\right]$$

$$N_3 - N_2 = R_3\left(\frac{1}{T_{32}+W_{32}}\right)\left[\frac{T_{21} - T_{32}}{T_{21}}\right]$$

for $N_3 - N_2 > 0$ or positive

$T_{21} >> T_{32}$

$A_{21} >> A_{32}$ $\quad \because A_{21} = \frac{1}{\tau_{21}}$

$\frac{1}{\tau_{21}} >> \frac{1}{\tau_{32}}$ $\quad A_{32} = \frac{1}{\tau_{32}}$

$\tau_{1} << \tau_{32}$

$\because T_{21} = A_{21} + S_{21}$

for $S_{21} \approx 0$

$T_{21} \approx A_{21}$