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Inner Product Spaces

- Inner product : represented by angle brackets $\langle \mathbf{u}, \mathbf{v} \rangle$

Let **u**, **v**, and **w** be vectors in a vector space *V*, and let *c* be any scalar. An inner product on *V* is a function that associates a real number $\langle \mathbf{u}, \mathbf{v} \rangle$ with each pair of vectors **u** and **v** and satisfies the following axioms (abstraction definition from the properties of dot product in Theorem 5.3 on Slide 5.12)

(1) $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ (commutative property of the inner product)

(2) $\langle \mathbf{u}, \mathbf{v}+\mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$ (distributive property of the inner product over vector addition)

(3) $k\langle \mathbf{u}, \mathbf{v} \rangle = \langle k\mathbf{u}, \mathbf{v} \rangle$ (associative property of the scalar multiplication and the inner product)

(4) $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$

(5) $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$

- Note:

$$\mathbf{u} \cdot \mathbf{v} = \text{dot product (Euclidean inner product for } R^n\text{)}$$

$$<\mathbf{u}\,,\,\mathbf{v}> = \text{general inner product for a vector space } V$$

- Note:

A vector space V with an inner product is called an inner product space

Vector space: $(V,\ +,\ \cdot)$

Inner product space: $(V,\ +,\ \cdot\ ,\ <,\ >)$

- Properties of inner products

Let **u**, **v**, and **w** be vectors in an inner product space V, and let c be any real number

(1) $\langle \mathbf{0}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{0} \rangle = 0$

(2) $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$

(3) $\langle \mathbf{u}, c\mathbf{v} \rangle = c \langle \mathbf{u}, \mathbf{v} \rangle$

- The definition of norm (or length), distance, angle, orthogonal, and normalizing for general inner product spaces closely parallel to those based on the dot product in Euclidean n-space

- Norm (length) of $\mathbf{u}$:

$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$$

- Distance between $\mathbf{u}$ and $\mathbf{v}$:

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{\langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle}$$

- Angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$:

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \, \|\mathbf{v}\|}, \; 0 \le \theta \le \pi$$

- Orthogonal: $(\mathbf{u} \perp \mathbf{v})$

$\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$

- Properties of norm:

1)

(2) $\|\mathbf{u}\| \geq 0$ if and only if

(3) $\|\mathbf{u}\| = 0$ $\mathbf{u} = \mathbf{0}$

$\|c\mathbf{u}\| = |c|\,\|\mathbf{u}\|$

- Properties of distance: (the same as the properties for the dot product in R^n on Slide 5.9)

(1) $d(\mathbf{u}, \mathbf{v}) \geq 0$

(2) $d(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$

(3) $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$

Let **u** and **v** be vectors in an inner product space V

(1) Cauchy-Schwarz inequality:

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|$$

(2) Triangle inequality:

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

(3) Pythagorean theorem:

u and **v** are orthogonal if and only if

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

Orthogonal and Orthogonal basis

- Orthogonal vectors:

  Two vectors **u** and **v** in R^n are orthogonal (perpendicular) if

$$\mathbf{u} \cdot \mathbf{v} = 0$$

- Note:

  The vector **0** is said to be orthogonal to every vector

- Ex : Finding orthogonal vectors

Determine all vectors in R^n that are orthogonal to $\mathbf{u} = (4, 2)$

Sol:

$$\mathbf{u} = (4\,,2) \quad \text{Let} \quad \mathbf{v} = (v_1\,, v_2)$$

$$\Rightarrow \quad \mathbf{u} \cdot \mathbf{v} = (4\,,2) \cdot (v_1\,, v_2)$$

$$= 4v_1 + 2v_2$$

$$= 0$$

$$\Rightarrow \quad v_1 = \frac{-t}{2} \quad , \quad v_2 = t$$

$$\therefore \quad \mathbf{v} = \left(\frac{-t}{2}, t \right), \; t \in R$$

Orthogonal basis

- Definition:
  - a,b in V, $a \perp b$ if $(a|b)=0$.
  - The zero vector is orthogonal to every vector.
  - An orthogonal set S is a set s.t. all pairs of distinct vectors are orthogonal.
  - An orthonormal set S is an orthogonal set of unit vectors.
  - Every nonzero finite dimension inner product space has an orthogonal basis

Theorems :

- If S={v1,v2,…,vn} is an orthogonal set of nonzero vectorin an inner product space V then S is linearly independent.
- Any orthogonal set of n nonzero vector in Rn is basis for Rn.
- If S={v1,v2,…,vn} is an orthonormal basis for an inner product space V, and u is any vector in V then it can be expressed as a linear combination of v1,v2,…,vn.

$$u=\langle u,v1\rangle v1 + \langle u,v2\rangle v2 + \ldots + \langle u,vn\rangle vn$$

- If S is an orthonormal basis for an n-dimensional inner product space, and if coordinate vectors of u & v with respect to S are [u]s=(a1,a2,…,an) and [v]s=(b1,b2,…,bn) then

$$\|u\|=\sqrt{(a_1)^2 + (a_2)^2 + \ldots + (a_n)^2}$$

$$d(u,v)=\sqrt{(a1-b1)2 + (a2-b2)2 + \ldots + (an-bn)2}$$

$$\langle u,v\rangle = a1b1 + a2b2 + \ldots + anbn$$

$$= [u]s.[v]s.$$

Gram Schmidt Process

- Gram schidt process to orthogonalisation of a set of vectors.
- Let $\{u_{1,} u_{2,} u_3\}$ be the given set of vector which is basis for vector space V.
- We shall constuct an orthogonal set$\{v_1 v_2 v_3\}$ of vector of V which becomes basis for V as under.
- Consider the vector space with the Euclidean inner product. Apply the Gram–Schmidt process to transform the basis vectors , , into an orthogonal basis ; then normalize the orthogonal basis vectors to obtain an orthonormal basis .

- Step 1. Let $V_1 = U_1$
- Step 2. $v_2 = u_2 - \frac{<u_2, v_1> v_1}{||v_1||^2}$
- Step 3. $v_3 = u_3 - \frac{<u_3, v_1> v_1}{||v_1||^2} - \frac{<u_3, v_2> v_2}{||v_2||^2}$ and so on

 ………………………

- And orthonormal basis are
- $\{\frac{v_1}{||v_1||}, \frac{v_2}{||v_2||}, \frac{v_3}{||v_3||}\}$

Use gram schmidt process the set u_1 $=(1,1,1) u_2= (-1,1,0) u_3=(1,2,1)$

- $v_1 = u_1 = (1,1,1)$ $\quad ||v_1||^2 = 1+1+1 = 3$
- $u_2 = (-1,1,0)$
- $<u_2, v_1>=<(-1,1,0)(1,1,1)>$
- $=-1+1+0$

$=0$

$v_2 = u_2 - \frac{<u_2, v_1> v_1}{||v_1||^2}$

$=(-1,1,0) - \frac{(0)(1,1,1)}{3}$

$v_2 = (-1,1,0)$ $\quad ||v_2||^2 = 2$

- Orthogonal basis set is
- $\{(1,1,1)(-1,1,0)\ (\frac{1}{6},\frac{1}{6},-\frac{1}{3})\}$
- Now,orthonormal basis are
- $\{\frac{v_1}{||v_1||},\frac{v_2}{||v_2||},\frac{v_3}{||v_3||}\}$
- $\{(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}})(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0)(\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}},-\frac{2}{\sqrt{6}})\}$

Orthogonal complements

- Let W be a subspace of on inner product space V. A vector u in V is orthogonal to W if is orthogonal to every vector in W. The set of all vector in V that are orthogonal to W is called orthogonal complement of W and is denoted by $W^{\perp}$.

Properties of Orthogonal complements

If W is a subspace of inner product space V than

- A vector u is in $W^{\perp}$ if and only if u is orthogonal to every spans W.

- The only vector common to W and $W^{\perp}$ is 0.

- $W^{\perp}$ is subspace of v.

- $W^{\perp} W^{\perp} = W$

Ex : Fine the basis for an orthogonal complement of the subspace of W span by the vector

u_1 =(2,0,-1)

u_2 =(4,0,-2)

- Let W=span $\{u_1, u_2\}$

  u_1 =(2,0,-1)

  u_2 =(4,0,-2)

We have

$$\begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \end{bmatrix}$$

The homo. system is

AX=0; $\quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

The A.M. is

$$\left[\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 4 & 0 & -2 & 0 \end{array}\right]$$

(1÷2)R1

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1/2 & 0 \\ 4 & 0 & -2 & 0 \end{array}\right]$$

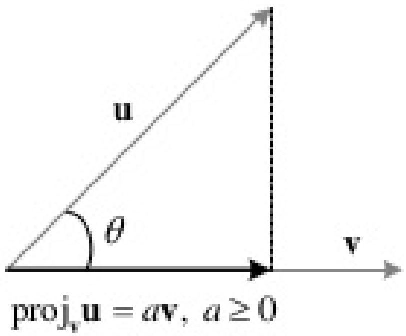
R2 +(-4)R1

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1/2 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$$\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1\div 2t_2 \\ t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ t_2 \end{bmatrix} t_1 + \begin{bmatrix} 1\div 2 \\ 0 \\ 1 \end{bmatrix} t_2$$

$w^{\perp}$ = { (0,1,0), (1÷2,0,1)

- Orthogonal projections : For the dot product function in R^n, we define the orthogonal projection of **u** onto **v** to be $\text{proj}_{\mathbf{v}}\mathbf{u} = a\mathbf{v}$ (a scalar multiple of **v**), and the coefficient a can be derived as follows



Consider $a \geq 0$, $\|a\mathbf{v}\| = |a|\,\|\mathbf{v}\| = a\|\mathbf{v}\| = \|\mathbf{u}\|\cos\theta$

$$= \frac{\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta}{\|\mathbf{v}\|} = \frac{\|\mathbf{u}\|\,\|\mathbf{v}\|}{\|\mathbf{v}\|}\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|}$$

$$\Rightarrow a = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}} \quad \Rightarrow \text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}$$

- For inner product spaces:

  Let **u** and **v** be two vectors in an inner product space V. If $\mathbf{v} \neq \mathbf{0}$, then the orthogonal projection of **u** onto **v** is given by

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\langle \mathbf{v}, \mathbf{v}\rangle}\mathbf{v}$$

- Ex : Finding an orthogonal projection in R^3

Use the Euclidean inner product in R^3 to find the orthogonal projection of $\mathbf{u} = (6, 2, 4)$ onto $\mathbf{v} = (1, 2, 0)$

Sol:

$$\Theta \ \langle \mathbf{u}\,,\,\mathbf{v}\rangle = (6)(1) + (2)(2) + (4)(0) = 10$$

$$\langle \mathbf{v}\,,\,\mathbf{v}\rangle = 1^2 + 2^2 + 0^2 = 5$$

$$\therefore \ \text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\langle \mathbf{u}\,,\,\mathbf{v}\rangle}{\langle \mathbf{v}\,,\,\mathbf{v}\rangle}\mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{10}{5}(1\,,2\,,0) = (2\,,4\,,0)$$

- Least Squares Approximation :

$$\underset{m\times n}{A}\underset{n\times 1}{\mathbf{x}} = \underset{m\times 1}{\mathbf{b}} \qquad \text{(A system of linear equations)}$$

(1) When the system is consistent, we can use the Gaussian elimination with the back substitution to solve for $\mathbf{x}$

(2) When the system is inconsistent, only the "best possible" solution of the system can be found, i.e., to find a solution of $\mathbf{x}$ for which the difference (or said the error) between $A\mathbf{x}$ and $\mathbf{b}$ is smallest

Note: the system of linear equations $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of A

- Least Squares Approximation :

  Given a system $A\mathbf{x} = \mathbf{b}$ of m linear equations in n unknowns, the least squares problem is to find a vector $\mathbf{x}$ in R^n that minimizes the distance between $A\mathbf{x}$ and $\mathbf{b}$, i.e., $\|A\mathbf{x} - \mathbf{b}\|$ with respect to the Euclidean inner product in R^n. Such vector is called a least squares solution of $A\mathbf{x} = \mathbf{b}$

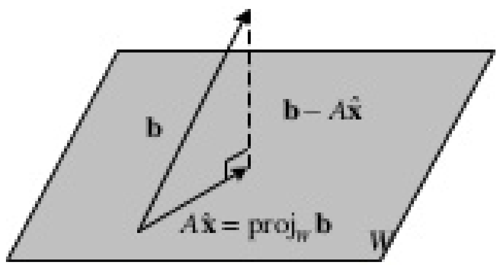
※ The term least squares comes from the fact that minimizing $\|A\mathbf{x} - \mathbf{b}\|$ is equivalent to minimizing $\|A\mathbf{x} - \mathbf{b}\|^2 = (A\mathbf{x} - \mathbf{b}) \cdot (A\mathbf{x} - \mathbf{b})$, which is a sum of squared errors

$A \in M_{m \times n}$

$\mathbf{x} \in R^n$

$A\mathbf{x} \in CS(A)$

Define $W = CS(A)$, and the problem to find $\hat{\mathbf{x}}$ such that $A\hat{\mathbf{x}}$ is closest to $\mathbf{b}$ is equivalent to find the vector in $CS(A)$ closest to $\mathbf{b}$, that is $\text{proj}_W \mathbf{b}$



Thus $A\hat{\mathbf{x}} = \text{proj}_W \mathbf{b}$ (To find the best solution $\hat{\mathbf{x}}$ which should satisfy this equation)

$\Rightarrow (\mathbf{b} - \text{proj}_W \mathbf{b}) = (\mathbf{b} - A\hat{\mathbf{x}}) \perp W \Rightarrow (\mathbf{b} - A\hat{\mathbf{x}}) \perp CS(A)$

$\Rightarrow \mathbf{b} - A\hat{\mathbf{x}} \in CS(A)^{\perp} = NS(A^T)$ (The nullspace of A^T is a solution space of the homogeneous system $A^T\mathbf{x}=\mathbf{0}$)

$\Rightarrow A^T(\mathbf{b} - A\hat{\mathbf{x}}) = \mathbf{0}$

$\Rightarrow A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$ (the $n \times n$ linear system of normal equations associated with $A\mathbf{x} = \mathbf{b}$)

Orthogonal Basis

- **Example 1**: $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$.

$$\mathbf{x}_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Find an orthogonal basis for W.

Gram-Schmidt

Algorithm to find an orthogonal basis, given a basis

1. Let first vector in orthogonal basis be first vector in original basis
2. Next vector in orthogonal basis is component of next vector in original basis orthogonal to the previously found vectors.

Next vector less the projection of that vector onto subspace defined by the set of vectors in the orthogonal set

Scaling may be convenient

1. Repeat step 2 for all other vectors in original basis

Gram-Schmidt - Example

- **Example 2**: $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$.

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Find an orthogonal basis for W.

Inner Product - Definition

Definition: An **inner product** on a vector space *V* is a function that to each pair of vectors **u** and **v** in *V*, associates a real number <**u**,**v**> and satisfies the following axioms for all **u**, **v**, **w** in ***V*** and all scalars *c*:

1. **<u,v> = <v,u>**
2. **<u+v,w> = <u,w> + <v,w>**
3. **<**c**u,v>** = c**<u,v>**
4. **<u,u>** ≥ 0 & <**u**,**u**>=0 iff **u=0**

Inner Product Space

- A vector space with an inner product is called an inner product space.
- Example - R*n* with the dot product is an inner product space

Inner Product - Example

u & **v** in R2, **u** = ($u1$, $u2$), **v**=($v1$, $v2$)

Show $\langle \mathbf{u},\mathbf{v} \rangle = 4u1u2 + 5v1v2$ defines an inner product