FINDING EIGENVALUES AND EIGENVECTORS

EXAMPLE 1: Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}.$$

SOLUTION:

- In such problems, we first find the **eigenvalues** of the matrix.

FINDING EIGENVALUES

- To do this, we find the values of λ which satisfy the **characteristic equation** of the matrix A, namely those values of λ for which

$$\det(A - \lambda I) = 0,$$

where I is the 3×3 **identity matrix**.

- Form the matrix $A - \lambda I$:

$$A - \lambda I = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{pmatrix}.$$

Notice that this matrix is just equal to A with λ subtracted from each entry on the main diagonal.

- Calculate $\det(A - \lambda I)$:

$$\begin{aligned} \det(A-\lambda I) &= (1-\lambda)\begin{vmatrix} -5-\lambda & 3 \\ -6 & 4-\lambda \end{vmatrix} - (-3)\begin{vmatrix} 3 & 3 \\ 6 & 4-\lambda \end{vmatrix} + 3\begin{vmatrix} 3 & -5-\lambda \\ 6 & -6 \end{vmatrix} \\ &= (1-\lambda)\left((-5-\lambda)(4-\lambda) - (3)(-6)\right) + 3(3(4-\lambda) - 3\times 6) + 3(3\times(-6) - (-5-\lambda)6) \\ &= (1-\lambda)(-20+5\lambda-4\lambda+\lambda^2+18) + 3(12-3\lambda-18) + 3(-18+30+6\lambda) \\ &= (1-\lambda)(-2+\lambda+\lambda^2) + 3(-6-3\lambda) + 3(12+6\lambda) \\ &= -2+\lambda+\lambda^2+2\lambda-\lambda^2-\lambda^3-18-9\lambda+36+18\lambda \\ &= 16+12\lambda-\lambda^3. \end{aligned}$$

- Therefore

$$\det(A - \lambda I) = -\lambda^3 + 12\lambda + 16.$$

REQUIRED: To find solutions to $\det(A - \lambda I) = 0$ i.e., to solve

$$\lambda^3 - 12\lambda - 16 = 0. \qquad (1)$$

  * Look for **integer** valued solutions.

- * Such solutions **divide** the **constant** term (-16). The list of possible integer solutions is

$$\pm 1, \pm 2, \pm 4, \pm 8, \pm 16.$$

- * Taking $\lambda = 4$, we find that $4^3 - 12.4 - 16 = 0$.
- * Now factor out $\lambda - 4$:

$$(\lambda - 4)(\lambda^2 + 4\lambda + 4) = \lambda^3 - 12\lambda^2 + 16.$$

- * Solving $\lambda^2 + 4\lambda + 4$ by formula[1] gives

$$\lambda = \frac{-4 \pm \sqrt{4^2 - 4.1.4}}{2} = \frac{-4 \pm 0}{2},$$

and so $\lambda = -2$ (a repeated root).

- Therefore, the eigenvalues of A are $\lambda = 4, -2$. ($\lambda = -2$ is a repeated root of the **characteristic equation**.)

FINDING EIGENVECTORS

- Once the **eigenvalues** of a matrix (A) have been found, we can find the **eigenvectors** by Gaussian Elimination.
- **STEP 1**: For each eigenvalue λ, we have

$$(A - \lambda I)\mathbf{x} = \mathbf{0},$$

where x is the **eigenvector** associated with **eigenvalue** λ.

- **STEP 2**: Find $\mathbf{x}$ by Gaussian elimination. That is, convert the augmented matrix

$$\left(A - \lambda I \vdots \mathbf{0} \right)$$

to row echelon form, and solve the resulting linear system by back substitution.

We find the **eigenvectors** associated with each of the **eigenvalues**

- **Case 1:** $\lambda = 4$

  – We must find vectors $\mathbf{x}$ which satisfy $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

[1] To find the roots of a quadratic equation of the form $ax^2 + bx + c = 0$ (with $a \neq 0$) first compute $\Delta = b^2 - 4ac$, then if $\Delta \geq 0$ the roots exist and are equal to $x = \frac{-b-\sqrt{\Delta}}{2a}$ and $x = \frac{-b+\sqrt{\Delta}}{2a}$.

– First, form the matrix $A - 4I$:

$$A - 4I = \begin{pmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{pmatrix}.$$

– Construct the augmented matrix $\left(A - \lambda I \vdots \mathbf{0}\right)$ and convert it to row echelon form

$$\begin{pmatrix} -3 & -3 & 3 & 0 \\ 3 & -9 & 3 & 0 \\ 6 & -6 & 0 & 0 \end{pmatrix} \begin{matrix} \text{R1} \\ \text{R2} \\ \text{R3} \end{matrix} \xrightarrow{R1 \to -1/3 \times R3} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 3 & -9 & 3 & 0 \\ 6 & -6 & 0 & 0 \end{pmatrix} \begin{matrix} \text{R1} \\ \text{R2} \\ \text{R3} \end{matrix}$$

$$\xrightarrow[R3 \to R3 - 6 \times R1]{R2 \to R2 - 3 \times R1} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & -12 & 6 & 0 \\ 0 & -12 & 6 & 0 \end{pmatrix} \begin{matrix} \text{R1} \\ \text{R2} \\ \text{R3} \end{matrix}$$

$$\xrightarrow{R2 \to -1/12 \times R2} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & -12 & 6 & 0 \end{pmatrix} \begin{matrix} \text{R1} \\ \text{R2} \\ \text{R3} \end{matrix}$$

$$\xrightarrow{R3 \to R3 + 12 \times R2} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \text{R1} \\ \text{R2} \\ \text{R3} \end{matrix}$$

$$\xrightarrow{R1 \to R1 - R2} \begin{pmatrix} 1 & 0 & -1/2 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \text{R1} \\ \text{R2} \\ \text{R3} \end{matrix}$$

– Rewriting this augmented matrix as a linear system gives

$$\begin{aligned} x_1 - 1/2x_3 &= 0 \\ x_2 - 1/2x_3 &= 0 \end{aligned}$$

So the eigenvector $\mathbf{x}$ is given by:

$$\mathbf{x} = \begin{pmatrix} x_1 = \frac{x_3}{2} \\ x_2 = \frac{x_3}{2} \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}$$

For any real number $x_3 \neq 0$. Those are the **eigenvectors of A associated with the eigenvalue** $\lambda = 4$.

- **Case 2**: $\lambda = -2$

  – We seek vectors $\mathbf{x}$ for which $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

  – Form the matrix $A - (-2)I = A + 2I$

$$A + 2I = \begin{pmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{pmatrix}.$$

– Now we construct the augmented matrix $\left(A - \lambda I \,\vdots\, \mathbf{0}\right)$ and convert it to row echelon form

$$\begin{pmatrix} 3 & -3 & 3 & 0 \\ 3 & -3 & 3 & 0 \\ 6 & -6 & 6 & 0 \end{pmatrix} \begin{matrix} \text{R1} \\ \text{R2} \\ \text{R3} \end{matrix} \quad \xrightarrow{R1 \to 1/3 \times R3} \quad \begin{pmatrix} 1 & -1 & 1 & 0 \\ 3 & -3 & 3 & 0 \\ 6 & -6 & 6 & 0 \end{pmatrix} \begin{matrix} \text{R1} \\ \text{R2} \\ \text{R3} \end{matrix}$$

$$\xrightarrow[R3 \to R3 - 6 \times R1]{R2 \to R2 - 3 \times R1} \quad \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \text{R1} \\ \text{R2} \\ \text{R3} \end{matrix}$$

– When this augmented matrix is rewritten as a linear system, we obtain

$$x_1 + x_2 - x_3 = 0,$$

so the eigenvectors $\mathbf{x}$ associated with the eigenvalue $\lambda = -2$ are given by:

$$\mathbf{x} = \begin{pmatrix} x_1 = x_3 - x_2 \\ x_2 \\ x_3 \end{pmatrix}$$

– Thus

$$\mathbf{x} = \begin{pmatrix} x_3 - x_2 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad \text{for any } x_2, x_3 \in \mathbb{R}\backslash\{0\}$$

are the **eigenvectors of A associated with the eigenvalue** $\lambda = -2$.