Properties of Matrix Transformations

Theorem 4.9.1: For every matrix A the matrix transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ has the following properties for all vectors u and v in $\mathbb{R}^n$ and for every scalar k:

(a) $T_A(0) = 0$

(b) $T_A(ku) = kT_A(u)$ (Homogeneity property)

(c) $T_A(u + v) = T_A(u) + T_A(v)$ (Additivity property)

(d) $T_A(u - v) = T_A(u) - T_A(v)$

Note: We can extend part (c) of Theorem 4.9.1 to three or more vectors.

$$T_A(c_1v_1 + c_2v_2) = c_1T_A(v_1) + c_2T_A(v_2) \quad \text{(Exercise: show this is true)}$$

In fact if $v_1, v_2, \ldots, v_k$ are vectors in $\mathbb{R}^n$ and $c_1, c_2, \ldots, c_k$ are any scalars then

$$T_A(c_1v_1 + \ldots + c_kv_k) = c_1T_A(v_1) + \ldots + c_kT_A(v_k)$$

Theorem 4.10.2 T:$\mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation if and only if the following relationships hold for all vectors u and v in $\mathbb{R}^n$ and for each scalar k:

(a) T(u+v)=T(u)+T(v) (Additivity property)

(b) T(ku)=kT(u) (Homogeneity property)

Theorem 4.9.2: If $T_A : \mathbb{R}^n \to \mathbb{R}^m$ and $T_B : \mathbb{R}^n \to \mathbb{R}^m$ are matrix transformations, and if $T_A(x) = T_B(x)$ for every vector x in $\mathbb{R}^n$, then A=B.

Given a matrix transformation we can find the matrix representing the transformation

Standard Matrix for a matrix transformation:

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a matrix transformation. To find the matrix A such that $T(x) = Ax$, consider the standard basis vectors of $\mathbb{R}^n$, $e_1, e_2, \ldots, e_n$. Next calculate the image of these vectors under the transformation T_A, that is $T_A(e_1) = Ae_1, T_A(e_2) = Ae_2, \ldots, T_A(e_n) = Ae_n$. Then matrix A is found by

$$A = \left[T_A(e_1)| \ldots |T_A(e_n)\right]$$

Examples:

1. $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $T(x) = 2x$. Find the standard matrix for this transformation.
   Solution: $e_1 = (1, 0), e_2 = (0, 1)$ standard basis vectors of $\mathbb{R}^2$. $T(e_1) = (2, 0), T(e_2) = (0, 2)$, then
   $$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

2. Find the standard matrix for the transformation defined by the equations

$$\begin{aligned} w_1 &= 3x_1 + 5x_2 - x_3 \\ w_2 &= 4x_1 - x_2 + x_3 \\ w_3 &= 3x_1 + 2x_2 - x_3 \end{aligned} \tag{1}$$

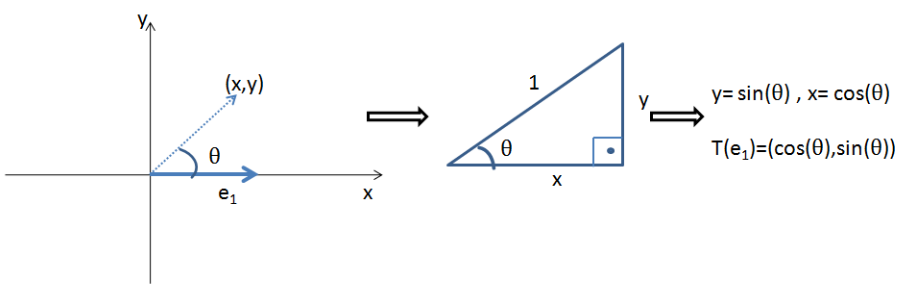
Solution: Note that domain and codomain of transformation are $\mathbb{R}^3$. Standar basis vectors of $\mathbb{R}^3$ are e_1, e_2, e_3.

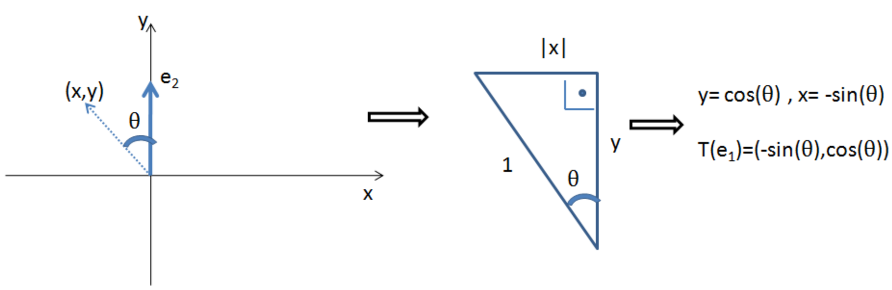
$$\begin{aligned} T(e_1) &= (3,4,3) \\ T(e_2) &= (5,-1,2) \\ T(e_3) &= (-1,1,-1) \end{aligned}$$

$$[T] = A = \begin{bmatrix} 3 & 5 & -1 \\ 4 & -1 & 1 \\ 3 & 2 & -1 \end{bmatrix}$$

Geometry of Matrix Transformations:

Rotation: Let θ be a fixed angle. Let T be an operator that rotates each vector x in $\mathbb{R}^2$ about the origin through the angle θ We can calculate the standard matrix $[T] = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix}$





Hence standard matrix of the rotation operator on $\mathbb{R}^2$

$$[T] = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Example: Use matrix multiplication to find the image of the vector $x = (3,4)$ under the rotation of $\frac{\pi}{4}$ about the origin.

Solution: $\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$. $[T] = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$. Then

$$T(x) = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{7}{\sqrt{2}} \end{bmatrix}$$

One can also write standard matrix for rotation operators in $\mathbb{R}^3$:

1)Counterclockwise rotation about the positive x-axis through an angle θ $[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$

2)Counterclockwise rotation about the positive y-axis through an angle θ $[T] = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$

3)Counterclockwise rotation about the positive z-axis through an angle θ $[T] = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Example: Use matrix multiplication to find the image of the vector $(-2, 1, 2)$ if it is rotated

(a) -30° about the x-axis

(b) 90° about the y-axis

(c) -45° about the z-axis

Solution: Recall trigonometric summation and substraction formulas

$$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$$

$$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(A)$$

1. To use the angle in the standard matrix formula we need positive angle, thus consider $\cos(2\pi - \frac{\pi}{6}) = \cos(\frac{\pi}{6})$, $\sin(2\pi - \frac{\pi}{6}) = -\sin(\frac{\pi}{6})$. Then

$$T_{\pi/6}(-2,1,2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ \frac{\sqrt{3}+2}{2} \\ \frac{2\sqrt{3}-1}{2} \end{bmatrix}$$

2.

$$T_{\pi/2}(-2,1,2) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

3. $\cos(2\pi - \frac{\pi}{4}) = \cos(\frac{\pi}{4}), \sin(2\pi - \frac{\pi}{4}) = -sin(\frac{\pi}{4})$

$$T_{\pi/4}(-2,1,2) = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{-3\sqrt{2}}{2} \\ \frac{-\sqrt{2}}{2} \\ 2 \end{bmatrix}$$