III.1 Galileo versus Maxwell

> I am convinced that the philosophers have had a harmful effect upon the progress of scientific thinking in removing certain fundamental concepts from the domain of empiricism, where they are under our control, to the intangible heights of the a priori. . . . This is particularly true of our concepts of time and space, which physicists have been obliged by the facts to bring down from the Olympus of the a priori in order to adjust them and put them in a serviceable condition.
>
> —A. Einstein[1]

Galilean transformation

Go back to the prelude, in which Galileo's ship was updated to Einstein's train. The observer on the train, Ms. Unprime, ascribed to some event the spatial coordinates (x, y, z) and temporal coordinate t. To the same event, the observer on the ground, Mr. Prime, assigns the coordinates (x', y', z') and t'. Denote the speed of the train by u, and choose the axis so that the train moves along the x-axis. Then the two sets of coordinates are related by

$$\begin{aligned} t' &= t \\ x' &= x + ut \\ y' &= y \\ z' &= z \end{aligned} \tag{1}$$

a set of relations known as the Galilean transformation. Consider a point on the train with $x = 0$. Plugging this into (1), we see that, for Mr. P on the ground, this point moves along according to $x' = ut = ut'$.

The innocuous looking equalities $y' = y$ and $z' = z$ actually represent an important consequence of Galileo's relativity principle. Call the y direction the vertical direction. We can supply sticks of a standard length L to Ms. U and Mr. P to build a fence.

To make sure that the sticks supplied to the two observers are identical, we can arrange for the woodcutter to ride in a train going by at speed $\frac{1}{2}u$ relative to Mr. P and $-\frac{1}{2}u$ relative to Ms. U. In other words, the coordinates of the woodcutter are given by $t' = t_w$

and $x' = x_w + \frac{1}{2}ut_w$. As far as the woodcutter is concerned, he is at rest, and Mr. P and Ms. U are going by him at the same speed but in opposite directions.[2] The woodcutter can toss the pre-cut sticks in identical ways to the two observers and their helpers. This long-winded digression is to answer any objection that the tossing of sticks from Mr. P to Ms. U, say, could have done something to the lengths of the sticks.

The top of the two fences is then given by $y = L$ and $y' = L$, respectively. The two lengths must agree, because as the two fences sweep past each other, the two observers could see whether one fence is taller than the other. In either case, Galileo's relativity principle, stating that two observers in relative uniform motion could not decide who is moving relative to the other, would be violated. Thus, we must have $y' = y$. Similarly, $z' = z$. The coordinates perpendicular to the direction of motion are unaffected by the motion.

The relation $x' = x + ut$ certainly does not violate Galileo's principle, since $x = x' + (-u)t'$. To Ms. U, she is at rest, but relative to her, Mr. P, sitting at $x' = 0$, is moving with speed $-u$ in the x direction.

We have set up the coordinates so that when $t' = t = 0$, we have $x' = x = 0$. Just as in chapter I.3, we can avoid having to line up the origins of the two coordinate systems by considering the separation between two events E_1 and E_2 in spacetime located at (t_1, x_1, y_1, z_1), (t'_1, x'_1, y'_1, z'_1), (t_2, x_2, y_2, z_2), and (t'_2, x'_2, y'_2, z'_2). Writing $\Delta t = t_2 - t_1$, $\Delta x = x_2 - x_1$, and so on, and $\Delta t' = t'_2 - t'_1$, $\Delta x' = x'_2 - x'_1$, and so forth, we have

$$\begin{aligned} \Delta t' &= \Delta t \\ \Delta x' &= \Delta x + u\Delta t \\ \Delta y' &= \Delta y \\ \Delta z' &= \Delta z \end{aligned} \qquad (2)$$

Since the y and z coordinates are just going along for the ride, we omit writing the transformation equations for them henceforth. Again, just as for rotations in chapter I.3, we can replace the finite differences Δt, Δx, and so on by infinitesimals dt, dx, and so forth:

$$\begin{aligned} dt' &= dt \\ dx' &= dx + udt \end{aligned} \qquad (3)$$

Adding velocities

The addition of velocities is so physically intuitive that almost everybody grasps it in everyday life. You are in a car speeding down the highway at 70 miles an hour. A fly trapped in the car flies forward at 3 miles an hour. To a hitchhiker standing by the roadside, the fly evidently moves forward at $70 + 3 = 73$ miles an hour, even though flies normally can't fly that fast. Indeed, if the hitchhiker also sees the fly moving forward at 3 miles an hour, it would have smashed into the rear window in an instant.

To formalize this intuitively obvious understanding, let us go back to the train. Ms. U tosses an object forward with velocity v; in other words, the object's trajectory is described by $x = vt$. (See figure 1, showing Ms. U as a stoker on Einstein's train.)

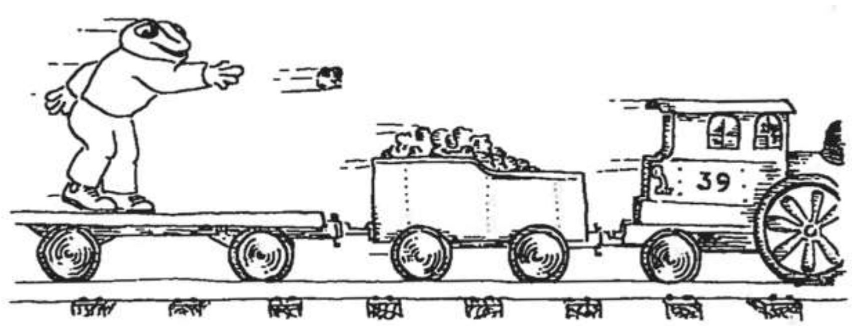

Figure 1 A lump of coal is tossed forward on a moving train. (Illustration adapted from *Fearful*.)

Simply plug this into (1) and we obtain, very slowly and carefully, the velocity seen by Mr. P:

$$v' \equiv \frac{dx'}{dt'} = \frac{d}{dt}(x + ut) = \frac{dx}{dt} + u = v + u \tag{4}$$

We just add the velocity of the object to the velocity of the train, as everybody would have felt intuitively. We can obtain the same result, perhaps a tad quicker, by going to (3) and dividing dx' by dt' to obtain

$$v' = \frac{dx'}{dt'} = \frac{dx + u dt}{dt} = \frac{dx}{dt} + u = v + u \tag{5}$$

The calculus book I read in high school warns the reader sternly that $\frac{dx}{dt}$ is a holistic (but of course that word did not become fashionable until much later) symbol of a single mathematical entity and is not to be thought of as dx divided by dt. I am telling you that at the level of rigor of theoretical physics it is okay. Just think of the differential dx as the difference Δx, divide by Δt, and then take the Newton-Leibniz limit. When we get to general relativity, we will be constantly manipulating differentials.

We now see that the invariance of Newtonian mechanics under the Galilean transformation follows merely because Newton's law involves the second derivative, so that

$$m\frac{d^2x'}{dt'^2} = m\frac{d}{dt'}\left(\frac{dx}{dt} + u\right) = m\frac{d^2x}{dt^2} \tag{6}$$

An important point here is that this derivation even tells us when Galilean invariance of Newtonian mechanics fails. If u changes in magnitude or in direction (we had chosen $\vec{u}$ to point in the x direction, but $\vec{u}$ is really a vector!), then (6) is changed to

$$\vec{F}' = m\vec{a}' = m\vec{a} + m\frac{d\vec{u}}{dt} \tag{7}$$

An ancient part of our brains interprets this extra term as an apparent additional force: our body feels it when the driver of the car (remember, the one with the fly trapped in it) speeding down the highway suddenly slams on the brake or zips around a sharp curve.

Even someone as dumb as a fly would feel the additional force $m\frac{d\vec{u}}{dt}$ as it smashes into the windshield. Unfortunate as well as dumb.

But for now, what the fly knows is advanced stuff for us; we will get to it when we discuss gravity. Let us check that the action for Newtonian mechanics is Galilean invariant. First, for simplicity, look at the action for a single free particle in one dimension:

$$S = \int dt' \tfrac{1}{2} m \left(\frac{dx'}{dt'} \right)^2 = \int dt \tfrac{1}{2} m \left(\frac{dx}{dt} + u \right)^2$$
$$= \int dt \tfrac{1}{2} m \left(\frac{dx}{dt} \right)^2 + u \int dt m \frac{dx}{dt} + u^2 \int dt \tfrac{1}{2} m \qquad (8)$$

The extra term linear in u in the Lagrangian is proportional to the integral of the derivative $\frac{dx}{dt}$. With fixed initial and final conditions, it is just an irrelevant additive constant. The term quadratic in u is also an additive constant. In other words, the change in the action S is just some additive constant whose variation vanishes.

This simple demonstration can be immediately generalized to the many-particle case with

$$S = \int dt \left\{ \sum_a \tfrac{1}{2} m_a \left(\frac{dx_a}{dt} \right)^2 - \sum_{a \neq b} V(x_a - x_b) \right\} \qquad (9)$$

Note that it is necessary for the interaction potential to depend on the difference $x_a - x_b = x'_a - x'_b$. The generalization to higher dimensional space is trivial.

Incidentally, you might have noticed that implicit in the argument is the assumption that the two observers in relative motion agree on the same mass. I have underlined this by writing m explicitly in (6) and (8). There is no m'. Galilean relativity requires that different observers measure the same mass.

Contrary to what the guy in the street might think, the principle of relativity did not start with Einstein, but, in a sense, was reestablished by Einstein's special relativity.

Showdown between Galileo and Maxwell

While the addition of velocities (4) is so intuitively obvious, even to a layperson not versed in physics (as in my everyday example of a speeding car), it came to play a central role in the looming crisis that confronted physics toward the end of the 19th century. In his monumental work, Maxwell finally gave a precise elucidation of the mystery of light, revealing it to be an undulating electromagnetic field. An electric field varying in space and time generates a magnetic field varying in space and time, which in turn generates an electric field varying in space and time, and thus the wave propagates through space and time. The speed of propagation c depends only on how oscillating electric and magnetic fields generate each other, and that, as the reader may recall or have heard, does not depend on the observer.

On that occasion with the fly in the car, I was riding in the back seat, and I had a camera[3] with me. I took a picture of a friend riding in the front seat next to the driver and the flash went off. Telling my friend that the speed of light is* $186{,}000 \times 3{,}600 = 669{,}600{,}000$ miles per hour, I asked my friend how fast a hitchhiker standing by the roadside would have seen the flash of light go by. Her answer, indeed the only intuitively reasonable and incontrovertible answer, was $70 + 669{,}600{,}000 = 669{,}600{,}070$ miles per hour.

But this contradicts Maxwell's equations.

To read this book, for the most part you do not need to have completely mastered Maxwell's theory of electromagnetism (although it would help). I will even derive it later. At this point in our development, the single most important point is that light does not obey the law of addition of velocities (4) that everyone took to be totally obvious. For light, both observers measure the same speed:

$$c = c \tag{10}$$

As I mentioned in chapter I.1, in the showdown between these two equations, (4) and (10), the law of addition of velocities blinked and had to be modified.

This great antinomy made him stuck

Various eminent physicists in the late 19th century realized that they could reconcile the contradiction between Maxwell's theory and the law of addition of velocities if they postulated that light, just like sound, had to propagate in a medium, an ether pervading the universe. The speed of light c determined by Maxwell's theory is the speed of light as seen by an observer at rest with respect to the ether. As the earth moves through the ether, the speed of light measured on earth would vary.

Notice that the existence of the ether would have profound implications for the foundation of physics, namely, that absolute rest could be defined as rest with respect to the ether. Ms. U and Mr. P could determine who is at rest and who is moving.

As you may have heard, the experimental evidence was against the infamous ether. In 1887 (when Einstein was 8 years old), Michelson and Morley performed a famous experiment to detect the ether and failed. By the way, Einstein claimed that he was guided solely by Maxwell's equations and had never heard of the experiment.

Indeed, Einstein even contemplated his own experimental setup to look for the ether. In an impromptu speech given in December 1922 in Kyoto, Japan, describing how he had discovered special relativity, he said that he had not doubted the existence of the ether and that he had even thought of an experiment using two thermocouples to measure the difference in the heat generated by two light rays, one moving in the same direction as the earth, the other in the opposite direction.

A pair of thermocouples to measure the difference in the heat generated by the two light rays, yeah right! You might have smiled: good old Albert was a far better theorist than experimentalist. Michelson and Morley had a far better idea, to interfere the two light rays. Putting that aside, you could sense Einstein's frustration. In 1922, he said something

* Since this true story took place in southern California, we use "royal" rather than "revolutionary" units here.