Functional Analysis	**Spring 2020**

Week 8-10

Instructor: Dr. Azhar Hussain — *Topic: Normed Spaces*

Definition 0.1 *Let X be a vector space (real or complex). A real-valued function $\|\cdot\| : X \to \mathbb{R}$ is said to be a norm on X if*

N1: $\|x\| \geq 0$;

N2: $\|x\| = 0 \;\Leftrightarrow\; x = 0$;

N3: $\|\alpha x\| = \alpha\|x\|$;

N4: $\|x + y\| \leq \|x\| + \|y\|$,

for all $x, y \in X$ and for all scalar α.

Remark 0.2 *The norm generalizes the concept of length of a vector in $\mathbb{R}^3$ to a general vector space X. In this case we write $|x| = \|x\|$.*

Definition 0.3 *A metric d on a vector space X can be defined by using the norm on X as*

$$d(x, y) = \|x - y\|.$$

The metric obtained in this way is called the metric induced by the norm.

M1: $d(x, y) = \|x - y\| \geq 0$ *(by N1), so* $d(x, y) \geq 0$;

M2: $d(x, y) = \|x - y\| = 0 \;\Leftrightarrow\; x - y = 0$ *(by N2),* $\Leftrightarrow\; x = y$;

M3: $d(x, y) = \|x - y\| = \| - (y - x)\| = | - 1|\|y - x\| = d(y, x)$ *(by N3)*;

M4:

$$\begin{aligned} d(x,y) = \|x-y\| &= \|x-z+z-y\| \\ &\leq \|x-z\| + \|z-y\| \\ &= d(x,z) + d(z,y), \end{aligned}$$

for all $x, y, z \in X$. Therefore, $d(x,y) = \|x-y\|$ is a metric on X. Hence (X,d) is a metric space.

Remark 0.4 *Every normed space is a metric space but the converse is not true in general.*

Example 0.5 *Consider $X = \{x : x \text{ is bounded or unbounded sequence}\}$ and the metric d on X is given by*

$$d(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i - \eta_i|}{1 + |\xi_i - \eta_i|}, \tag{1}$$

where $x = (\xi_i)$ and $y = (\eta_i)$. It is easy to verify that d is a metric on X. Now suppose that d is induced by norm $\|\cdot\|$ on X, i.e. $d(x,y) = \|x-y\|$. Then

$$\|x\| = d(x,0) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i|}{1 + |\xi_i|}. \tag{2}$$

Now

$$\begin{aligned} \|\alpha x\| &= \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\alpha \xi_i|}{1 + |\alpha \xi_i|} \\ &\neq |\alpha| \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i|}{1 + |\xi_i|} \\ &= |\alpha| \|x\|. \end{aligned} \tag{3}$$

This implies $\|\alpha x\| \neq \alpha \|x\|$. Therefore, X is not a normed space.

Definition 0.6 *A normed space is complete if it is complete as metric space. A complete norm space is called a Banach space.*

2.2-2 Euclidean space $\mathbf{R}^n$ and unitary space $\mathbf{C}^n$. These spaces were defined in 1.1-5. They are Banach spaces with norm defined by

$$\|x\| = \left(\sum_{j=1}^{n} |\xi_j|^2\right)^{1/2} = \sqrt{|\xi_1|^2 + \cdots + |\xi_n|^2}. \tag{3}$$

In fact, $\mathbf{R}^n$ and $\mathbf{C}^n$ are complete (cf. 1.5-1), and (3) yields the metric (7) in Sec. 1.1:

$$d(x, y) = \|x - y\| = \sqrt{|\xi_1 - \eta_1|^2 + \cdots + |\xi_n - \eta_n|^2}.$$

We note in particular that in $\mathbf{R}^3$ we have

$$\|x\| = |x| = \sqrt{\xi_1{}^2 + \xi_2{}^2 + \xi_3{}^2}.$$

This confirms our previous remark that the norm generalizes the elementary notion of the length $|x|$ of a vector.

2.2-3 Space l^p. This space was defined in 1.2-3. It is a Banach space with norm given by

$$\|x\| = \left(\sum_{j=1}^{\infty} |\xi_j|^p\right)^{1/p}. \tag{4}$$

In fact, this norm induces the metric in 1.2-3:

$$d(x, y) = \|x - y\| = \left(\sum_{j=1}^{\infty} |\xi_j - \eta_j|^p\right)^{1/p}.$$

Completeness was shown in 1.5-4.

2.2-4 Space l^∞. This space was defined in 1.1-6 and is a Banach space since its metric is obtained from the norm defined by

$$\|x\| = \sup_j |\xi_j|$$

and completeness was shown in 1.5-2.

2.2-5 Space $C[a, b]$. This space was defined in 1.1-7 and is a Banach space with norm given by

$$\|x\| = \max_{t \in J} |x(t)| \tag{5}$$

where $J = [a, b]$. Completeness was shown in 1.5-5.

Lemma 0.7 *Show that a metric d induced by a norm on a normed space X satisfy*

1. $d(x + a, y + a) = d(x, y)$;
2. $d(\alpha x, \alpha y) = |\alpha| d(x, y)$,

for all $x, y, a \in X$ and for any scalar α. Proof: *1. Since d is induced by the norm $\|\cdot\|$, therefore,*

$$d(x, y) = \|x - y\|.$$

Now,

$$\begin{aligned} d(x + a, y + a) &= \|x + a - y - a\| \\ &= \|x - y\| = d(x, y). \end{aligned}$$

2. For any scalar α

$$\begin{aligned} d(\alpha x, \alpha y) &= \|\alpha x - \alpha y\| \\ &= |\alpha| \|x - y\| \\ &= |\alpha| d(x, y). \end{aligned}$$

■

1 Properties of Normed Spaces

Definition 1.1 *Let X be a normed space and Y a nonempty subset of X. We say that Y is a subspace of X if it is a subspace as a vector space with the norm obtained by restricting the norm on X to the subset Y. The norm on Y is said to be induced by the norm on X.*

Note 1.2 *If Y is closed in X, then Y is a called a closed subspace of X.*

Remark 1.3 *A subspace Y of a Banach space X is a subspace of X considered as a normed space. i.e. the completeness of Y is not essential.*

Theorem 1.4 *Show that a subspace Y of a Banach space X is complete if and only if the set Y is closed in X.*

Proof: *Suppose that Y is complete. We show that Y is closed. For this, we show that $Y = \bar{Y}$.*

Let $y \in \bar{Y}$, then there is a sequence (y_n) in Y such that $y_n \to y$ as $n \to \infty$. Now, the sequence (y_n) being convergent is Cauchy and Y is complete, so (y_n) converges in y. That is, $y \in Y$. Hence $\bar{Y} \subseteq Y$. But $Y \subseteq \bar{Y}$. Therefore, $Y = \bar{Y}$. Hence Y is closed.

Conversely, suppose that Y is closed. We show that Y is complete. For this, let (y_n) ba a Cauchy sequence in Y. Then it is Cauchy in X (as Y is subspace of X). But X is complete, so there is $y \in X$ such that $y_n \to y$. That is, Y contains infinite number of terms of the sequence (y_n). This shows that y is a limit point of Y, i.e. $y \in \bar{Y}$. Since Y is closed, so $y \in \bar{Y} = Y$. Hence Y is complete. ■

By using the concept of metric induced by norm, we now define the convergence of a sequence in normed spaces.

Definition 1.5 *A sequence (x_n) in a normed space X*

1. *is convergent to a point* $x \in X$ *if*

$$\lim_{n\to\infty} \|x_n - x\| = 0. \tag{4}$$

We write it as $x_n \to x$ *as* $n \to \infty$.

2. *is Cauchy if for every* $\epsilon > 0$ *there is an* $n_0 \in \mathbb{N}$ *such that for all* $m, n > n_0$, *we have*

$$\|x_m - x_n\| < \epsilon. \tag{5}$$

d

Definition 1.6 *Let* x_n *be a sequence in a normed space* X. *Associate with* (x_n), *a sequence* (S_n) *of partial sums*

$$S_n = x_1 + x_2 + \cdots + x_n,$$

where $n = 1, 2, 3, \cdots$. *If* (S_n) *is convergent, say*

$$S_n \to s, \qquad \text{that is} \qquad \|S_n - s\| \to 0,$$

then the series

$$\sum_{n=1}^{\infty} x_n = x_1 + x_2 + \cdots \tag{6}$$

is said to converge or to be convergent, s *is called the sum of the series and we write*

$$s = \sum_{n=1}^{\infty} x_n = x_1 + x_2 + \cdots .$$

Remark 1.7 *If* $\|x_1\| + \|x_2\| + \cdots$ *converges, the series (6) is said to be absolutely convergent. This happens only in case of* $\mathbb{R}$ *or* $\mathbb{C}$. *In other words,*

$$\sum_{n=1}^{\infty} \|x_n\| < \infty \Rightarrow \sum_{n=1}^{\infty} x_n < \infty.$$

But note that, in a normed space X, *this is not true in general. That is, absolute convergence does not implies convergence. However, absolute convergence implies convergence if and only if* X *is complete.*

We can use the concept of convergence of a series to define a "basis" of normed space.

Definition 1.8 *Let (e_n) be a sequence in normed space X such that for every $x \in X$ there is a unique sequence of scalars (α_n) such that*

$$\|x - (\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n)\| \to 0 \qquad as \quad n \to \infty, \tag{7}$$

then (e_n) is called a Schauder basis (or basis) for X. The series

$$\sum_{k=1}^{\infty} \alpha_k e_k \tag{8}$$

which has the sum x is then called the expansion of x with respect to (e_n), and we write

$$x = \sum_{k=1}^{\infty} \alpha_k e_k \tag{9}$$

Example 1.9 *Consider the normed space l^p, The Schauder basis (e_n) of l^p space are given by*

$$e_n = \delta_{nj} = \begin{cases} 1 & n = j \\ 0 & n \neq j \end{cases} \tag{10}$$

that is, (e_n) is the sequence whose nth term is 1 and all other terms are zero. Thus, we have

$$\begin{aligned} e_1 &= (1, 0, 0, 0, \cdots) \\ e_2 &= (0, 1, 0, 0, \cdots) \\ e_3 &= (0, 0, 1, 0, \cdots) \\ &\ldots \end{aligned}$$

Exercise 1.10 *Show that if a normed space X has a Schauder basis, then X is separable.*

Definition 1.11 *Let $(X, \|.\|)$ and $(X', \| * \|$ be two normed spaces. A mapping T of X into X' is said to be isometric or an isometry if T preserves norm, that* *is, if for all* $x, y \in X$*,*

$$\|Tx, Ty\| = \|x, y\|. \tag{11}$$

Theorem 1.12 *Let* $(X, \|\|)$ *be a normed space. Then show that there is a Banach space* X *and an isometry* A *from* X *onto a subspace* W *of* X *which is dense in* X*. The space* X *is unique, except for isometries.* Proof: ■

2 Finite Dimensional Normed Spaces and Subspaces

In this section, we will study some properties of finite dimensional normed spaces. First recall that a normed space X is called finite dimensional if it is that of as vector space and it is well known that a vector space is finite dimensional if there are finite number of linearly independent vectors in the spanning set of X. We start with the following important lemma:

Lemma 2.1 *Let* $\{x_1, \cdots, x_n\}$ *be a linearly independent set of vectors in a normed space* X *(of any dimension). Then there is a number* $c > 0$ *such that for every choice of scalars* $\alpha_l, \cdots, \alpha_n$*, we have*

$$\|\alpha_1 x_1 + \cdots + \alpha_n x_n\| \geq c(|\alpha_1| + \cdots + |\alpha_n|). \tag{12}$$

Proof: *For simplicity, put* $s = |\alpha_1| + \cdots + |\alpha_n|$*. If* $s = 0$*, then* $\alpha_j = 0$ *for all* $j = 1, 2, \cdots, n$*. So that (12) holds for all* $c > 0$*. Assume now that* $s > 0$*. Then dividing (12) by on both sides, we get*

$$\frac{1}{s}\|\alpha_1 x_1 + \cdots + \alpha_n x_n\| \geq c. \tag{13}$$

Since $s > 0$*, putting* $\frac{\alpha_j}{s} = \beta_j$ *in (13), we have*

$$\|\beta_1 x_1 + \cdots + \beta_n x_n\| < c, \tag{14}$$

where

$$\sum_{j=1}^{n} |\beta_j| = \sum_{j=1}^{n} \frac{|\alpha_j|}{s} = 1 \qquad as \quad s = |\alpha_1| + \cdots |\alpha_n|.$$

Now in order to show that (12) holds, it is enough to show that (13) holds for every $c > 0$ and for every n-tuple of scalars $\beta_1, \cdots, \beta_n$ with $\sum_{j=1}^{n} |\beta_j| = 1$. Suppose on contrary that

$$\|\beta_1 x_1 + \cdots + \beta_n x_n\| \leq c. \tag{15}$$

Then there exists a sequence (y_m) of vectors

$$y_m = \beta_1^m x_1 + \cdots + \beta_n^m x_n \tag{16}$$

with $\sum_{j=1}^{n} |\beta_j^m| = 1$, such that

$$\|y_m\| \to 0 \quad as \quad m \to \infty. \tag{17}$$

Now since, $\sum_{j=1}^{n} |\beta_j^m| = 1$, so we have $|\beta_j^m| \leq 1$. Hence for each fixed j the sequence

$$(\beta_j^m) = (\beta_j^1, \beta_j^2, \cdots)$$

is bounded. Consequently, by the Bolzano-Weierstrass theorem, (β_1^m) has a convergent subsequence. Denote by β_1, the limit of the sequence (β_1^m) and let $(y_{1,m})$ denote the corresponding subsequence of (y_m).
By the same argument, $(y_{l,m})$ has a subsequence $(y_{2,m})$ for which the corresponding subsequence of scalars (β_2^m) converges and let β_2 is the limit of (β_2^m). Continuing in this way, after n steps we obtain a subsequence $(y_{n,m}) = (y_{n,1}, y_{n,2}, \cdots)$ of (y_m) whose terms are of the form

$$y_{n,m} = \sum_{j=1}^{n} \gamma_j x_j, \qquad \sum_{j=1}^{n} |\gamma_j^m| = 1 \tag{18}$$

with scalars γ_j satisfying $\gamma_j \to \beta_j$ as $m \to \infty$. Hence

$$y_{n,m} \to y = \sum_{j=1}^{n} \beta_j^m x_j, \tag{19}$$

where $\sum_{j=1}^{n} |\beta_j^m| = 1$, so that not all β_j can be zero. Since $\{x_1, \cdots, x_n\}$ is a linearly independent set, we thus have $y \neq 0$. Now, since

$$y_{n,m} \to y,$$

this by the continuity of norm implies

$$\|y_{n,m}\| \to \|y\|.$$

Now since, $y_{n,m}$ *is a subsequence of* (y_m) *and from (17), we have* $\|y_m\| \to 0$, *therefore,* $\|y_{n,m}\| \to 0$. *By the uniqueness of limit, we have* $\|y\| = 0$, *so that* $y = 0$. *Which is a contradiction to the fact that* $y \neq 0$. *Hence our assumption in (15) is wrong. Consequently,*

$$\|\beta_1 x_1 + \cdots + \beta_n x_n\| \geq c, \tag{20}$$

which corresponds

$$\|\alpha_1 x_1 + \cdots + \alpha_n x_n\| \geq c(|\alpha_1| + \cdots + |\alpha_n|) \tag{21}$$

for every $c > 0$. ■

Theorem 2.2 *Every finite dimensional subspace* Y *of a normed space* X *is complete. In particular, every finite dimensional normed space is complete.*

Proof: *Suppose that* (y_m) *is a Cauchy sequence in* Y. *Since* Y *is a finite dimensional subspace of a normed space. Assume that dim(*Y*)=* n *and* $\{e_1, \cdots, e_n\}$ *any basis for* Y. *Then each* y_m *has a unique representation of the form*

$$y_m = \alpha_1^m e_1 + \cdots + \alpha_n^m e_n. \tag{22}$$

Since (y_m) *is a Cauchy sequence, for every* $\epsilon > 0$ *there is an* $N \in \mathbb{N}$ *such that for all* $m, r > N$, *we have*

$$\|y_m - y_r\| < \epsilon. \tag{23}$$

Now, by Lemma 2.1, there is a $c > 0$ *such that*

$$\begin{aligned} \epsilon > \|y_m - y_r\| &= \left\| \sum_{i=1}^{n} \alpha_j^m e_j - \sum_{i=1}^{\infty} \alpha_j^r e_j \right\| \\ &= \left\| \sum_{i=1}^{n} (\alpha_j^m - \alpha_j^r) e_j \right\| \\ &\geq c \sum_{i=1}^{n} |\alpha_j^m - \alpha_j^r|. \end{aligned} \tag{24}$$

This gives

$$\sum_{i=1}^{n} |\alpha_j^m - \alpha_j^r| < \frac{\epsilon}{c}, \qquad (m, r > N), \tag{25}$$

which implies

$$|\alpha_j^m - \alpha_j^r| \leq \sum_{i=1}^{n} |\alpha_j^m - \alpha_j^r| < \frac{\epsilon}{c} \qquad (m, r > N). \tag{26}$$

Hence

$$|\alpha_j^m - \alpha_j^r| < \frac{\epsilon}{c}, \qquad (m, r > N). \tag{27}$$

This shows that each of the n sequences

$$(\alpha_j^m) = (\alpha_j^1, \alpha_j^2, \cdots) \qquad j = 1, 2, \cdots, n$$

is Cauchy in $\mathbb{R}$ or $\mathbb{C}$. By the completeness of $\mathbb{R}$ or $\mathbb{C}$, it converges and let α_j denotes the limit. Using these n limits $a_1, \cdots, a_n$, we define

$$y = \alpha_1 e_1 + \cdots + \alpha_n e_n.$$

Clearly, $y \in Y$. Furthermore, since $\alpha_j^m \to \alpha_j$, we have

$$\begin{aligned} \|y_m - y\| &= \left\| \sum_{i=1}^{n} \alpha_j^m e_j - \sum_{i=1}^{\infty} \alpha_j e_j \right\| \\ &= \left\| \sum_{i=1}^{n} (\alpha_j^m - \alpha_j) e_j \right\| \\ &\leq \sum_{i=1}^{n} |\alpha_j^m - \alpha_j| \|e_j\| \to 0. \end{aligned} \tag{28}$$

That is, $y_m \to y$. This shows that (y_m) is convergent in Y. Since (y_m) was an arbitrary Cauchy sequence in Y, this proves that Y is complete. ■

Theorem 2.3 *Show that every finite dimensional subspace Y of a normed space X is closed in X.*

Proof: *We have proved that*

"Every finite dimensional subspace Y of a normed space X is complete."

Also we know that

"A subspace Y of a complete normed space X is itself complete if and only if the set Y is closed in X."

Therefore, we conclude that

"every finite dimensional subspace Y of a normed space X is closed in X".

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Note 2.4 *In case of infinite dimensional subspace, Theorem 2.3 need not to be true.*

Example 2.5 *Let $X = C[O, 1]$ and the norm $\|\cdot\|$ is defined by*

$$\|x(t)\| = \max_{t\in[0,1]} |x(t)|.$$

Under this norm X is complete. Let $Y = Span(1, t, t^2, \cdots)$ be the set of polynomials. Let (y_n) be a sequence in Y such that

$$y_n = 1 + t + \frac{t^2}{2!} + \cdots + \frac{t^n}{n!} + \cdots .$$

Now

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} 1 + t + \frac{t^2}{2!} + \cdots + \frac{t^n}{n!} + \cdots = e^t \notin Y.$$

This shows that Y is not closed and hence not complete.

3 Equivalent Norm

Definition 3.1 *A norm $\|\cdot\|$ on a vector space X is said to be equivalent to a norm $\|\cdot\|_0$ on X if there are positive numbers a and b such that for all $x \in X$ we have*

$$a\|x\|_0 \leq \|x\| \leq b\|x\|_0. \tag{29}$$

Note 3.2 *Equivalent norms on X define the same topology for X.*

Theorem 3.3 *Show that on a finite dimensional vector space X, any norm $\|\cdot\|$ is equivalent to any other norm $\|\cdot\|_0$.*

Proof: *Suppose $dim(X) = n$ and $\{e_1, e_2, \cdots, e_n\}$ any basis for X. Then every $x \in X$ has a unique representation*

$$x = \alpha_1 e_1 + \cdots + \alpha_n e_n \tag{30}$$

for any scalars $\alpha_1, \cdots, \alpha_n$. Then by Lemma, there is a positive constant c such that

$$\|x\| = \|\alpha_1 e_1 + \cdots + \alpha_n e_n\| \geq c(|\alpha_1| + \cdots + |\alpha_n|). \tag{31}$$

On the other hand

$$\begin{aligned} \|x\|_0 &= \|\alpha_1 e_1 + \cdots + \alpha_n e_n\|_0 \\ &\leq \|\alpha_1 e_1\|_0 + \|\alpha_2 e_2\|_0 + \cdots + \|\alpha_n e_n\|_0 \\ &= |\alpha_1|\|e_1\|_0 + |\alpha_2|\|e_2\|_0 + \cdots + |\alpha_n|\|e_n\|_0 \\ &= \sum_{j=1}^{n} |\alpha_j|\|e_j\|_0 \\ &\leq k\sum_{j=1}^{n} |\alpha_j|, \end{aligned} \tag{32}$$

where $k = \max_j \|e_j\|_0$. This gives

$$\|x\|_0 \leq k(|\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|). \tag{33}$$

From (33) and (31), we have

$$\|x\|_0 \leq k(|\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|) \leq \frac{k}{c}\|x\| \tag{34}$$

implies

$$\frac{c}{k}\|x\|_0 \leq \|x\|. \tag{35}$$

Hence

$$a\|x\|_0 \leq \|x\|, \tag{36}$$

where $a = \frac{c}{k} > 0$.

Again, by interchanging the role of $\|\cdot\|$ *and* $\|\cdot\|_0$ *in above arguments, we get that*

$$\|x\| \leq b\|x\|_0. \tag{37}$$

Combining (36) and (37), we have that

$$a\|x\|_0 \leq \|x\| \leq b\|x\|_0. \tag{38}$$

■

4 Compactness and Finite Dimension

Definition 4.1 *A metric space* X *is said to be compact if every sequence in* X *has a convergent subsequence. A subset* M *of* X *is said to be compact if* M *is compact considered as a subspace of* X*, that is, if every sequence in* M *has a convergent subsequence whose limit is an element of* M*.*

Lemma 4.2 *Show that a compact subset* M *of a metric space is closed and bounded.*

Proof: *Suppose that* M *is a compact subset of a metric space* X*. We first show that* M *is closed. For this, we show that* $M = \bar{M}$*.*

Let $x \in \bar{M}$*, then there is a sequence* (x_n) *in* M *such that* $x_n \to x$*. Since* M *is* *compact and* (x_n) *is a convergent subsequence of itself, so* $x \in M$*, But* $x \in \bar{M}$*. So,*

$$\bar{M} \subseteq M.$$

But

$$M \subseteq \bar{M}.$$

Hence

$$M = \bar{M}.$$

Therefore, M *is closed.*

To show that M *is bounded, assume that it is not. Then there is an unbounded sequence* (y_n) *in* M *such that* $d(y_n, b) > n$*, where* b *is any fixed element. So, there exist no convergent subsequence of* (y_n) *because convergent subsequence must be bounded. Which is a contradiction. Hence* M *is bounded.* ■

Theorem 4.3 *Show that in a finite dimensional normed space* X*, any subset* $M \subset X$ *is compact if and only if* M *is closed and bounded.*

Proof: *Suppose that* M *is a compact subset of a metric space* X*. We first show that* M *is closed. For this, we show that* $M = \bar{M}$*.*

Let $x \in \bar{M}$*, then there is a sequence* (x_n) *in* M *such that* $x_n \to x$*. Since* M *is compact and* (x_n) *is a convergent subsequence of itself, so* $x \in M$*, But* $x \in \bar{M}$*. So,*

$$\bar{M} \subseteq M.$$

But

$$M \subseteq \bar{M}.$$

Hence

$$M = \bar{M}.$$

Therefore, M *is closed.*

To show that M *is bounded, assume that it is not. Then there is an unbounded sequence* (y_n) *in* M *such that* $d(y_n, b) > n$*, where* b *is any fixed element. So,* *there exist no convergent subsequence of* (y_n) *because convergent subsequence must be bounded. Which is a contradiction. Hence* M *is bounded.*

Conversely: *Suppose now that* M *is closed and bounded subset of a finite dimensional normed space* X*. Let* $dim(X) = n$ *and* $\{e_1, e_2, \cdots, e_n\}$ *any basis for* X*. Let* (x_m) *be a sequence in* M *then each* x_m *has a representation*

$$x_m = \alpha_1^m e_1 + \cdots + \alpha_n^m e_n. \tag{39}$$

Since M *is bounded, so is the sequence* (x_m)*. Then there is* $k > 0$ *such that for all* m*, we have*

$$\|x_m\| \leq k. \tag{40}$$

From (39), (40) and by Lemma, we have

$$k \geq \|x_m\| = \|\alpha_1^m e_1 + \cdots + \alpha_n^m e_n\| \geq c \sum_{j=1}^{n} |\alpha_j^m|, \tag{41}$$

where $c > 0$*. This gives*

$$\sum_{j=1}^{n} |\alpha_j^m| \leq \frac{k}{c}. \tag{42}$$

Hence the sequence of numbers (α_j^m) *is bounded and, by the Bolzano-Weierstrass theorem, has a point of accumulation* α_j, $1 \leq j \leq n$*. Therefore,* (x_m) *has a subsequence* (z_m) *which converges to* $z = \sum \alpha_j e_j$*. Since* M *is closed,* $z \in M$*. This shows that the arbitrary sequence* (x_m) *in* M *has a subsequence which converges in* M*. Hence* M *is compact.* ■