2.18 NORMALISATION OF WAVE FUNCTIONS

According to Born the probability of finding a particle is represented by $\Psi\Psi^*$ dx dy dz in a volume element dxdydz. If the probability for a particle having wave function Ψ were evaluated over the entire space in which the particle exists, then the probability should be equal to 1 or 100%. Mathematically it can be stated as

$$\int_{-\infty}^{+\infty} \Psi\Psi^* dxdydz = 1$$

or

$$\int_{-\infty}^{+\infty} \Psi\Psi^* d\tau = 1 \qquad (2.59)$$

(where $d\tau$ = dxdydz is small volume element).

When the integral of the wave function times its complex conjugate over the entire space available is equal to unity, then the wave function is said to be normalised and this condition is known as normlisation of wave function. The limits $-\alpha$ and $+\alpha$ are conventionally used to represent "all space" although the entire space of a system may not actually extend to infinity in both directions. The integral must be equal to unity since the particle must exist somewhere in that interval if it is to exist at all. For one dimension the above equation can be written as

$$\int_{-\infty}^{+\infty} \Psi\Psi^*(x)\, dx = 1 \qquad (2.60)$$

The SWE is a homogenous differential equation whose solution gives a value for Ψ. Very often Ψ is not a normalised wave function. But it can be shown that multiplication of a wave function Ψ by any constant A is also a solution to the wave equation.

Let us assume that a wave function for a system exists and is $\Psi_{(x)} = \sin(\pi x/2)$ where x is the only variable. If the region of interest is from x = 0, to x = 1, then normalisation of wave function is carried out as below.

According to equation (2.59) the function must be multiplied by some constant so that

$$\int_0^1 \Psi\Psi^* \, dx = 1$$

Note that the limits are 0 to 1, not $-\alpha$ to $+\alpha$ and that $d\tau$ is simply dx for this one-dimensional example. Let us assume that Ψ is multiplied by some constant A.

$$\Psi \longrightarrow A\Psi$$

Substituting for Ψ into integral, we get

$$\int_0^1 (A\Psi)(A\Psi)^* \, dx = \int_0^1 AA^* \left(\sin\frac{\pi x}{2}\right)\left(\sin\frac{\pi x}{2}\right)^* dx$$

Sine A is a constant, it can be pulled out of the integral, and since this function is a real function, the * has no effect on the function.

Therefore, we get

$$\int_0^1 AA^* \left(\sin\frac{\pi x}{2}\right)\left(\sin\frac{\pi x}{2}\right)^* dx = A^2 \int_0^1 \sin^2\frac{\pi x}{2} \, dx$$

Normalization requires that this expression equals to 1:

$$A^2 \int_0^1 \sin^2\frac{\pi x}{2} \, dx = 1$$

The integral in this expression has a known form and it can be solved and the definite interval from limits 0 to 1 can be evaluated.

$$\int \sin^2 bx dx = \frac{x}{2} - \frac{1}{4b}\sin^2 bx$$

In this case $b = \pi/2$, evaluating the integral between limits,

$$A^2\left[\frac{x}{2} - \frac{2}{4\pi}\sin^2\frac{2\pi x}{2}\right]_0^1 = 1$$

$$A^2\left[\frac{x}{2} - \frac{1}{2\pi}\sin^2 \pi x\right]_0^1 = 1$$

$$A^2\left(\frac{1}{2}\right) = 1$$

$$A = \sqrt{2}$$

Hence the correctly normalized wave function is therefore

$$\Psi_{(x)} = \sqrt{2}\left(\sin\frac{\pi x}{2}\right)$$