to the other end.

2.17 EIGENFUNCTIONS AND EIGENVALUES

The time independent Schrodinger wave equation is usually written in the compact form

$$\hat{H}\Psi = E\Psi$$

It belongs to the class of equations called *eigenvalue equations*. The word "eigenvalue" is a partial translation of the German word 'Eignewert'. A full translation is "characteristic value". An eigenvalue equation has on one side an operator operating on a function, and on the other side a constant *called* the *eignevalue* multiplying, the same function, which is called the *eigenfunction*. In the above equation $\hat{H}$ represents the Hamiltonian operator, E the eigenvalue and Ψ the eigenfunction.

(operator) (function) = (constant factor) × (same function)

When an operator operates on a function and the same function is reproduced with same numerical value, then the function is called an eigenfunction and numerical value is called its eigenvalue. An equation that contains both eigenfunction and eigenvalue is called an *eigenequation*.

The SWE is the eigenvalue equation for Hamiltonian operator. The coordinate wavefunction is the eigenfunction of the Hamiltonian operator, and is often called energy eigenfunction. The eigenvalue of the Hamiltonian operator E, is the value of energy, and is called the *energy eigenvalue*. Solving an equation means finding not only the set of eigenfunctions that satisfy the equation, but also the eigenvalue that belongs to each eigenfunction. Two common cases occur. The first case is that the eigenvalue can take an any value within some range of values (a continuous spectrum of eigenvalues). The second case is that there is a discrete set of eigenvalues with the values between the members of the set not permitted (a discrete spectrum of eigenvalues). The occurrence of a discrete spectrum of eighenvalues corresponds to quantization.

In addition to satisfying the SWE, a wave function must satisfy other conditions. Since it represents a wave, we assume that it has following properties, which are generally shared by waves.

(i) The wave function is single-valued

(ii) The wave function is continuous, and

(iii) The wave function is finite

These properties will lead to boundary conditions that have important consequences.

Example 2.10

Which of the following operator/function combinations would yield eigenvalue equations? What are eigenvalues of the eigenfunctions?

(i) $\frac{d}{dx}(\sin x)$

(ii) $\frac{d}{dx}(\cos x)$

(iii) $\frac{d}{dx}(e^{ax})$

(iv) $\frac{d}{dx}(e^{x})$

(v) $\frac{d^2}{dx^2}\left(\cos\frac{x}{4}\right)$

(vi) $\frac{d}{dx}(e^{-4x})$

(vii) $\frac{d}{dx}(e^{-4x^2})$

(viii) $\frac{d^2}{dx^2}(\sin 4x)$

(xi) $\frac{d^n}{dx^n}(e^{ax})$

(x) $\frac{d}{dx}(5x^3)$

Solution

(i) $\frac{d}{dx}(\sin x) = \cos x$

This is not an eigenequation as function sin x is not generated.

(ii) $\frac{d}{dx}(\cos x) = -\sin x$

This is not an eigenequation as function cos x is not generated.

(iii) $\frac{d}{dx}(e^{ax}) = a \cdot e^{ax}$

This is an eigen equation with an eigenvalue of a.

(iv) $\frac{d}{dx}(e^{x}) = 1 \cdot e^{x}$

This is eigen equation with an eigenvalue of 1.

(v) $\frac{d^2}{dx^2}\left(\cos\frac{x}{4}\right) = -\frac{1}{16}\cos\frac{x}{4}$

This is an eigenvalue equation with an eigenvalue of − 1/16.

(vi) $\frac{d}{dx}(c^{-4x}) = -4(e^{-4x})$

This is an eigenvalue equation with an eigenvalue of −4.

(vii) $\frac{d}{dx}(e^{-4x^2}) = -8x(e^{-4x^2})$

This is not an eigenvalue equation because although the original function is reproduced, it is not multiplied by a constant, it is multiplied by another function –8x.

(viii) $\frac{d^2}{dx^2}(\sin 4x) = \frac{d}{dx}(4 \cos 4x) = -16 \sin 4x$

This is an eigenvalue equation with an eigenvalue if –16.

(ix) $\frac{d^n}{dx^n}(e^{\alpha n}) = \alpha^n e^{\alpha n}$

This is an eigenvalue equation with an eigenvalue of α^n.

(x) $\frac{d}{dx}(5x^3) = 3.5 \cdot x^2 = 15 \times 2$

This is not an eigenvalue equation as the function is not reproduced.