Cauchy's Integral Theorem for Multiply Connected Domains

Cauchy's theorem applies to multiply connected domains. We first explain this for a **doubly connected domain** D with outer boundary curve C_1 and inner C_2 (Fig. 350). If a function $f(z)$ is analytic in any domain D^* that contains D and its boundary curves, we claim that

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz \qquad \text{(Fig. 350)} \tag{6}$$

both integrals being taken counterclockwise (or both clockwise, and regardless of whether or not the full interior of C_2 belongs to D^*).

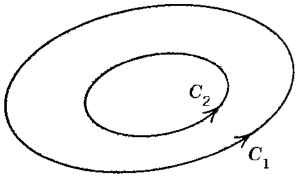


Fig. 350. Paths in (5)

PROOF By two cuts $\tilde{C}_1$ and $\tilde{C}_2$ (Fig. 351) we cut D into two simply connected domains D_1 and D_2 in which and on whose boundaries $f(z)$ is analytic. By Cauchy's integral theorem the integral over the entire boundary of D_1 (taken in the sense of the arrows in Fig. 351) is zero, and so is the integral over the boundary of D_2, and thus their sum. In this sum the integrals over the cuts $\tilde{C}_1$ and $\tilde{C}_2$ cancel because we integrate over them in both directions—this is the key—and we are left with the integrals over C_1 (counterclockwise) and C_2 (clockwise; see Fig. 351); hence by reversing the integration over C_2 (to counterclockwise) we have

$$\oint_{C_1} f\,dz - \oint_{C_2} f\,dz = 0$$

and (6) follows. ■

For domains of higher connectivity the idea remains the same. Thus, for a **triply connected domain** we use three cuts $\tilde{C}_1$, $\tilde{C}_2$, $\tilde{C}_3$ (Fig. 352). Adding integrals as before, the integrals over the cuts cancel and the sum of the integrals over C_1 (counterclockwise) and C_2, C_3 (clockwise) is zero. Hence the integral over C_1 equals the sum of the integrals over C_2 and C_3, all three now taken counterclockwise. Similarly for quadruply connected domains, and so on.

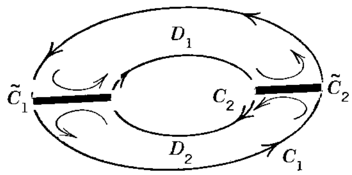


Fig. 351. Doubly connected domain

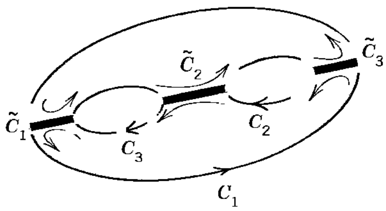


Fig. 352. Triply connected domain

[illegible]

1–11 CAUCHY'S INTEGRAL THEOREM APPLICABLE?

Integrate $f(z)$ counterclockwise around the unit circle, indicating whether Cauchy's integral theorem applies. (Show the details of your work.)

1. $f(z) = \text{Re}\, z$ **2.** $f(z) = 1/(3z - \pi i)$

3. $f(z) = e^{z^2/2}$ **4.** $f(z) = 1/\bar{z}$

5. $f(z) = \tan z^2$ **6.** $f(z) = \sec(z/2)$

7. $f(z) = 1/(z^8 - 1.2)$ **8.** $f(z) = 1/(4z - 3)$

9. $f(z) = 1/(2|z|^3)$ **10.** $f(z) = \bar{z}^2$

11. $f(z) = z^2 \cot z$

12–17 COMMENTS ON TEXT AND EXAMPLES

12. (Singularities) Can we conclude in Example 2 that the integral of $1/(z^2 + 4)$ taken over (a) $|z - 2| = 2$, (b) $|z - 2| = 3$ is zero? Give reasons.

13. (Cauchy's integral theorem) Verify Theorem 1 for the integral of z^2 over the boundary of the square with vertices $1 + i$, $-1 + i$, $-1 - i$, and $1 - i$ (counterclockwise).

14. (Cauchy's integral theorem) For what contours C will it follow from Theorem 1 that

(a) $\displaystyle\oint_C \frac{dz}{z} = 0$, (b) $\displaystyle\oint_C \frac{\cos z}{z^6 - z^2}\, dz = 0$,

(c) $\displaystyle\oint_C \frac{e^{1/z}}{z^2 + 9}\, dz = 0$?

15. (Deformation principle) Can we conclude from Example 4 that the integral is also zero over the contour in Problem 13?

16. (Deformation principle) If the integral of a function $f(z)$ over the unit circle equals 3 and over the circle $|z| = 2$ equals 5, can we conclude that $f(z)$ is analytic everywhere in the annulus $1 < |z| < 2$?

17. (Path independence) Verify Theorem 2 for the integral of $\cos z$ from 0 to $(1 + i)\pi$ (a) over the shortest path, (b) over the x-axis to π and then straight up to $(1 + i)\pi$.

18. TEAM PROJECT. Cauchy's Integral Theorem.
(a) Main Aspects. Each of the problems in Examples 1–5 explains a basic fact in connection with Cauchy's theorem. Find five examples of your own, more complicated ones if possible, each illustrating one of those facts.

(b) Partial fractions. Write $f(z)$ in terms of partial fractions and integrate it counterclockwise over the unit circle, where

(i) $f(z) = \dfrac{2z + 3i}{z^2 + \frac{1}{4}}$ (ii) $f(z) = \dfrac{z + 1}{z^2 + 2z}$.

(c) Deformation of path. Review (c) and (d) of Team Project 34, Sec. 14.1, in the light of the principle of deformation of path. Then consider another family of paths with common endpoints, say, $z(t) = t + ia(t - t^2)$, $0 \leq t \leq 1$, and experiment with the integration of analytic and nonanalytic functions of your choice over these paths (e.g., z, $\text{Im}\, z$, z^2, $\text{Re}\, z^2$, $\text{Im}\, z^2$, etc).

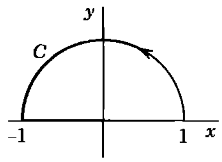
19–30 FURTHER CONTOUR INTEGRALS

Evaluate (showing the details and using partial fractions if necessary)

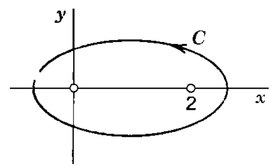
19. $\displaystyle\oint_C \frac{dz}{2z - i}$, C the circle $|z| = 3$ (counterclockwise)

20. $\displaystyle\oint_C \tanh z\, dz$, C the circle $|z - \frac{1}{4}\pi i| = \frac{1}{2}$ (clockwise)

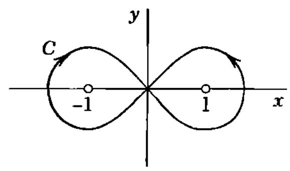
21. $\displaystyle\oint_C \text{Re}\, 2z\, dz$, C as shown



22. $\displaystyle\oint_C \frac{7z - 6}{z^2 - 2z}\, dz$, C as shown



23. $\displaystyle\oint_C \frac{dz}{z^2 - 1}$, C as shown



24. $\displaystyle\oint_C \frac{e^{2z}}{4z}\, dz$, C consists of $|z| = 2$ (clockwise) and $|z| = \frac{1}{2}$ (counterclockwise)

25. $\oint_C \frac{\cos z}{z}\, dz$, C consists of $|z| = 1$ (counterclockwise) and $|z| = 3$ (clockwise)

26. $\oint_C \operatorname{Ln}(2 + z)\, dz$, C the boundary of the square with vertices $\pm 1, \pm i$

27. $\oint_C \frac{dz}{z^2 + 1}$, C: (a) $|z| = \frac{1}{2}$, (b) $|z - i| = \frac{3}{2}$ (counterclockwise)

28. $\oint_C \frac{dz}{z^2 + 1}$, C: (a) $|z + i| = 1$, (b) $|z - i| = 1$ (counterclockwise)

29. $\oint_C \frac{\sin z}{z + 2i}\, dz$, C: $|z - 4 - 2i| = 5.5$ (clockwise)

30. $\oint_C \frac{\tan(z/2)}{z^4 - 16}\, dz$, C the boundary of the square with vertices $\pm 1, \pm i$ (clockwise)

14.3 Cauchy's Integral Formula

The most important consequence of Cauchy's integral theorem is Cauchy's integral formula. This formula is useful for evaluating integrals, as we show below. Even more important is its key role in proving the surprising fact that analytic functions have derivatives of all orders (Sec. 14.4), in establishing Taylor series representations (Sec. 15.4), and so on. Cauchy's integral formula and its conditions of validity may be stated as follows.

THEOREM 1

Cauchy's Integral Formula

Let $f(z)$ be analytic in a simply connected domain D. Then for any point z_0 in D and any simple closed path C in D that encloses z_0 (Fig. 353),

$$\oint_C \frac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0) \qquad (1) \quad \textbf{(Cauchy's integral formula)}$$

the integration being taken counterclockwise. Alternatively (for representing $f(z_0)$ by a contour integral, divide (1) by $2\pi i$),

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\, dz \qquad (1^*) \quad \textbf{(Cauchy's integral formula).}$$

PROOF By addition and subtraction, $f(z) = f(z_0) + [f(z) - f(z_0)]$. Inserting this into (1) on the left and taking the constant factor $f(z_0)$ out from under the integral sign, we have

$$\oint_C \frac{f(z)}{z - z_0}\, dz = f(z_0)\oint_C \frac{dz}{z - z_0} + \oint_C \frac{f(z) - f(z_0)}{z - z_0}\, dz. \qquad (2)$$

The first term on the right equals $f(z_0)\cdot 2\pi i$ (see Example 6 in Sec. 14.2 with $m = -1$). This proves the theorem, provided the second integral on the right is zero. This is what we are now going to show. Its integrand is analytic, except at z_0. Hence by (6) in Sec. 14.2 we can replace C by a small circle K of radius ρ and center z_0 (Fig. 354), without

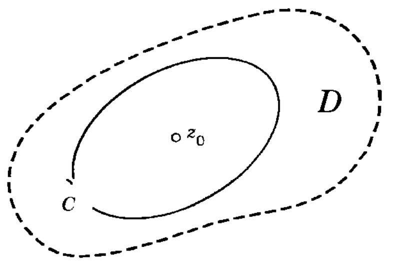


Fig. 353. Cauchy's integral formula

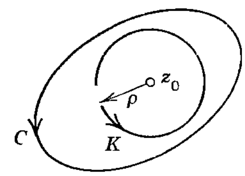


Fig. 354. Proof of Cauchy's integral formula

altering the value of the integral. Since $f(z)$ is analytic, it is continuous (Team Project 26, Sec. 13.3). Hence an $\epsilon > 0$ being given, we can find a $\delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$ for all z in the disk $|z - z_0| < \delta$. Choosing the radius ρ of K smaller than δ, we thus have the inequality

$$\left| \frac{f(z) - f(z_0)}{z - z_0} \right| < \frac{\epsilon}{\rho}$$

at each point of K. The length of K is $2\pi\rho$. Hence, by the ML-inequality in Sec. 14.1,

$$\left| \oint_K \frac{f(z) - f(z_0)}{z - z_0} \, dz \right| < \frac{\epsilon}{\rho} 2\pi\rho = 2\pi\epsilon.$$

Since ϵ (> 0) can be chosen arbitrarily small, it follows that the last integral in (2) must have the value zero, and the theorem is proved. ■

EXAMPLE 1 Cauchy's Integral Formula

$$\oint_C \frac{e^z}{z - 2} \, dz = 2\pi i e^z \Big|_{z=2} = 2\pi i e^2 = 46.4268i$$

for any contour enclosing $z_0 = 2$ (since e^z is entire), and zero for any contour for which $z_0 = 2$ lies outside (by Cauchy's integral theorem). ■

EXAMPLE 2 Cauchy's Integral Formula

$$\begin{aligned} \oint_C \frac{z^3 - 6}{2z - i} \, dz &= \oint_C \frac{\frac{1}{2}z^3 - 3}{z - \frac{1}{2}i} \, dz \\ &= 2\pi i \left[\tfrac{1}{2}z^3 - 3\right]\Big|_{z=i/2} \\ &= \frac{\pi}{8} - 6\pi i \qquad (z_0 = \tfrac{1}{2}i \text{ inside } C). \end{aligned}$$ ■

EXAMPLE 3 Integration Around Different Contours

Integrate

$$g(z) = \frac{z^2 + 1}{z^2 - 1} = \frac{z^2 + 1}{(z + 1)(z - 1)}$$

counterclockwise around each of the four circles in Fig. 355.

Solution. $g(z)$ is not analytic at -1 and 1. These are the points we have to watch for. We consider each circle separately.

(a) The circle $|z - 1| = 1$ encloses the point $z_0 = 1$ where $g(z)$ is not analytic. Hence in (1) we have to write

$$g(z) = \frac{z^2+1}{z^2-1} = \frac{z^2+1}{z+1}\,\frac{1}{z-1};$$

thus

$$f(z) = \frac{z^2+1}{z+1}$$

and (1) gives

$$\oint_C \frac{z^2+1}{z^2-1}\,dz = 2\pi i f(1) = 2\pi i\left[\frac{z^2+1}{z+1}\right]_{z=1} = 2\pi i.$$

(b) gives the same as (a) by the principle of deformation of path.

(c) The function $g(z)$ is as before, but $f(z)$ changes because we must take $z_0 = -1$ (instead of 1). This gives a factor $z - z_0 = z + 1$ in (1). Hence we must write

$$g(z) = \frac{z^2+1}{z-1}\,\frac{1}{z+1};$$

thus

$$f(z) = \frac{z^2+1}{z-1}.$$

Compare this for a minute with the previous expression and then go on:

$$\oint_C \frac{z^2+1}{z^2-1}\,dz = 2\pi i f(-1) = 2\pi i\left[\frac{z^2+1}{z-1}\right]_{z=-1} = -2\pi i$$

(d) gives 0. Why? ■

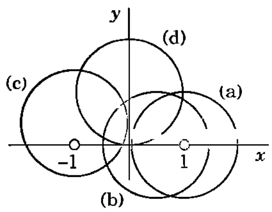


Fig. 355. Example 3

Multiply connected domains may be handled as in Sec. 14.2. For instance, if $f(z)$ is analytic on C_1 and C_2 and in the ring-shaped domain bounded by C_1 and C_2 (Fig. 356) and z_0 is any point in that domain, then

$$f(z_0) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{z-z_0}\,dz + \frac{1}{2\pi i}\oint_{C_2} \frac{f(z)}{z-z_0}\,dz, \tag{3}$$

where the outer integral (over C_1) is taken counterclockwise and the inner clockwise, as indicated in Fig. 356.

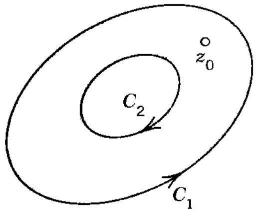


Fig. 356. Formula (3)

Our discussion in this section has illustrated the use of Cauchy's integral formula in integration. In the next section we show that the formula plays the key role in proving the surprising fact that an analytic function has derivatives of all orders, which are thus analytic functions themselves.

1–4 CONTOUR INTEGRATION

Integrate $(z^2 - 4)/(z^2 + 4)$ counterclockwise around the circle:

1. $|z - i| = 2$ **2.** $|z - 1| = 2$

3. $|z + 3i| = 2$ **4.** $|z| = \pi/2$

5–17 CONTOUR INTEGRATION

Using Cauchy's integral formula (and showing the details), integrate counterclockwise (or as indicated)

5. $\oint_C \frac{z+2}{z-2}\,dz$, C: $|z - 1| = 2$

6. $\oint_C \frac{e^{3z}}{3z - i}\,dz$, C: $|z| = 1$

7. $\oint_C \frac{\sinh \pi z}{z^2 - 3z}\,dz$, C: $|z| = 1$

8. $\oint_C \frac{dz}{z^2 - 1}$, C: $|z - 1| = \pi/2$

9. $\oint_C \frac{dz}{z^2 - 1}$, C: $|z + 1| = 1$

10. $\oint_C \frac{e^z}{z - 2i}\,dz$, C: $|z - 2i| = 4$

11. $\oint_C \frac{\cos z}{2z}\,dz$, C: $|z| = \frac{1}{2}$

12. $\oint_C \frac{\tan z}{z - i}\,dz$, C the boundary of the triangle with vertices 0 and $\pm 1 + 2i$

13. $\oint_C \frac{e^{-3\pi z}}{2z + i}\,dz$, C the boundary of the square with vertices ± 1, $\pm i$

14. $\oint_C \frac{\text{Ln}(z + 1)}{z^2 + 1}\,dz$, C consists of $|z - 2i| = 2$ (counterclockwise) and $|z - 2i| = \frac{1}{2}$ (clockwise)

15. $\oint_C \frac{\text{Ln}(z - 1)}{z - 5}\,dz$, C: $|z - 4| = 2$

16. $\oint_C \frac{\sin z}{z^2 - 2iz}\,dz$, C consists of $|z| = 3$ (counterclockwise) and $|z| = 1$ (clockwise)

17. $\oint_C \frac{\cosh^2 z}{(z - 1 - i)z^2}\,dz$, C as in Prob. 16

18. Show that $\oint_C (z - z_1)^{-1}(z - z_2)^{-1}\,dz = 0$ for a simple closed path C enclosing z_1 and z_2, which are arbitrary.

19. CAS PROJECT. Contour Integration. Experiment to find out to what extent your CAS can do contour integration (a) by using the second method in Sec. 14.1, (b) by Cauchy's integral formula.

20. TEAM PROJECT. Cauchy's Integral Theorem. Gain additional insight into the proof of Cauchy's integral theorem by producing (2) with a contour enclosing z_0 (as in Fig. 353) and taking the limit as in the text. Choose

(a) $\oint_C \frac{z^3 - 6}{z - \frac{1}{2}i}\,dz$, (b) $\oint_C \frac{\sin z}{z - \frac{1}{2}\pi}\,dz$,

and (c) two other examples of your choice.