9. $\sum_{n=0}^{\infty} \left[\binom{n+k}{k}\right]^{-1} z^{n+k}$

10. $\sum_{n=0}^{\infty} \binom{n+m}{m} z^n$

11. (Addition and subtraction) Write out the details of the proof on termwise addition and subtraction of power series.

12. (Cauchy product) Show that $(1-z)^{-2} = \sum_{n=0}^{\infty} (n+1)z^n$ (a) by using the Cauchy product, (b) by differentiating a suitable series.

13. (Cauchy product) Show that the Cauchy product of $\sum_{n=0}^{\infty} z^n/n!$ multiplied by itself gives $\sum_{n=0}^{\infty} (2z)^n/n!$.

14. (On Theorem 3) Prove that $\sqrt[n]{n} \to 1$ as $n \to \infty$ (as claimed in the proof of Theorem 3).

15. (On Theorems 3 and 4) Find further examples of your own.

16–20 APPLICATIONS OF THE IDENTITY THEOREM

State clearly and explicitly where and how you are using Theorem 2.

16. (Bionomial coefficients) Using $(1+z)^p(1+z)^q = (1+z)^{p+q}$, obtain the basic relation

$$\sum_{n=0}^{r} \binom{p}{n}\binom{q}{r-n} = \binom{p+q}{r}.$$

17. (Odd function) If $f(z)$ in (1) is *odd* (i.e., $f(-z) = -f(z)$), show that $a_n = 0$ for even n. Give examples.

18. (Even functions) If $f(z)$ in (1) is *even* (i.e., $f(-z) = f(z)$), show that $a_n = 0$ for odd n. Give examples.

19. Find applications of Theorem 2 in differential equations and elsewhere

20. TEAM PROJECT. Fibonacci numbers.[2] **(a)** The Fibonacci numbers are recursively defined by $a_0 = a_1 = 1$, $a_{n+1} = a_n + a_{n-1}$ if $n = 1, 2, \cdots$. Find the limit of the sequence (a_{n+1}/a_n).

(b) Fibonacci's rabbit problem. Compute a list of $a_1, \cdots, a_{12}$. Show that $a_{12} = 233$ is the number of pairs of rabbits after 12 months if initially there is 1 pair and each pair generates 1 pair per month, beginning in the second month of existence (no deaths occurring).

(c) Generating function. Show that the *generating function* of the **Fibonacci numbers** is $f(z) = 1/(1 - z - z^2)$; that is, if a power series (1) represents this $f(z)$, its coefficients must be the Fibonacci numbers and conversely. *Hint.* Start from $f(z)(1 - z - z^2) = 1$ and use Theorem 2.

15.4 Taylor and Maclaurin Series

The **Taylor series**[3] of a function $f(z)$, the complex analog of the real Taylor series is

$$(1) \qquad f(z) = \sum_{n=1}^{\infty} a_n(z - z_0)^n \qquad \text{where} \qquad a_n = \frac{1}{n!} f^{(n)}(z_0)$$

or, by (1), Sec. 14.4,

$$(2) \qquad a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}}\, dz^*.$$

In (2) we integrate counterclockwise around a simple closed path C that contains z_0 in its interior and is such that $f(z)$ is analytic in a domain containing C and every point inside C.

A **Maclaurin series**[3] is a Taylor series with center $z_0 = 0$.

[2]LEONARDO OF PISA, called FIBONACCI (= son of Bonaccio), about 1180–1250, Italian mathematician, credited with the first renaissance of mathematics on Christian soil.

[3]BROOK TAYLOR (1685–1731), English mathematician who introduced real Taylor series. COLIN MACLAURIN (1698–1746), Scots mathematician, professor at Edinburgh.

The **remainder** of the Taylor series (1) after the term $a_n(z - z_0)^n$ is

$$R_n(z) = \frac{(z - z_0)^{n+1}}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}(z^* - z)} \, dz^* \tag{3}$$

(proof below). Writing out the corresponding partial sum of (1), we thus have

$$\begin{aligned} f(z) = f(z_0) &+ \frac{z - z_0}{1!} f'(z_0) + \frac{(z - z_0)^2}{2!} f''(z_0) + \cdots \\ &+ \frac{(z - z_0)^n}{n!} f^{(n)}(z_0) + R_n(z). \end{aligned} \tag{4}$$

This is called **Taylor's formula** *with remainder.*

We see that ***Taylor series are power series.*** From the last section we know that power series represent analytic functions. And we now show that *every* analytic function can be represented by power series, namely, by Taylor series (with various centers). This makes Taylor series very important in complex analysis. Indeed, they are more fundamental in complex analysis than their real counterparts are in calculus.

THEOREM 1

Taylor's Theorem

Let $f(z)$ be analytic in a domain D, and let $z = z_0$ be any point in D. Then there exists precisely one Taylor series (1) *with center z_0 that represents $f(z)$. This representation is valid in the largest open disk with center z_0 in which $f(z)$ is analytic. The remainders $R_n(z)$ of* (1) *can be represented in the form* (3). *The coefficients satisfy the inequality*

$$|a_n| \leqq \frac{M}{r^n} \tag{5}$$

where M is the maximum of $|f(z)|$ on a circle $|z - z_0| = r$ in D whose interior is also in D.

PROOF The key tool is Cauchy's integral formula in Sec. 14.3; writing z and z^* instead of z_0 and z (so that z^* is the variable of integration), we have

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{z^* - z} \, dz^*. \tag{6}$$

z lies inside C, for which we take a circle of radius r with center z_0 and interior in D (Fig. 364). We develop $1/(z^* - z)$ in (6) in powers of $z - z_0$. By a ***standard algebraic manipulation*** (worth remembering!) we first have

$$\frac{1}{z^* - z} = \frac{1}{z^* - z_0 - (z - z_0)} = \frac{1}{(z^* - z_0)\left(1 - \dfrac{z - z_0}{z^* - z_0}\right)}. \tag{7}$$

For later use we note that since z^* is on C while z is inside C, we have

$$\left|\frac{z - z_0}{z^* - z_0}\right| < 1 \qquad \text{(Fig. 364).} \tag{7*}$$

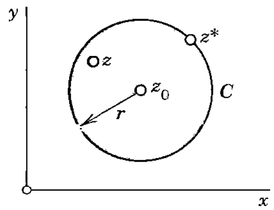

Fig. 364. Cauchy formula (6)

To (7) we now apply the sum formula for a finite geometric sum

$$1 + q + \cdots + q^n = \frac{1 - q^{n+1}}{1 - q} = \frac{1}{1 - q} - \frac{q^{n+1}}{1 - q} \qquad (q \neq 1), \tag{8*}$$

which we use in the form (take the last term to the other side and interchange sides)

$$\frac{1}{1 - q} = 1 + q + \cdots + q^n + \frac{q^{n+1}}{1 - q}. \tag{8}$$

Applying this with $q = (z - z_0)/(z^* - z_0)$ to the right side of (7), we get

$$\frac{1}{z^* - z} = \frac{1}{z^* - z_0}\left[1 + \frac{z - z_0}{z^* - z_0} + \left(\frac{z - z_0}{z^* - z_0}\right)^2 + \cdots + \left(\frac{z - z_0}{z^* - z_0}\right)^n\right]$$
$$+ \frac{1}{z^* - z}\left(\frac{z - z_0}{z^* - z_0}\right)^{n+1}$$

We insert this into (6). Powers of $z - z_0$ do not depend on the variable of integration z^*, so that we may take them out from under the integral sign. This yields

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(z^*)}{z^* - z_0}\,dz^* + \frac{z - z_0}{2\pi i}\oint_C \frac{f(z^*)}{(z^* - z_0)^2}\,dz^* + \cdots$$
$$\cdots + \frac{(z - z_0)^n}{2\pi i}\oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}}\,dz^* + R_n(z)$$

with $R_n(z)$ given by (3). The integrals are those in (2) related to the derivatives, so that we have proved the Taylor formula (4).

Since analytic functions have derivatives of all orders, we can take n in (4) as large as we please. If we let n approach infinity, we obtain (1). Clearly, (1) will converge and represent $f(z)$ if and only if

$$\lim_{n\to\infty} R_n(z) = 0. \tag{9}$$

We prove (9) as follows. Since z^* lies on C, whereas z lies inside C (Fig. 364), we have $|z^* - z| > 0$. Since $f(z)$ is analytic inside and on C, it is bounded, and so is the function $f(z^*)/(z^* - z)$, say,

$$\left| \frac{f(z^*)}{z^* - z} \right| \leqq \widetilde{M}$$

for all z^* on C. Also, C has the radius $r = |z^* - z_0|$ and the length $2\pi r$. Hence by the ML-inequality (Sec. 14.1) we obtain from (3)

$$\begin{aligned} |R_n| &= \frac{|z - z_0|^{n+1}}{2\pi} \left| \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}(z^* - z)} \, dz^* \right| \\ &\leqq \frac{|z - z_0|^{n+1}}{2\pi} \widetilde{M} \frac{1}{r^{n+1}} 2\pi r = \widetilde{M} \left| \frac{z - z_0}{r} \right|^{n+1} \end{aligned} \tag{10}$$

Now $|z - z_0| < r$ because z lies *inside* C. Thus $|z - z_0|/r < 1$, so that the right side approaches 0 as $n \to \infty$. This proves the convergence of the Taylor series. Uniqueness follows from Theorem 2 in the last section. Finally, (5) follows from (1) and the Cauchy inequality in Sec. 14.4. This proves Taylor's theorem. ■

Accuracy of Approximation. We can achieve any preassinged accuracy in approximating $f(z)$ by a partial sum of (1) by choosing n large enough. This is the practical aspect of formula (9).

Singularity, Radius of Convergence. On the circle of convergence of (1) there is at least one **singular point** of $f(z)$, that is, a point $z = c$ at which $f(z)$ is not analytic (but such that every disk with center c contains points at which $f(z)$ *is* analytic). We also say that $f(z)$ **is singular** at c or **has a singularity** at c. Hence the radius of convergence R of (1) is usually equal to the distance from z_0 to the nearest singular point of $f(z)$.

(Sometimes R can be greater than that distance: $\operatorname{Ln} z$ is singular on the negative real axis, whose distance from $z_0 = -1 + i$ is 1, but the Taylor series of $\operatorname{Ln} z$ with center $z_0 = -1 + i$ has radius of convergence $\sqrt{2}$.)

Power Series as Taylor Series

Taylor series are power series—of course! Conversely, we have

THEOREM 2 **Relation to the Last Section**

A power series with a nonzero radius of convergence is the Taylor series of its sum.

PROOF Given the power series

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + \cdots .$$

Then $f(z_0) = a_0$. By Theorem 5 in Sec. 15.3 we obtain

$$f'(z) = a_1 + 2a_2(z - z_0) + 3a_3(z - z_0)^2 + \cdots, \qquad \text{thus} \qquad f'(z_0) = a_1$$

$$f''(z) = 2a_2 + 3 \cdot 2(z - z_0) + \cdots, \qquad \text{thus} \qquad f''(z_0) = 2!a_2$$

and in general $f^{(n)}(z_0) = n!a_n$. With these coefficients the given series becomes the Taylor series of $f(z)$ with center z_0. ■

Comparison with Real Functions. One surprising property of complex analytic functions is that they have derivatives of all orders, and now we have discovered the other surprising property that they can always be represented by power series of the form (1). This is not true in general for ***real functions;*** there are real functions that have derivatives of all orders but cannot be represented by a power series. (Example: $f(x) = \exp(-1/x^2)$ if $x \neq 0$ and $f(0) = 0$; this function cannot be represented by a Maclaurin series in an open disk with center 0 because all its derivatives at 0 are zero.)

.mportant Special Taylor Series

These are as in calculus, with x replaced by complex z. Can you see why? (*Answer.* The coefficient formulas are the same.)

XAMPLE 1 Geometric Series

Let $f(z) = 1/(1-z)$. Then we have $f^{(n)}(z) = n!/(1-z)^{n+1}$, $f^{(n)}(0) = n!$. Hence the Maclaurin expansion of $1/(1-z)$ is the geometric series

(11)
$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots \qquad (|z| < 1).$$

$f(z)$ is singular at $z = 1$; this point lies on the circle of convergence. ■

. . LE Exponential Function

We know that the exponential function e^z (Sec. 13.5) is analytic for all z, and $(e^z)' = e^z$. Hence from (1) with $z_0 = 0$ we obtain the Maclaurin series

(12)
$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \cdots$$

This series is also obtained if we replace x in the familiar Maclaurin series of e^x by z.

Furthermore, by setting $z = iy$ in (12) and separating the series into the real and imaginary parts (see Theorem 2, Sec. 15.1) we obtain

$$e^{iy} = \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} = \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k+1}}{(2k+1)!}.$$

Since the series on the right are the familiar Maclaurin series of the real functions $\cos y$ and $\sin y$, this shows that we have rediscovered the **Euler formula**

(13)
$$e^{iy} = \cos y + i \sin y.$$

Indeed, one may use (12) for ***defining*** e^z and derive from (12) the basic properties of e^z. For instance, the differentiation formula $(e^z)' = e^z$ follows readily from (12) by termwise differentiation. ■

EXAMPLE 3 Trigonometric and Hyperbolic Functions

By substituting (12) into (1) of Sec. 13.6 we obtain

$$\text{(14)}\quad \begin{aligned} \cos z &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - + \cdots \\ \sin z &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - + \cdots . \end{aligned}$$

When $z = x$ these are the familiar Maclaurin series of the real functions $\cos x$ and $\sin x$. Similarly, by substituting (12) into (11), Sec. 13.6, we obtain

$$\text{(15)}\quad \begin{aligned} \cosh z &= \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots \\ \sinh z &= \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = z + \frac{z^3}{3!} + \frac{z^5}{5!} \cdots . \end{aligned}$$ ■

XAMPLE 4 Logarithm

From (1) it follows that

$$\text{(16)}\quad \operatorname{Ln}(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - + \cdots \qquad (|z| < 1).$$

Replacing z by $-z$ and multiplying both sides by -1, we get

$$\text{(17)}\quad -\operatorname{Ln}(1 - z) = \operatorname{Ln}\frac{1}{1 - z} = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots \qquad (|z| < 1).$$

By adding both series we obtain

$$\text{(18)}\quad \operatorname{Ln}\frac{1 + z}{1 - z} = 2\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \cdots\right) \qquad (|z| < 1).$$ ■

Practical Methods

The following examples show ways of obtaining Taylor series more quickly than by the use of the coefficient formulas. Regardless of the method used, the result will be the same. This follows from the uniqueness (see Theorem 1).

PLE 5 Substitution

Find the Maclaurin series of $f(z) = 1/(1 + z^2)$.

Solution. By substituting $-z^2$ for z in (11) we obtain

$$\text{(19)}\quad \frac{1}{1 + z^2} = \frac{1}{1 - (-z^2)} = \sum_{n=0}^{\infty} (-z^2)^n = \sum_{n=0}^{\infty} (-1)^n z^{2n} = 1 - z^2 + z^4 - z^6 + \cdots \qquad (|z| < 1).$$ ■

EXAMPLE 6 **Integration**

Find the Maclaurin series of $f(z) = \arctan z$.

Solution. We have $f'(z) = 1/(1 + z^2)$. Integrating (19) term by term and using $f(0) = 0$ we get

$$\arctan z = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} z^{2n+1} = z - \frac{z^3}{3} + \frac{z^5}{5} - + \cdots \qquad (|z| < 1);$$

this series represents the principal value of $w = u + iv = \arctan z$ defined as that value for which $|u| < \pi/2$.

EXAMPLE 7 **Development by Using the Geometric Series**

Develop $1/(c - z)$ in powers of $z - z_0$, where $c - z_0 \neq 0$.

Solution. This was done in the proof of Theorem 1, where $c = z^*$. The beginning was simple algebra and then the use of (11) with z replaced by $(z - z_0)/(c - z_0)$:

$$\frac{1}{c-z} = \frac{1}{c - z_0 - (z - z_0)} = \frac{1}{(c - z_0)\left(1 - \dfrac{z - z_0}{c - z_0}\right)} = \frac{1}{c - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{c - z_0}\right)^n$$

$$= \frac{1}{c - z_0}\left(1 + \frac{z - z_0}{c - z_0} + \left(\frac{z - z_0}{c - z_0}\right)^2 + \cdots\right).$$

This series converges for

$$\left|\frac{z - z_0}{c - z_0}\right| < 1, \qquad \text{that is,} \qquad |z - z_0| < |c - z_0|. \qquad \blacksquare$$

EXAMPLE 8 **Binomial Series, Reduction by Partial Fractions**

Find the Taylor series of the following function with center $z_0 = 1$.

$$f(z) = \frac{2z^2 + 9z + 5}{z^3 + z^2 - 8z - 12}$$

Solution. We develop $f(z)$ in partial fractions and the first fraction in a **binomial series**

$$\text{(20)} \qquad \begin{aligned} \frac{1}{(1+z)^m} &= (1+z)^{-m} = \sum_{n=0}^{\infty} \binom{-m}{n} z^n \\ &= 1 - mz + \frac{m(m+1)}{2!} z^2 - \frac{m(m+1)(m+2)}{3!} z^3 + \cdots \end{aligned}$$

with $m = 2$ and the second fraction in a geometric series, and then add the two series term by term. This gives

$$f(z) = \frac{1}{(z+2)^2} + \frac{2}{z-3} = \frac{1}{[3 + (z-1)]^2} - \frac{2}{2 - (z-1)} = \frac{1}{9}\left(\frac{1}{[1 + \frac{1}{3}(z-1)]^2}\right) - \frac{1}{1 - \frac{1}{2}(z-1)}$$

$$= \frac{1}{9}\sum_{n=0}^{\infty} \binom{-2}{n} \left(\frac{z-1}{3}\right)^n - \sum_{n=0}^{\infty} \left(\frac{z-1}{2}\right)^n = \sum_{n=0}^{\infty} \left[\frac{(-1)^n(n+1)}{3^{n+2}} - \frac{1}{2^n}\right](z-1)^n$$

$$= -\frac{8}{9} - \frac{31}{54}(z-1) - \frac{23}{108}(z-1)^2 - \frac{275}{1944}(z-1)^3 - \cdots.$$

We see that the first series converges for $|z - 1| < 3$ and the second for $|z - 1| < 2$. This had to be expected because $1/(z + 2)^2$ is singular at -2 and $2/(z - 3)$ at 3, and these points have distance 3 and 2, respectively, from the center $z_0 = 1$. Hence the whole series converges for $|z - 1| < 2$. $\blacksquare$

PROBLEM SET 15.4

1–12 TAYLOR AND MACLAURIN SERIES

Find the Taylor or Maclaurin series of the given function with the given point as center and determine the radius of convergence.

1. e^{-2z}, 0
2. $1/(1 - z^3)$, 0
3. e^z, $-2i$
4. $\cos^2 z$, 0
5. $\sin z$, $\pi/2$
6. $1/z$, 1
7. $1/(1 - z)$, i
8. $\mathrm{Ln}\,(1 - z)$, i
9. $e^{-z^2/2}$, 0
10. $e^{z^2} \int_0^z e^{-t^2}\, dt$, 0
11. $z^6 - z^4 + z^2 - 1$, 1
12. $\sinh (z - 2i)$, $2i$

13–16 HIGHER TRANSCENDENTAL FUNCTIONS

Find the Maclaurin series by termwise integrating the integrand. (The integrals cannot be evaluated by the usual methods of calculus. They define the **error function** erf z, **sine integral** Si(z), and **Fresnel integrals**[4] S(z) and C(z), which occur in statistics, heat conduction, optics, and other applications. These are special so-called higher transcendental functions.)

13. $\operatorname{erf} z = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^z e^{-t^2}\, dt$
14. $\mathrm{Si}(z) = \displaystyle\int_0^z \frac{\sin t}{t}\, dt$
15. $\mathrm{S}(z) = \displaystyle\int_0^z \sin t^2\, dt$
16. $\mathrm{C}(z) = \displaystyle\int_0^z \cos t^2\, dt$

17. **CAS PROJECT. sec, tan, arcsin. (a) Euler numbers.** The Maclaurin series

$$(21) \quad \sec z = E_0 - \frac{E_2}{2!} z^2 + \frac{E_4}{4!} z^4 - + \cdots$$

defines the *Euler numbers* E_{2n}. Show that $E_0 = 1$, $E_2 = -1$, $E_4 = 5$, $E_6 = -61$. Write a program that computes the E_{2n} from the coefficient formula in (1) or extracts them as a list from the series. (For tables see Ref. [GR1], p. 810, listed in App. 1.)

(b) Bernoulli numbers. The Maclaurin series

$$(22) \quad \frac{z}{e^z - 1} = 1 + B_1 z + \frac{B_2}{2!} z^2 + \frac{B_3}{3!} z^3 + \cdots$$

defines the *Bernoulli numbers* B_n. Using undetermined coefficients, show that

$$(23) \quad \begin{aligned} &B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6} \quad B_3 = 0. \\ &B_4 = -\frac{1}{30}, \quad B_5 = 0, \quad B_6 = \frac{1}{42}, \cdots. \end{aligned}$$

Write a program for computing B_n.

(c) Tangent. Using (1), (2), Sec. 13.6, and (22), show that $\tan z$ has the following Maclaurin series and calculate from it a table of $B_0, \cdots, B_{20}$:

$$(24) \quad \tan z = \frac{2i}{e^{2iz} - 1} - \frac{4i}{e^{4iz} - 1} - i = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{2n}(2^{2n} - 1)}{(2n)!} B_{2n} z^{2n-1}.$$

18. **(Inverse sine)** Developing $1/\sqrt{1 - z^2}$ and integrating, show that

$$\arcsin z = z + \left(\frac{1}{2}\right)\frac{z^3}{3} + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)\frac{z^5}{5} + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)\frac{z^7}{7} + \cdot \qquad (|z| < 1).$$

Show that this series represents the principal value of $\arcsin z$ (defined in Team Project 30, Sec. 13.7).

19. **(Undetermined coefficients)** Using the relation $\sin z = \tan z \cos z$ and the Maclaurin series of $\sin z$ and $\cos z$, find the first four nonzero terms of the Maclaurin series of $\tan z$. (Show the details.)

20. **TEAM PROJECT. Properties from Maclaurin Series.** Clearly, from series we can compute function values. In this project we show that properties of functions can often be discovered from their Taylor or Maclaurin series. Using suitable series, prove the following.

(a) The formulas for the derivatives of e^z, $\cos z$, $\sin z$, $\cosh z$, $\sinh z$, and $\mathrm{Ln}\,(1 + z)$

(b) $\frac{1}{2}(e^{iz} + e^{-iz}) = \cos z$

(c) $\sin z \neq 0$ for all pure imaginary $z = iy \neq 0$

[4] AUGUSTIN FRESNEL (1788–1827), French physicist and engineer, known for his work in optics