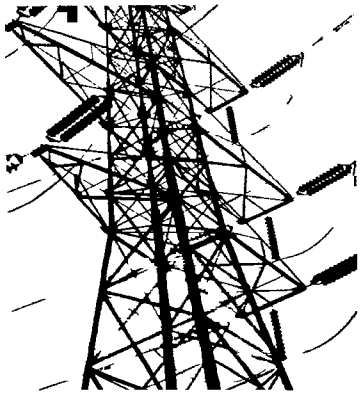

CHAPTER 16

Laurent Series. Residue Integration

Laurent series generalize Taylor series. Indeed, whereas a Taylor series has positive integer powers (and a constant term) and converges in a disk, a **Laurent series** (Sec. 16.1) is a series of positive ***and negative*** integer powers of $z - z_0$ and converges in an annulus (a circular ring) with center z_0. Hence by a Laurent series we can represent a given function $f(z)$ that is analytic in an annulus and may have singularities outside the ring as well as in the "hole" of the annulus.

We know that for a given function the Taylor series with a given center z_0 is unique. We shall see that, in contrast, a function $f(z)$ can have several Laurent series with the same center z_0 and valid in several concentric annuli. The most important of these series is that which converges for $0 < |z - z_0| < R$, that is, everywhere near the center z_0 except at z_0 itself, where z_0 is a singular point of $f(z)$. The series (or finite sum) of the negative powers of *this* Laurent series is called the **principal part** of the singularity of $f(z)$ at z_0, and is used to classify this singularity (Sec. 16.2). The coefficient of the power $1/(z - z_0)$ of *this* series is called the **residue** of $f(z)$ at z_0. Residues are used in an elegant and powerful integration method, called **residue integration,** for complex contour integrals (Sec. 16.3) as well as for certain complicated real integrals (Sec. 16.4).

Prerequisite: Chaps. 13, 14, Sec. 15.2.
Sections that may be omitted in a shorter course: 16.2, 16.4.
References and Answers to Problems: App. 1. Part D, App. 2.

16.1 Laurent Series

Laurent series generalize Taylor series. If in an application we want to develop a function $f(z)$ in powers of $z - z_0$ when $f(z)$ is singular at z_0 (as defined in Sec. 15.4), we cannot use a Taylor series. Instead we may use a new kind of series, called **Laurent series,**[1] consisting of positive integer powers of $z - z_0$ (and a constant) as well as ***negative integer powers*** of $z - z_0$; this is the new feature.

Laurent series are also used for classifying singularities (Sec. 16.2) and in a powerful integration method ("residue integration", Sec. 16.3).

A Laurent series of $f(z)$ converges in an annulus (in the "hole" of which $f(z)$ may have singularities), as follows.

[1]PIERRE ALPHONSE LAURENT (1813–1854), French military engineer and mathematician, published the theorem in 1843.

THEOREM 1 **Laurent's Theorem**

Let $f(z)$ be analytic in a domain containing two concentric circles C_1 and C_2 with center z_0 and the annulus between them (blue in Fig. 367). *Then $f(z)$ can be represented by the Laurent series*

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

$$= a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots \quad (1)$$

$$\cdots + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots$$

consisting of nonnegative and negative powers. The coefficients of this Laurent series are given by the integrals

$$(2) \qquad a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}}\, dz^*, \quad b_n = \frac{1}{2\pi i} \oint_C (z^* - z_0)^{n-1} f(z^*)\, dz^*,$$

taken counterclockwise around any simple closed path C that lies in the annulus and encircles the inner circle, as in Fig. 367. [The variable of integration is denoted by z^* since z is used in (1).]

This series converges and represents $f(z)$ in the enlarged open annulus obtained from the given annulus by continuously increasing the outer circle C_1 and decreasing C_2 until each of the two circles reaches a point where $f(z)$ is singular.

In the important special case that z_0 is the only singular point of $f(z)$ inside C_2, this circle can be shrunk to the point z_0, giving convergence in a disk except at the center. In this case the series (or finite sum) of the negative powers of (1) *is called the* **principal part** *of the singularity of $f(z)$ at z_0.*

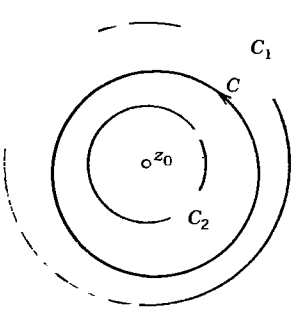

Fig. 367. Laurent's theorem

COMMENT. Obviously, instead of (1), (2) we may write (denoting b_n by a_{-n})

$$(1') \qquad f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$$

where all the coefficients are now given by a single integral formula, namely,

$$(2') \qquad a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}}\, dz^* \qquad (n = 0, \pm 1, \pm 2, \cdots).$$

PROOF We prove Laurent's theorem. **(a)** ***The nonnegative powers*** are those of a Taylor series. To see this, we use Cauchy's integral formula (3) in Sec. 14.3 with z^* (instead of z) as the variable of integration and z instead of z_0. Let $g(z)$ and $h(z)$ denote the functions represented by the two terms in (3), Sec. 14.3. Then

$$(3) \qquad f(z) = g(z) + h(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z^*)}{z^* - z}\, dz^* - \frac{1}{2\pi i} \oint_{C_2} \frac{f(z^*)}{z^* - z}\, dz^*.$$

Here z is any point in the given annulus and we integrate counterclockwise over both C_1 and C_2, so that the minus sign appears since in (3) of Sec. 14.3 the integration over C_2 is taken clockwise. We transform each of these two integrals as in Sec. 15.4. The first integral is precisely as in Sec. 15.4. Hence we get precisely the same result, namely, the Taylor series of $g(z)$,

$$(4) \qquad g(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z^*)}{z^* - z}\, dz^* = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

with coefficients [see (2), Sec. 15.4, counterclockwise integration]

$$(5) \qquad a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z^*)}{(z^* - z_0)^{n+1}}\, dz^*.$$

Here we can replace C_1 by C (see Fig. 367), by the principle of deformation of path, since z_0, the point where the integrand in (5) is not analytic, is not a point of the annulus. This proves the formula for the a_n in (2).

(b) ***The negative powers*** in (1) and the formula for b_n in (2) are obtained if we consider $h(z)$ (the second integral times $-1/(2\pi i)$ in (3). Since z lies in the annulus, it lies in the exterior of the path C_2. Hence the situation differs from that for the first integral. The essential point is that instead of [see (7*) in Sec. 15.4]

$$(6) \quad \text{(a)} \quad \left| \frac{z - z_0}{z^* - z_0} \right| < 1 \qquad \text{we now have} \qquad \text{(b)} \quad \left| \frac{z^* - z_0}{z - z_0} \right| < 1.$$

Consequently, we must develop the expression $1/(z^* - z)$ in the integrand of the second integral in (3) in powers of $(z^* - z_0)/(z - z_0)$ (instead of the reciprocal of this) to get a *convergent* series. We find

$$\frac{1}{z^* - z} = \frac{1}{z^* - z_0 - (z - z_0)} = \frac{-1}{(z - z_0)\left(1 - \dfrac{z^* - z_0}{z - z_0}\right)}.$$

Compare this for a moment with (7) in Sec. 15.4, to really understand the difference. Then go on and apply formula (8), Sec. 15.4, for a finite geometric sum, obtaining

$$\frac{1}{z^* - z} = -\frac{1}{z - z_0}\left\{1 + \frac{z^* - z_0}{z - z_0} + \left(\frac{z^* - z_0}{z - z_0}\right)^2 + \cdots + \left(\frac{z^* - z_0}{z - z_0}\right)^n\right\} - \frac{1}{z - z^*}\left(\frac{z^* - z_0}{z - z_0}\right)^{n+1}.$$

Multiplication by $-f(z^*)/2\pi i$ and integration over C_2 on both sides now yield

$$\begin{aligned} h(z) &= -\frac{1}{2\pi i}\oint_{C_2}\frac{f(z^*)}{z^* - z}\,dz^* \\ &= \frac{1}{2\pi i}\left\{\frac{1}{z - z_0}\oint_{C_2} f(z^*)\,dz^* + \frac{1}{(z - z_0)^2}\oint_{C_2}(z^* - z_0)f(z^*)\,dz^* + \cdots \right. \\ &\qquad + \frac{1}{(z - z_0)^n}\oint_{C_2}(z^* - z_0)^{n-1}f(z^*)\,dz^* \\ &\qquad \left. + \frac{1}{(z - z_0)^{n+1}}\oint_{C_2}(z^* - z_0)^n f(z^*)\,dz^*\right\} + R_n^*(z) \end{aligned}$$

with the last term on the right given by

$$R_n^*(z) = \frac{1}{2\pi i(z - z_0)^{n+1}}\oint_{C_2}\frac{(z^* - z_0)^{n+1}}{z - z^*} f(z^*)\,dz^*. \tag{7}$$

As before, we can integrate over C instead of C_2 in the integrals on the right. We see that on the right, the power $1/(z - z_0)^n$ is multiplied by b_n as given in (2). This establishes Laurent's theorem, provided

$$\lim_{n\to\infty} R_n^*(z) = 0. \tag{8}$$

(c) ***Convergence proof of* (8).** Very often (1) will have only finitely many negative powers. Then there is nothing to be proved. Otherwise, we begin by noting that $f(z^*)/(z - z^*)$ in (7) is bounded in absolute value, say,

$$\left|\frac{f(z^*)}{z - z^*}\right| < \tilde{M} \qquad \text{for all } z^* \text{ on } C_2$$

because $f(z^*)$ is analytic in the annulus and on C_2, and z^* lies on C_2 and z outside, so that $z - z^* \neq 0$. From this and the *ML*-inequality (Sec. 14.1) applied to (7) we get the inequality (L = length of C_2, $|z^* - z_0|$ = radius of C_2 = *const*)

$$|R_n^*(z)| \leqq \frac{1}{2\pi|z - z_0|^{n+1}}\,|z^* - z_0|^{n+1}\,\tilde{M}L = \frac{\tilde{M}L}{2\pi}\left|\frac{z^* - z_0}{z - z_0}\right|^{n+1}.$$

From (6b) we see that the expression on the right approaches zero as n approaches infinity. This proves (8). The representation (1) with coefficients (2) is now established in the given annulus.

(d) ***Convergence of* (1) *in the enlarged annulus.*** The first series in (1) is a Taylor series [representing $g(z)$]; hence it converges in the disk D with center z_0 whose radius equals the distance of the singularity (or singularities) closest to z_0. Also, $g(z)$ must be singular at all points outside C_1 where $f(z)$ is singular.

The second series in (1), representing $h(z)$, is a power series in $Z = 1/(z - z_0)$. Let the given annulus be $r_2 < |z - z_0| < r_1$, where r_1 and r_2 are the radii of C_1 and C_2, respectively (Fig. 367). This corresponds to $1/r_2 > |Z| > 1/r_1$. Hence this power series in Z must converge at least in the disk $|Z| < 1/r_2$. This corresponds to the exterior $|z - z_0| > r_2$ of C_2, so that $h(z)$ is analytic for all z outside C_2. Also, $h(z)$ must be singular inside C_2 where $f(z)$ is singular, and the series of the negative powers of (1) converges for all z in the exterior E of the circle with center z_0 and radius equal to the maximum distance from z_0 to the singularities of $f(z)$ inside C_2. The domain common to D and E is the enlarged open annulus characterized near the end of Laurent's theorem, whose proof is now complete. ■

Uniqueness. *The Laurent series of a given analytic function $f(z)$ in its annulus of convergence is unique* (see Team Project 24). *However, $f(z)$ may have different Laurent series in two annuli with the same center;* see the examples below. The uniqueness is essential. As for a Taylor series, to obtain the coefficients of Laurent series, we do not generally use the integral formulas (2); instead, we use various other methods, some of which we shall illustrate in our examples. If a Laurent series has been found by any such process, the uniqueness guarantees that it must be ***the*** Laurent series of the given function in the given annulus.

EXAMPLE 1 Use of Maclaurin Series

Find the Laurent series of $z^{-5} \sin z$ with center 0.

Solution. By (14). Sec. 15.4. we obtain

$$z^{-5} \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n-4} = \frac{1}{z^4} - \frac{1}{6z^2} + \frac{1}{120} - \frac{1}{5040} z^2 + - \cdots \qquad (|z| > 0).$$

Here the "annulus" of convergence is the whole complex plane without the origin and the principal part of the series at 0 is $z^{-4} - \frac{1}{6} z^{-2}$. ■

EXAMPLE 2 Substitution

Find the Laurent series of $z^2 e^{1/z}$ with center 0.

Solution. From (12) in Sec. 15.4 with z replaced by $1/z$ we obtain a Laurent series whose principal part is an infinite series,

$$z^2 e^{1/z} = z^2 \left(1 + \frac{1}{1!z} + \frac{1}{2!z^2} + \cdots\right) = z^2 + z + \frac{1}{2} + \frac{1}{3!z} + \frac{1}{4!z^2} + \cdots \qquad (|z| > 0). \blacksquare$$

EXAMPLE 3 Development of $1/(1 - z)$

Develop $1/(1 - z)$ (a) in nonnegative powers of z, (b) in negative powers of z.

Solution.

(a) $$\frac{1}{1 - z} = \sum_{n=0}^{\infty} z^n \qquad \text{(valid if } |z| < 1).$$

(b) $$\frac{1}{1 - z} = \frac{-1}{z(1 - z^{-1})} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} = -\frac{1}{z} - \frac{1}{z^2} - \cdots \qquad \text{(valid if } |z| > 1). \blacksquare$$

EXAMPLE 4 Laurent Expansions in Different Concentric Annuli

Find all Laurent series of $1/(z^3 - z^4)$ with center 0.

Solution. Multiplying by $1/z^3$, we get from Example 3

(I) $$\frac{1}{z^3 - z^4} = \sum_{n=0}^{\infty} z^{n-3} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + \cdots \qquad (0 < |z| < 1),$$

(II) $$\frac{1}{z^3 - z^4} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+4}} = -\frac{1}{z^4} - \frac{1}{z^5} - \cdots \qquad (|z| > 1).$$ ■

EXAMPLE 5 Use of Partial Fractions

Find all Taylor and Laurent series of $f(z) = \dfrac{-2z + 3}{z^2 - 3z + 2}$ with center 0.

Solution. In terms of partial fractions,

$$f(z) = -\frac{1}{z - 1} - \frac{1}{z - 2}.$$

(a) and (b) in Example 3 take care of the first fraction. For the second fraction,

(c) $$-\frac{1}{z - 2} = \frac{1}{2\left(1 - \frac{1}{2}z\right)} = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n \qquad (|z| < 2),$$

(d) $$-\frac{1}{z - 2} = -\frac{1}{z\left(1 - \frac{2}{z}\right)} = -\sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} \qquad (|z| > 2).$$

(I) From (a) and (c), valid for $|z| < 1$ (see Fig. 368),

$$f(z) = \sum_{n=0}^{\infty} \left(1 + \frac{1}{2^{n+1}}\right) z^n = \frac{3}{2} + \frac{5}{4} z + \frac{9}{8} z^2 + \cdots.$$

(II) From (c) and (b), valid for $1 < |z| < 2$,

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n - \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} = \frac{1}{2} + \frac{1}{4} z + \frac{1}{8} z^2 + \cdots - \frac{1}{z} - \frac{1}{z^2} - \cdots.$$

(III) From (d) and (b), valid for $|z| > 2$,

$$f(z) = -\sum_{n=0}^{\infty} (2^n + 1) \frac{1}{z^{n+1}} = -\frac{2}{z} - \frac{3}{z^2} - \frac{5}{z^3} - \frac{9}{z^4} - \cdots.$$ ■

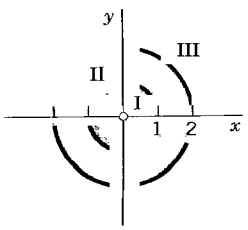

Fig. 368. Regions of convergence in Example 5

If $f(z)$ in Laurent's theorem is analytic inside C_2, the coefficients b_n in (2) are zero by Cauchy's integral theorem, so that the Laurent series reduces to a Taylor series. Examples 3(a) and 5(I) illustrate this.

1–6 LAURENT SERIES NEAR A SINGULARITY AT 0

Expand the given function in a Laurent series that converges for $0 < |z| < R$ and determine the precise region of convergence. (Show the details of your work.)

1. $\dfrac{1}{z^4 - z^5}$
2. $z \cos \dfrac{1}{z}$
3. $\dfrac{e^{-z}}{z^3}$
4. $\dfrac{\cosh 2z}{z^2}$
5. $z^{-3}e^{1/z^2}$
6. $\dfrac{e^z}{z^2 - z^3}$

7–14 LAURENT SERIES NEAR A SINGULARITY AT z_0

Expand the given function in a Laurent series that converges for $0 < |z - z_0| < R$ and determine the precise region of convergence. (Show details.)

7. $\dfrac{e^z}{z - 1}$, $z_0 = 1$
8. $\dfrac{\sin z}{(z - \frac{1}{4}\pi)^3}$, $z_0 = \frac{1}{4}\pi$
9. $\dfrac{1}{z^2 + 1}$, $z_0 = i$
10. $\dfrac{\cos z}{(z - \pi)^4}$, $z_0 = \pi$
11. $\dfrac{1}{(z + i)^2 - (z + i)}$, $z_0 = -i$
12. $\dfrac{z^3}{(z + i)^2}$, $z_0 = -i$
13. $\dfrac{z^2 - 4}{z - 1}$, $z_0 = 1$
14. $z^2 \sinh \dfrac{1}{z}$, $z_0 = 0$

15–23 TAYLOR AND LAURENT SERIES

Find all Taylor and Laurent series with center $z = z_0$ and determine the precise regions of convergence.

15. $\dfrac{1}{1 - z^3}$, $z_0 = 0$
16. $\dfrac{1}{1 - z^2}$, $z_0 = 1$
17. $\dfrac{z^2}{1 - z^4}$, $z_0 = 0$
18. $\dfrac{1}{z}$, $z_0 = 1$
19. $\dfrac{z^3 - 2iz^2}{(z - i)^2}$, $z_0 = i$
20. $\dfrac{\sinh z}{(z - 1)^4}$, $z_0 = 1$
21. $\dfrac{4z - 1}{z^4 - 1}$, $z_0 = 0$
22. $\dfrac{1}{z^2}$, $z_0 = i$
23. $\dfrac{\sin z}{z + \frac{1}{2}\pi}$, $z_0 = -\frac{1}{2}\pi$

24. **TEAM PROJECT. Laurent Series. (a) Uniqueness.** Prove that the Laurent expansion of a given analytic function in a given annulus is unique.

    (b) Accumulation of singularities. Does $\tan(1/z)$ have a Laurent series that converges in a region $0 < |z| < R$? (Give a reason.)

    (c) Integrals. Expand the following functions in a Laurent series that converges for $|z| > 0$:

$$\frac{1}{z^2}\int_0^z \frac{e^t - 1}{t}\,dt, \qquad \frac{1}{z^3}\int_0^z \frac{\sin t}{t}\,dt.$$

25. **CAS PROJECT. Partial Fractions.** Write a program for obtaining Laurent series by the use of partial fractions. Using the program, verify the calculations in Example 5 of the text. Apply the program to two other functions of your choice.

16.2 Singularities and Zeros. Infinity

Roughly, a *singular point* of an analytic function $f(z)$ is a z_0 at which $f(z)$ ceases to be analytic, and a *zero* is a z at which $f(z) = 0$. Precise definitions follow below. In this section we show that Laurent series can be used for classifying singularities and Taylor series for discussing zeros.

Singularities were defined in Sec. 15.4, as we shall now recall and extend. We also remember that. by definition, a function is a ***single-valued*** relation, as was emphasized in Sec. 13.3.

We say that a function $f(z)$ **is singular** or **has a singularity** at a point $z = z_0$ if $f(z)$ is not analytic (perhaps not even defined) at $z = z_0$, but every neighborhood of $z = z_0$ contains points at which $f(z)$ is analytic. We also say that $z = z_0$ is a **singular point** of $f(z)$.

We call $z = z_0$ an **isolated singularity** of $f(z)$ if $z = z_0$ has a neighborhood without further singularities of $f(z)$. *Example:* $\tan z$ has isolated singularities at $\pm\pi/2$, $\pm 3\pi/2$, etc.; $\tan(1/z)$ has a nonisolated singularity at 0. (Explain!)

Isolated singularities of $f(z)$ at $z = z_0$ can be classified by the Laurent series

$$(1) \qquad f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \qquad \text{(Sec. 16.1)}$$

valid ***in the immediate neighborhood*** of the singular point $z = z_0$, except at z_0 itself, that is, in a region of the form

$$0 < |z - z_0| < R.$$

The sum of the first series is analytic at $z = z_0$, as we know from the last section. The second series, containing the negative powers, is called the **principal part** of (1), as we remember from the last section. If it has only finitely many terms, it is of the form

$$(2) \qquad \frac{b_1}{z - z_0} + \cdots + \frac{b_m}{(z - z_0)^m} \qquad (b_m \neq 0).$$

Then the singularity of $f(z)$ at $z = z_0$ is called a **pole,** and m is called its **order.** Poles of the first order are also known as **simple poles.**

If the principal part of (1) has infinitely many terms, we say that $f(z)$ has at $z = z_0$ an **isolated essential singularity.**

We leave aside nonisolated singularities.

EXAMPLE 1 Poles. Essential Singularities

The function

$$f(z) = \frac{1}{z(z - 2)^5} + \frac{3}{(z - 2)^2}$$

has a simple pole at $z = 0$ and a pole of fifth order at $z = 2$. Examples of functions having an isolated essential singularity at $z = 0$ are

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!z^n} = 1 + \frac{1}{z} + \frac{1}{2!z^2} +$$

and

$$\sin \frac{1}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)!z^{2n+1}} = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - + \cdots.$$

Section 16.1 provides further examples. For instance, Example 1 shows that $z^{-5} \sin z$ has a fourth-order pole at 0. Example 4 shows that $1/(z^3 - z^4)$ has a third-order pole at 0 and a Laurent series with infinitely many negative powers. This is no contradiction, since this series is valid for $|z| > 1$; it merely tells us that in classifying singularities it is quite important to consider the Laurent series valid ***in the immediate neighborhood*** of a singular point. In Example 4 this is the series (1), which has three negative powers. ■

The classification of singularities into poles and essential singularities is not merely a formal matter, because the behavior of an analytic function in a neighborhood of an essential singularity is entirely different from that in the neighborhood of a pole.

EXAMPLE 2 Behavior Near a Pole

$f(z) = 1/z^2$ has a pole at $z = 0$, and $|f(z)| \to \infty$ as $z \to 0$ in any manner. This illustrates the following theorem. ■

THEOREM 1 **Poles**

If $f(z)$ is analytic and has a pole at $z = z_0$, then $|f(z)| \to \infty$ as $z \to z_0$ in any manner.

The proof is left to the student (see Prob. 12).

EXAMPLE 3 **Behavior Near an Essential Singularity**

The function $f(z) = e^{1/z}$ has an essential singularity at $z = 0$. It has no limit for approach along the imaginary axis; it becomes infinite if $z \to 0$ through positive real values, but it approaches zero if $z \to 0$ through negative real values. It takes on any given value $c = c_0 e^{i\alpha} \neq 0$ in an arbitrarily small ϵ-neighborhood of $z = 0$. To see the letter, we set $z = re^{i\theta}$, and then obtain the following complex equation for r and θ, which we must solve:

$$e^{1/z} = e^{(\cos\theta - i\sin\theta)/r} = c_0 e^{i\alpha}.$$

Equating the absolute values and the arguments, we have $e^{(\cos\theta)/r} = c_0$, that is

$$\cos\theta = r \ln c_0, \qquad \text{and} \qquad -\sin\theta = \alpha r$$

respectively. From these two equations and $\cos^2\theta + \sin^2\theta = r^2(\ln c_0)^2 + \alpha^2 r^2 = 1$ we obtain the formulas

$$r^2 = \frac{1}{(\ln c_0)^2 + \alpha^2} \qquad \text{and} \qquad \tan\theta = -\frac{\alpha}{\ln c_0}.$$

Hence r can be made arbitrarily small by adding multiples of 2π to α, leaving c unaltered. This illustrates the very famous *Picard's theorem* (with $z = 0$ as the exceptional value). For the rather complicated proof, see Ref. [D4], vol. 2, p. 258. For Picard, see Sec. 1.7. ■

THEOREM 2 **Picard's Theorem**

If $f(z)$ is analytic and has an isolated essential singularity at a point z_0, it takes on every value, with at most one exceptional value, in an arbitrarily small ϵ-neighborhood of z_0.

Removable Singularities. We say that a function $f(z)$ has a *removable singularity* at $z = z_0$ if $f(z)$ is not analytic at $z = z_0$, but can be made analytic there by assigning a suitable value $f(z_0)$. Such singularities are of no interest since they can be removed as just indicated. *Example:* $f(z) = (\sin z)/z$ becomes analytic at $z = 0$ if we define $f(0) = 1$.

Zeros of Analytic Functions

A **zero** of an analytic function $f(z)$ in a domain D is a $z = z_0$ in D such that $f(z_0) = 0$. A zero has **order** n if not only f but also the derivatives $f', f'', \cdots, f^{(n-1)}$ are all 0 at $z = z_0$ but $f^{(n)}(z_0) \neq 0$. A first-order zero is also called a **simple zero.** For a second-order zero, $f(z_0) = f'(z_0) = 0$ but $f''(z_0) \neq 0$. And so on.

EXAMPLE 4 **Zeros**

The function $1 + z^2$ has simple zeros at $\pm i$. The function $(1 - z^4)^2$ has second-order zeros at ± 1 and $\pm i$. The function $(z - a)^3$ has a third-order zero at $z = a$. The function e^z has no zeros (see Sec. 13.5). The function $\sin z$ has simple zeros at $0, \pm\pi, \pm 2\pi, \cdots$, and $\sin^2 z$ has second-order zeros at these points. The function $1 - \cos z$ has second-order zeros at $0, \pm 2\pi, \pm 4\pi, \cdots$, and the function $(1 - \cos z)^2$ has fourth-order zeros at these points. ■

Taylor Series at a Zero. At an nth-order zero $z = z_0$ of $f(z)$, the derivatives $f'(z_0), \cdots, f^{(n-1)}(z_0)$ are zero, by definition. Hence the first few coefficients $a_0, \cdots, a_{n-1}$ of the Taylor series (1), Sec. 15.4, are zero, too, whereas $a_n \neq 0$, so that this series takes the form

$$\begin{aligned} f(z) &= a_n(z - z_0)^n + a_{n+1}(z - z_0)^{n+1} + \cdots \\ &= (z - z_0)^n [a_n + a_{n+1}(z - z_0) + a_{n+2}(z - z_0)^2 + \cdots] \qquad (a_n \neq 0). \end{aligned} \tag{3}$$

This is characteristic of such a zero, because if $f(z)$ has such a Taylor series, it has an nth-order zero at $z = z_0$, as follows by differentiation.

Whereas nonisolated singularities may occur, for zeros we have

THEOREM 3 **Zeros**

The zeros of an analytic function $f(z)$ ($\not\equiv 0$) are isolated; that is, each of them has a neighborhood that contains no further zeros of $f(z)$.

PROOF The factor $(z - z_0)^n$ in (3) is zero only at $z = z_0$. The power series in the brackets $[\cdots]$ represents an analytic function (by Theorem 5 in Sec. 15.3), call it $g(z)$. Now $g(z_0) = a_n \neq 0$, since an analytic function is continuous, and because of this continuity, also $g(z) \neq 0$ in some neighborhood of $z = z_0$. Hence the same holds of $f(z)$. ■

This theorem is illustrated by the functions in Example 4.

Poles are often caused by zeros in the denominator. (*Example:* $\tan z$ has poles where $\cos z$ is zero.) This is a major reason for the importance of zeros. The key to the connection is the following theorem, whose proof follows from (3) (see Team Project 24).

THEOREM 4 **Poles and Zeros**

Let $f(z)$ be analytic at $z = z_0$ and have a zero of nth order at $z = z_0$. Then $1/f(z)$ has a pole of nth order at $z = z_0$; and so does $h(z)/f(z)$, provided $h(z)$ is analytic at $z = z_0$ and $h(z_0) \neq 0$.

Riemann Sphere. Point at Infinity

When we want to study complex functions for large $|z|$, the complex plane will generally become rather inconvenient. Then it may be better to use a representation of complex numbers on the so-called **Riemann sphere.** This is a sphere S of diameter 1 touching the complex z-plane at $z = 0$ (Fig. 369), and we let the image of a point P (a number z in the plane) be the intersection P^* of the segment PN with S, where N is the "North Pole" diametrically opposite to the origin in the plane. Then to each z there corresponds a point on S.

Conversely, each point on S represents a complex number z, except for N, which does not correspond to any point in the complex plane. This suggests that we introduce an additional point, called the **point at infinity** and denoted ∞ ("infinity") and let its image be N. The complex plane together with ∞ is called the **extended complex plane.** The complex plane is often called the *finite complex plane,* for distinction, or simply the

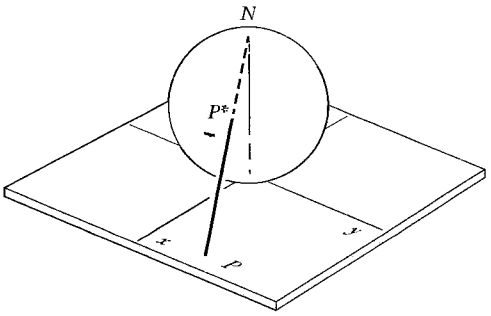

Fig. 369. Riemann sphere

complex plane as before. The sphere S is called the **Riemann sphere.** The mapping of the extended complex plane onto the sphere is known as a **stereographic projection.** (What is the image of the Northern Hemisphere? Of the Western Hemisphere? Of a straight line through the origin?)

Analytic or Singular at Infinity

If we want to investigate a function $f(z)$ for large $|z|$, we may now set $z = 1/w$ and investigate $f(z) = f(1/w) \equiv g(w)$ in a neighborhood of $w = 0$. We define $f(z)$ to be **analytic** or **singular at infinity** if $g(w)$ is analytic or singular, respectively, at $w = 0$. We also define

$$g(0) = \lim_{w \to 0} g(w) \tag{4}$$

if this limit exists.

Furthermore, we say that $f(z)$ has an *nth-order zero at infinity* if $f(1/w)$ has such a zero at $w = 0$. Similarly for poles and essential singularities.

EXAMPLE 5 Functions Analytic or Singular at Infinity. Entire and Meromorphic Functions

The function $f(z) = 1/z^2$ is analytic at ∞ since $g(w) = f(1/w) = w^2$ is analytic at $w = 0$, and $f(z)$ has a second-order zero at ∞. The function $f(z) = z^3$ is singular at ∞ and has a third-order pole there since the function $g(w) = f(1/w) = 1/w^3$ has such a pole at $w = 0$. The function e^z has an essential singularity at ∞ since $e^{1/w}$ has such a singularity at $w = 0$. Similarly, $\cos z$ and $\sin z$ have an essential singularity at ∞.

Recall that an **entire function** is one that is analytic everywhere in the (finite) complex plane. Liouville's theorem (Sec. 14.4) tells us that the only *bounded* entire functions are the constants, hence any nonconstant entire function must be unbounded. Hence it has a singularity at ∞, a pole if it is a polynomial or an essential singularity if it is not. The functions just considered are typical in this respect.

An analytic function whose only singularities in the finite plane are poles is called a **meromorphic function.** Examples are rational functions with nonconstant denominator, $\tan z$, $\cot z$, $\sec z$, and $\csc z$. ■

In this section we used Laurent series for investigating singularities. In the next section we shall use these series for an elegant integration method.

[illegible]

1–10 SINGULARITIES

Determine the location and kind of the singularities of the following functions in the finite plane and at infinity. In the case of poles also state the order.

1. $\tan^2 \pi z$
2. $z + \dfrac{2}{z} - \dfrac{3}{z^2}$
3. $\cot z^2$
4. $z^3 e^{1/(z-1)}$
5. $\cos z - \sin z$
6. $1/(\cos z - \sin z)$

7. $\dfrac{\sin 3z}{(z^4 - 1)^4}$

8. $\dfrac{4}{z - 1} + \dfrac{2}{(z - 1)^2} - \dfrac{8}{(z - 1)^3}$

9. $\cosh [1/(z^2 + 1)]$

10. $e^{1/(z-1)}/(e^z - 1)$

11. **(Essential singularity)** Discuss e^{1/z^2} in a similar way as $e^{1/z}$ is discussed in Example 3.

12. **(Poles)** Verify Theorem 1 for $f(z) = z^{-3} - z^{-1}$. Prove Theorem 1.

13–22 ZEROS

Determine the location and order of the zeros.

13. $(z + 16i)^4$

14. $(z^4 - 16)^4$

15. $z^{-3} \sin^3 \pi z$

16. $\cosh^2 z$

17. $(3z^2 + 1)e^{-z}$

18. $(z^2 - 1)^2(e^{z^2} - 1)$

19. $(z^2 + 4)(e^z - 1)^2$

20. $(\sin z - 1)^3$

21. $(1 - \cos z)^2$

22. $e^z - e^{2z}$

23. **(Zeros)** If $f(z)$ is analytic and has a zero of order n at $z = z_0$, show that $f^2(z)$ has a zero of order $2n$.

24. **TEAM PROJECT. Zeros. (a) Derivative.** Show that if $f(z)$ has a zero of order $n > 1$ at $z = z_0$, then $f'(z)$ has a zero of order $n - 1$ at z_0.

(b) Poles and zeros. Prove Theorem 4.

(c) Isolated k-points. Show that the points at which a nonconstant analytic function $f(z)$ has a given value k are isolated.

(d) Identical functions. If $f_1(z)$ are analytic in a domain D and equal at a sequence of points z_n in D that converges in D, show that $f_1(z) \equiv f_2(z)$ in D.

25. **(Riemann sphere)** Assuming that we let the image of the x-axis be meridians 0° and 180°, describe and sketch (or graph) the images of the following regions on the Riemann sphere: (a) $|z| > 100$, (b) the lower half-plane, (c) $\frac{1}{2} \leqq |z| \leqq 2$.

16.3 Residue Integration Method

The purpose of Cauchy's residue integration method is the evaluation of integrals

$$\oint_C f(z)\, dz$$

taken around a simple close path C. The idea is as follows.

If $f(z)$ is analytic everywhere on C and inside C, such an integral is zero by Cauchy's integral theorem (Sec. 14.2), and we are done.

If $f(z)$ has a singularity at a point $z = z_0$ inside C, but is otherwise analytic on C and inside C, then $f(z)$ has a Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots$$

that converges for all points near $z = z_0$ (except at $z = z_0$ itself), in some domain of the form $0 < |z - z_0| < R$ (sometimes called a **deleted neighborhood,** an old-fashioned term that we shall not use). Now comes the key idea. The coefficient b_1 of the first negative power $1/(z - z_0)$ of this Laurent series is given by the integral formula (2) in Sec. 16.1 with $n = 1$, namely,

$$b_1 = \frac{1}{2\pi i} \oint_C f(z)\, dz.$$

Now, since we can obtain Laurent series by various methods, without using the integral formulas for the coefficients (see the examples in Sec. 16.1), we can find b_1 by one of those methods and then use the formula for b_1 for evaluating the integral, that is,

(1)
$$\oint_C f(z)\, dz = 2\pi i b_1.$$

Here we integrate counterclockwise around a simple closed path C that contains $z = z_0$ in its interior (but no other singular points of $f(z)$ on or inside C!).

The coefficient b_1 is called the **residue** of $f(z)$ at $z = z_0$ and we denote it by

$$b_1 = \operatorname*{Res}_{z=z_0} f(z). \tag{2}$$

EXAMPLE 1 Evaluation of an Integral by Means of a Residue

Integrate the function $f(z) = z^{-4} \sin z$ counterclockwise around the unit circle C.

Solution. From (14) in Sec. 15.4 we obtain the Laurent series

$$f(z) = \frac{\sin z}{z^4} = \frac{1}{z^3} - \frac{1}{3!z} + \frac{z}{5!} - \frac{z^3}{7!} + - \cdots$$

which converges for $|z| > 0$ (that is, for all $z \neq 0$). This series shows that $f(z)$ has a pole of third order at $z = 0$ and the residue $b_1 = -1/3!$. From (1) we thus obtain the answer

$$\oint_C \frac{\sin z}{z^4}\, dz = 2\pi i b_1 = -\frac{\pi i}{3}$$ ■

EXAMPLE 2 CAUTION! Use the Right Laurent Series!

Integrate $f(z) = 1/(z^3 \quad z^4)$ clockwise around the circle C: $|z| = 1/2$.

Solution. $z^3 - z^4 = z^3(1 - z)$ shows that $f(z)$ is singular at $z = 0$ and $z = 1$. Now $z = 1$ lies outside C. Hence it is of no interest here. So we need the residue of $f(z)$ at 0. We find it from the Laurent series that converges for $0 < |z| < 1$. This is series (I) in Example 4, Sec. 16.1,

$$\frac{1}{z^3 - z^4} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + \cdots \qquad (0 < |z| < 1).$$

We see from it that this residue is 1. Clockwise integration thus yields

$$\oint_C \frac{dz}{z^3 - z^4} = -2\pi i \operatorname*{Res}_{z=0} f(z) = -2\pi i.$$

CAUTION! Had we used the wrong series (II) in Example 4, Sec. 16.1,

$$\frac{1}{z^3 - z^4} = -\frac{1}{z^4} - \frac{1}{z^5} - \frac{1}{z^6} - \cdots \qquad (|z| > 1),$$

we would have obtained the wrong answer, 0, because this series has no power $1/z$. ■

Formulas for Residues

To calculate a residue at a pole, we need not produce a whole Laurent series, but, more economically, we can derive formulas for residues once and for all.

Simple Poles. Two formulas for the residue of $f(z)$ at a simple pole at z_0 are

$$\operatorname*{Res}_{z=z_0} f(z) = b_1 = \lim_{z \to z_0} (z - z_0) f(z) \tag{3}$$

and, assuming that $f(z) = p(z)/q(z)$, $p(z_0) \neq 0$, and $q(z)$ has a simple zero at z_0 (so that $f(z)$ has at z_0 a simple pole, by Theorem 4 in Sec. 16.2),

$$\operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}. \tag{4}$$

PROOF For a simple pole at $z = z_0$ the Laurent series (1), Sec. 16.1, is

$$f(z) = \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots \qquad (0 < |z - z_0| < R).$$

Here $b_1 \neq 0$. (Why?) Multiplying both sides by $z - z_0$ and then letting $z \to z_0$, we obtain the formula (3):

$$\lim_{z \to z_0} (z - z_0) f(z) = b_1 + \lim_{z \to z_0} (z - z_0)[a_0 + a_1(z - z_0) + \cdots] = b_1$$

where the last equality follows from continuity (Theorem 1, Sec. 15.3).

We prove (4). The Taylor series of $q(z)$ at a simple zero z_0 is

$$q(z) = (z - z_0)q'(z_0) + \frac{(z - z_0)^2}{2!} q''(z_0) + \cdots.$$

Substituting this into $f = p/q$ and then f into (3) gives

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z \to z_0} (z - z_0) \frac{p(z)}{q(z)} = \lim_{z \to z_0} \frac{(z - z_0)p(z)}{(z - z_0)[q'(z_0) + (z - z_0)q''(z_0)/2 + \cdots]}$$

$z - z_0$ cancels. By continuity, the limit of the denominator is $q'(z_0)$ and (4) follows. ■

EXAMPLE 3 Residue at a Simple Pole

$f(z) = (9z + i)/(z^3 + z)$ has a simple pole at i because $z^2 + 1 = (z + i)(z - i)$, and (3) gives the residue

$$\operatorname*{Res}_{z=i} \frac{9z + i}{z(z^2 + 1)} = \lim_{z \to i} (z - i) \frac{9z + i}{z(z + i)(z - i)} = \left[\frac{9z + i}{z(z + i)} \right]_{z=i} = \frac{10i}{-2} = -5i.$$

By (4) with $p(i) = 9i + i$ and $q'(z) = 3z^2 + 1$ we confirm the result,

$$\operatorname*{Res}_{z=i} \frac{9z + i}{z(z^2 + 1)} = \left[\frac{9z + i}{3z^2 + 1} \right]_{z=i} = \frac{10i}{-2} = -5i.$$ ■

Poles of Any Order. The residue of $f(z)$ at an mth-order pole at z_0 is

$$\text{(5)} \qquad \operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m - 1)!} \lim_{z \to z_0} \left\{ \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z) \right] \right\}.$$

In particular, for a second-order pole ($m = 2$),

$$\text{(5*)} \qquad \operatorname*{Res}_{z=z_0} f(z) = \lim_{z \to z_0} \left\{ [(z - z_0)^2 f(z)]' \right\}.$$

PROOF The Laurent series of $f(z)$ converging near z_0 (except at z_0 itself) is (Sec. 16.2)

$$f(z) = \frac{b_m}{(z - z_0)^m} + \frac{b_{m-1}}{(z - z_0)^{m-1}} + \cdots + \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + \cdots$$

where $b_m \neq 0$. The residue wanted is b_1. Multiplying both sides by $(z - z_0)^m$ gives

$$(z - z_0)^m f(z) = b_m + b_{m-1}(z - z_0) + \cdots + b_1(z - z_0)^{m-1} + a_0(z - z_0)^m + \cdots.$$

We see that b_1 is now the coefficient of the power $(z - z_0)^{m-1}$ of the power series of $g(z) = (z - z_0)^m f(z)$. Hence Taylor's theorem (Sec. 15.4) gives (5):

$$b_1 = \frac{1}{(m-1)!} g^{(m-1)}(z_0)$$

$$= \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)]. \quad \blacksquare$$

EXAMPLE 4 Residue at a Pole of Higher Order

$f(z) = 50z/(z^3 + 2z^2 - 7z + 4)$ has a pole of second order at $z = 1$ because the denominator equals $(z + 4)(z - 1)^2$ (verify!). From (5*) we obtain the residue

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z \to 1} \frac{d}{dz} [(z-1)^2 f(z)]$$

$$= \lim_{z \to 1} \frac{d}{dz} \left(\frac{50z}{z+4} \right)$$

$$= \frac{200}{5^2} = 8. \quad \blacksquare$$

Several Singularities Inside the Contour. Residue Theorem

Residue integration can be extended from the case of a single singularity to the case of several singularities within the contour C. This is the purpose of the residue theorem. The extension is surprisingly simple.

THEOREM 1 Residue Theorem

Let $f(z)$ be analytic inside a simple closed path C and on C, except for finitely many singular points $z_1, z_2, \cdots, z_k$ inside C. Then the integral of $f(z)$ taken counterclockwise around C equals $2\pi i$ times the sum of the residues of $f(z)$ at $z_1, \cdots, z_k$:

$$\oint_C f(z)\, dz = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z). \tag{6}$$

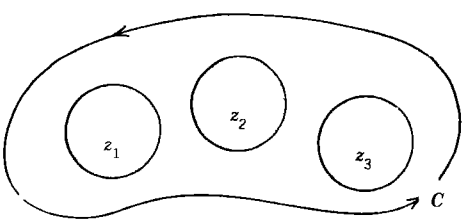

Fig. 370. Residue theorem

PROOF We enclose each of the singular points z_j in a circle C_j with radius small enough that those k circles and C are all separated (Fig. 370). Then $f(z)$ is analytic in the multiply connected domain D bounded by C and $C_1, \cdots, C_k$ and on the entire boundary of D. From Cauchy's integral theorem we thus have

$$(7) \qquad \oint_C f(z)\,dz + \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \cdots + \oint_{C_k} f(z)\,dz = 0,$$

the integral along C being taken *counterclockwise* and the other integrals *clockwise* (as in Figs. 351 and 352, Sec. 14.2). We take the integrals over $C_1, \cdots, C_k$ to the right and compensate the resulting minus sign by reversing the sense of integration. Thus,

$$(8) \qquad \oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \cdots + \oint_{C_k} f(z)\,dz$$

where all the integrals are now taken counterclockwise. By (1) and (2),

$$\oint_{C_j} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=z_j} f(z), \qquad j = 1, \cdots, k,$$

so that (8) gives (6) and the residue theorem is proved. ■

This important theorem has various applications in connection with complex and real integrals. Let us first consider some complex integrals. (Real integrals follow in the next section.)

EXAMPLE 5 Integration by the Residue Theorem. Several Contours

Evaluate the following integral counterclockwise around any simple closed path such that (a) 0 and 1 are inside C, (b) 0 is inside, 1 outside, (c) 1 is inside, 0 outside, (d) 0 and 1 are outside.

$$\oint_C \frac{4 - 3z}{z^2 - z}\,dz$$

Solution. The integrand has simple poles at 0 and 1, with residues [by (3)]

$$\operatorname*{Res}_{z=0} \frac{4 - 3z}{z(z - 1)} = \left[\frac{4 - 3z}{z - 1}\right]_{z=0} = -4. \qquad \operatorname*{Res}_{z=1} \frac{4 - 3z}{z(z - 1)} = \left[\frac{4 - 3z}{z}\right]_{z=1} = 1.$$

[Confirm this by (4).] *Ans.* (a) $2\pi i(-4 + 1) = -6\pi i$, (b) $-8\pi i$, (c) $2\pi i$, (d) 0. ■

EXAMPLE 6 Another Application of the Residue Theorem

Integrate $(\tan z)/(z^2 - 1)$ counterclockwise around the circle C: $|z| = 3/2$.

Solution. $\tan z$ is not analytic at $\pm\pi/2, \pm 3\pi/2, \cdots$, but all these points lie outside the contour C. Because of the denominator $z^2 - 1 = (z - 1)(z + 1)$ the given function has simple poles at ± 1. We thus obtain from (4) and the residue theorem

$$\oint_C \frac{\tan z}{z^2 - 1}\,dz = 2\pi i\left(\operatorname*{Res}_{z=1} \frac{\tan z}{z^2 - 1} + \operatorname*{Res}_{z=-1} \frac{\tan z}{z^2 - 1}\right)$$

$$= 2\pi i\left(\frac{\tan z}{2z}\bigg|_{z=1} + \frac{\tan z}{2z}\bigg|_{z=-1}\right)$$

$$= 2\pi i \tan 1 = 9.7855i.$$

■

EXAMPLE 7 Poles and Essential Singularities

Evaluate the following integral, where C is the ellipse $9x^2 + y^2 = 9$ (counterclockwise, sketch it).

$$\oint_C \left(\frac{ze^{\pi z}}{z^4 - 16} + ze^{\pi/z} \right) dz$$

Solution. Since $z^4 - 16 = 0$ at $\pm 2i$ and ± 2, the first term of the integrand has simple poles at $\pm 2i$ inside C, with residues [by (4); note that $e^{2\pi i} = 1$]

$$\operatorname*{Res}_{z=2i} \frac{ze^{\pi z}}{z^4 - 16} = \left[\frac{ze^{\pi z}}{4z^3} \right]_{z=2i} = -\frac{1}{16}$$

$$\operatorname*{Res}_{z=-2i} \frac{ze^{\pi z}}{z^4 - 16} = \left[\frac{ze^{\pi z}}{4z^3} \right]_{z=-2i} = -\frac{1}{16}$$

and simple poles at ± 2, which lie outside C, so that they are of no interest here. The second term of the integrand has an essential singularity at 0, with residue $\pi^2/2$ as obtained from

$$ze^{\pi/z} = z\left(1 + \frac{\pi}{z} + \frac{\pi^2}{2!z^2} + \frac{\pi^3}{3!z^3} + \cdots\right) = z + \pi + \frac{\pi^2}{2} \cdot \frac{1}{z} + \cdot \qquad (|z| > 0).$$

Ans. $2\pi i(-\frac{1}{16} - \frac{1}{16} + \frac{1}{2}\pi^2) = \pi(\pi^2 - \frac{1}{4})i = 30.221i$ by the residue theorem. ■

1. Verify the calculations in Example 3 and find the other residues.
2. Verify the calculations in Example 4 and find the other residue.

3–12 RESIDUES

Find all the singular points and the corresponding residues. (Show the details of your work.)

3. $\dfrac{1}{4 + z^2}$
4. $\dfrac{\cos z}{z^6}$
5. $\dfrac{\sin z}{z^6}$
6. $\dfrac{z^2 + 1}{z^2 - z}$
7. $\cot z$
8. $\sec z$
9. $\dfrac{1}{(z^2 - 1)^2}$
10. $\dfrac{1/3}{z^4 - 1}$
11. $\tan z$
12. $\dfrac{z^2}{z^4 - 1}$
13. **CAS PROJECT. Residue at a Pole.** Write a program for calculating the residue at a pole of any order. Use it for solving Probs. 3–8.

4–25 RESIDUE INTEGRATION

Evaluate (counterclockwise). (Show the details.)

14. $\displaystyle\oint_C \frac{\sin \pi z}{z^4}\, dz, \quad C: |z - i| = 2$
15. $\displaystyle\oint_C e^{1/z}\, dz, \quad C: |z| = 1$
16. $\displaystyle\oint_C \frac{dz}{\sinh \frac{1}{2}\pi z}, \quad C: |z - 1| = 1.4$
17. $\displaystyle\oint_C \tan \pi z\, dz, \quad C: |z| = 1$
18. $\displaystyle\oint_C \tan \pi z\, dz, \quad C: |z| = 2$
19. $\displaystyle\oint_C \frac{e^z}{\cos z}\, dz, \quad C: |z| = 4.5$
20. $\displaystyle\oint_C \coth z\, dz, \quad C: |z| = 1$
21. $\displaystyle\oint_C \frac{e^z}{\cos \pi z}\, dz, \quad C: |z - i| = 1.5$
22. $\displaystyle\oint_C \frac{\cosh z}{z^2 - 3iz}\, dz, \quad C: |z| = 1$
23. $\displaystyle\oint_C \frac{\tan \pi z}{z^3}\, dz, \quad C: |z + \tfrac{1}{2}i| = 1$
24. $\displaystyle\oint_C \frac{1 - 4z + 6z^2}{(z^2 + \frac{1}{4})(2 - z)}\, dz, \quad C: |z| = 1$
25. $\displaystyle\oint_C \frac{30z^2 - 23z + 5}{(2z - 1)^2(3z - 1)}\, dz, \quad C: |z| = 1$

16.4 Residue Integration of Real Integrals

It is quite surprising that certain classes of complicated ***real*** integrals can be integrated by the residue theorem, as we shall see.

Integrals of Rational Functions of cos θ and sin θ

We first consider integrals of the type

$$J = \int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta \tag{1}$$

where $F(\cos\theta, \sin\theta)$ is a real rational function of $\cos\theta$ and $\sin\theta$ [for example, $(\sin^2\theta)/(5 - 4\cos\theta)$] and is finite (does not become infinite) on the interval of integration. Setting $e^{i\theta} = z$, we obtain

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$
$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}\left(z - \frac{1}{z}\right) \tag{2}$$

Since F is rational in $\cos\theta$ and $\sin\theta$, Eq. (2) shows that F is now a rational function of z, say, $f(z)$. Since $dz/d\theta = ie^{i\theta}$, we have $d\theta = dz/iz$ and the given integral takes the form

$$J = \oint_C f(z)\,\frac{dz}{iz} \tag{3}$$

and, as θ ranges from 0 to 2π in (1), the variable $z = e^{i\theta}$ ranges counterclockwise once around the unit circle $|z| = 1$. (Review Sec. 13.5 if necessary.)

EXAMPLE 1 An Integral of the Type (1)

Show by the present method that $\displaystyle\int_0^{2\pi} \frac{d\theta}{\sqrt{2} - \cos\theta} = 2\pi$.

Solution. We use $\cos\theta = \frac{1}{2}(z + 1/z)$ and $d\theta = dz/iz$. Then the integral becomes

$$\oint_C \frac{dz/iz}{\sqrt{2} - \frac{1}{2}\left(z + \frac{1}{z}\right)} = \oint_C \frac{dz}{-\frac{i}{2}(z^2 - 2\sqrt{2}z + 1)}$$
$$= -\frac{2}{i}\oint_C \frac{dz}{(z - \sqrt{2} - 1)(z - \sqrt{2} + 1)}.$$

We see that the integrand has a simple pole at $z_1 = \sqrt{2} + 1$ outside the unit circle C, so that it is of no interest here, and another simple pole at $z_2 = \sqrt{2} - 1$ (where $z - \sqrt{2} + 1 = 0$) inside C with residue [by (3), Sec. 16.3]

$$\operatorname*{Res}_{z=z_2} \frac{1}{(z - \sqrt{2} - 1)(z - \sqrt{2} + 1)} = \left[\frac{1}{z - \sqrt{2} - 1}\right]_{z=\sqrt{2}-1}$$
$$= -\frac{1}{2}$$

Answer: $2\pi i(-2/i)(-1/2) = 2\pi$. (Here $-2/i$ is the factor in front of the last integral.) ■

As another large class, let us consider real integrals of the form

$$\int_{-\infty}^{\infty} f(x)\,dx. \tag{4}$$

Such an integral, whose interval of integration is not finite is called an **improper integral,** and it has the meaning

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a\to-\infty} \int_{a}^{0} f(x)\,dx + \lim_{b\to\infty} \int_{0}^{b} f(x)\,dx. \tag{5'}$$

If both limits exist, we may couple the two independent passages to $-\infty$ and ∞, and write

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx. \tag{5}$$

The limit in (5) is called the **Cauchy principal value** of the integral. It is written

$$\text{pr. v.} \int_{-\infty}^{\infty} f(x)\,dx.$$

It may exist even if the limits in (5′) do not. *Example:*

$$\lim_{R\to\infty} \int_{-R}^{R} x\,dx = \lim_{R\to\infty} \left(\frac{R^2}{2} - \frac{R^2}{2}\right) = 0, \qquad \text{but} \qquad \lim_{b\to\infty} \int_{0}^{b} x\,dx = \infty.$$

We assume that the function $f(x)$ in (4) is a real rational function whose denominator is different from zero for all real x and is of degree at least two units higher than the degree of the numerator. Then the limits in (5′) exist, and we may start from (5). We consider the corresponding contour integral

$$\oint_C f(z)\,dz \tag{5*}$$

around a path C in Fig. 371. Since $f(x)$ is rational, $f(z)$ has finitely many poles in the upper half-plane, and if we choose R large enough, then C encloses all these poles. By the residue theorem we then obtain

$$\oint_C f(z)\,dz = \int_S f(z)\,dz + \int_{-R}^{R} f(x)\,dx = 2\pi i \sum \text{Res } f(z)$$

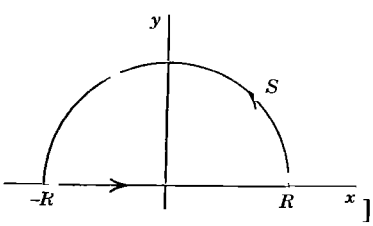

Fig. 371. Path C of the contour integral in (5*)

where the sum consists of all the residues of $f(z)$ at the points in the upper half-plane at which $f(z)$ has a pole. From this we have

$$\int_{-R}^{R} f(x)\,dx = 2\pi i \sum \operatorname{Res} f(z) - \int_S f(z)\,dz. \tag{6}$$

We prove that, if $R \to \infty$, the value of the integral over the semicircle S approaches zero. If we set $z = Re^{i\theta}$, then S is represented by $R = const$, and as z ranges along S, the variable θ ranges from 0 to π. Since, by assumption, the degree of the denominator of $f(z)$ is at least two units higher than the degree of the numerator, we have

$$|f(z)| < \frac{k}{|z|^2} \qquad (|z| = R > R_0)$$

for sufficiently large constants k and R_0. By the *ML*-inequality in Sec. 14.1,

$$\left|\int_S f(z)\,dz\right| < \frac{k}{R^2}\,\pi R = \frac{k\pi}{R} \qquad (R > R_0).$$

Hence, as R approaches infinity, the value of the integral over S approaches zero, and (5) and (6) yield the result

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum \operatorname{Res} f(z) \tag{7}$$

where we sum over all the residues of $f(z)$ at the poles of $f(z)$ in the upper half-plane.

EXAMPLE 2 An Improper Integral from 0 to ∞

Using (7), show that

$$\int_0^{\infty} \frac{dx}{1 + x^4} = \frac{\pi}{2\sqrt{2}}.$$

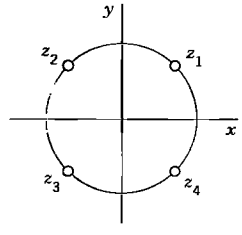

Fig. 372. Example 2

Solution. Indeed, $f(z) = 1/(1 + z^4)$ has four simple poles at the points (make a sketch)

$$z_1 = e^{\pi i/4}, \qquad z_2 = e^{3\pi i/4}, \qquad z_3 = e^{-3\pi i/4}, \qquad z_4 = e^{-\pi i/4}.$$

The first two of these poles lie in the upper half-plane (Fig. 372). From (4) in the last section we find the residues

$$\operatorname*{Res}_{z=z_1} f(z) = \left[\frac{1}{(1+z^4)'}\right]_{z=z_1} = \left[\frac{1}{4z^3}\right]_{z=z_1} = \frac{1}{4}\,e^{-3\pi i/4} = -\frac{1}{4}\,e^{\pi i/4}.$$

$$\operatorname*{Res}_{z=z_2} f(z) = \left[\frac{1}{(1+z^4)'}\right]_{z=z_2} = \left[\frac{1}{4z^3}\right]_{z=z_2} = \frac{1}{4}\,e^{-9\pi i/4} = \frac{1}{4}\,e^{-\pi i/4}.$$

(Here we used $e^{\pi i} = -1$ and $e^{-2\pi i} = 1$.) By (1) in Sec. 13.6 and (7) in this section,

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = -\frac{2\pi i}{4}\,(e^{\pi i/4} - e^{-\pi i/4}) = -\frac{2\pi i}{4}\cdot 2i\cdot \sin\frac{\pi}{4} = \pi \sin\frac{\pi}{4} = \frac{\pi}{\sqrt{2}}.$$

Since $1/(1+x^4)$ is an even function, we thus obtain, as asserted,

$$\int_0^{\infty} \frac{dx}{1+x^4} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}. \qquad \blacksquare$$

Fourier Integrals

The method of evaluating (4) by creating a closed contour (Fig. 371) and "blowing it up" extends to integrals

$$(8) \qquad \int_{-\infty}^{\infty} f(x)\cos sx\,dx \qquad \text{and} \qquad \int_{-\infty}^{\infty} f(x)\sin sx\,dx \qquad (s \text{ real})$$

as they occur in connection with the Fourier integral (Sec. 11.7).

If $f(x)$ is a rational function satisfying the assumption on the degree as for (4), we may consider the corresponding integral

$$\oint_C f(z)e^{isz}\,dz \qquad (s \text{ real and positive})$$

over the contour C in Fig. 371 on p. 719. Instead of (7) we now get

$$(9) \qquad \int_{-\infty}^{\infty} f(x)e^{isx}\,dx = 2\pi i \sum \operatorname{Res}\left[f(z)e^{isz}\right] \qquad (s > 0)$$

where we sum the residues of $f(z)e^{isz}$ at its poles in the upper half-plane. Equating the real and the imaginary parts on both sides of (9), we have

$$(10) \qquad \begin{aligned} \int_{-\infty}^{\infty} f(x)\cos sx\,dx &= -2\pi \sum \operatorname{Im}\operatorname{Res}\left[f(z)e^{isz}\right], \\ \int_{-\infty}^{\infty} f(x)\sin sx\,dx &= 2\pi \sum \operatorname{Re}\operatorname{Res}\left[f(z)e^{isz}\right]. \end{aligned} \qquad (s > 0)$$

To establish (9), we must show [as for (4)] that the value of the integral over the semicircle S in Fig. 371 approaches 0 as $R \to \infty$. Now $s > 0$ and S lies in the upper half-plane $y \geqq 0$. Hence

$$|e^{isz}| = |e^{is(x+iy)}| = |e^{isx}|\,|e^{-sy}| = 1\cdot e^{-sy} \leqq 1 \qquad (s > 0, \quad y \geqq 0).$$

From this we obtain the inequality $|f(z)e^{isz}| = |f(z)|\,|e^{isz}| \leqq |f(z)|$ $(s > 0, \quad y \geqq 0)$. This reduces our present problem to that for (4). Continuing as before gives (9) and (10). $\blacksquare$

EXAMPLE 3 An Application of (10)

Show that $\int_{-\infty}^{\infty} \frac{\cos sx}{k^2 + x^2}\, dx = \frac{\pi}{k}\, e^{-ks}, \qquad \int_{-\infty}^{\infty} \frac{\sin sx}{k^2 + x^2}\, dx = 0 \qquad (s > 0,\ k > 0).$

Solution. In fact, $e^{isz}/(k^2 + z^2)$ has only one pole in the upper half-plane, namely, a simple pole at $z = ik$, and from (4) in Sec. 16.3 we obtain

$$\operatorname*{Res}_{z=ik} \frac{e^{isz}}{k^2 + z^2} = \left[\frac{e^{isz}}{2z}\right]_{z=ik} = \frac{e^{-ks}}{2ik}.$$

Thus

$$\int_{-\infty}^{\infty} \frac{e^{isx}}{k^2 + x^2}\, dx = 2\pi i\, \frac{e^{-ks}}{2ik} = \frac{\pi}{k}\, e^{-ks}.$$

Since $e^{isx} = \cos sx + i \sin sx$, this yields the above results [see also (15) in Sec. 11.7.] ■

Another Kind of Improper Integral

We consider an improper integral

$$\int_A^B f(x)\, dx \tag{11}$$

whose integrand becomes infinite at a point a in the interval of integration,

$$\lim_{x \to a} |f(x)| = \infty.$$

By definition, this integral (11) means

$$\int_A^B f(x)\, dx = \lim_{\epsilon \to 0} \int_A^{a-\epsilon} f(x)\, dx + \lim_{\eta \to 0} \int_{a+\eta}^B f(x)\, dx \tag{12}$$

where both ϵ and η approach zero independently and through positive values. It may happen that neither of these two limits exists if ϵ and η go to 0 independently, but the limit

$$\lim_{\epsilon \to 0} \left[\int_A^{a-\epsilon} f(x)\, dx + \int_{a+\epsilon}^B f(x)\, dx\right] \tag{13}$$

exists. This is called the **Cauchy principal value** of the integral. It is written

$$\text{pr. v.} \int_A^B f(x)\, dx.$$

For example,

$$\text{pr. v.} \int_{-1}^1 \frac{dx}{x^3} = \lim_{\epsilon \to 0} \left[\int_{-1}^{-\epsilon} \frac{dx}{x^3} + \int_{\epsilon}^1 \frac{dx}{x^3}\right] = 0:$$

the principal value exists, although the integral itself has no meaning.

In the case of simple poles on the real axis we shall obtain a formula for the principal value of an integral from $-\infty$ to ∞. This formula will result from the following theorem.

THEOREM 1

Simple Poles on the Real Axis

If $f(z)$ has a simple pole at $z = a$ on the real axis, then (Fig. 373)

$$\lim_{r\to 0} \int_{C_2} f(z)\,dz = \pi i \operatorname*{Res}_{z=a} f(z).$$

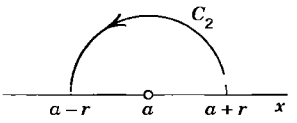

Fig. 373. Theorem 1

PROOF By the definition of a simple pole (Sec. 16.2) the integrand $f(z)$ has for $0 < |z - a| < R$ the Laurent series

$$f(z) = \frac{b_1}{z-a} + g(z), \qquad b_1 = \operatorname*{Res}_{z=a} f(z).$$

Here $g(z)$ is analytic on the semicircle of integration (Fig. 373)

$$C_2\colon \quad z = a + re^{i\theta}, \qquad 0 \leqq \theta \leqq \pi$$

and for all z between C_2 and the x-axis, and thus bounded on C_2, say, $|g(z)| \leqq M$. By integration,

$$\int_{C_2} f(z)\,dz = \int_0^{\pi} \frac{b_1}{re^{i\theta}}\, ire^{i\theta}\,d\theta + \int_{C_2} g(z)\,dz = b_1\pi i + \int_{C_2} g(z)\,dz.$$

The second integral on the right cannot exceed $M\pi r$ in absolute value, by the ML-inequality (Sec. 14.1), and $ML = M\pi r \to 0$ as $r \to 0$. ■

Figure 374 shows the idea of applying Theorem 1 to obtain the principal value of the integral of a rational function $f(x)$ from $-\infty$ to ∞. For sufficiently large R the integral over the entire contour in Fig. 374 has the value J given by $2\pi i$ times the sum of the residues of $f(z)$ at the singularities in the upper half-plane. We assume that $f(x)$ satisfies the degree

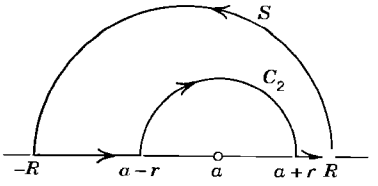

Fig. 374. Application of Theorem 1

condition imposed in connection with (4). Then the value of the integral over the large semicircle S approaches 0 as $R \to \infty$. For $r \to 0$ the integral over C_2 (clockwise!) approaches the value

$$K = -\pi i \operatorname*{Res}_{z=a} f(z)$$

by Theorem 1. Together this shows that the principal value P of the integral from $-\infty$ to ∞ plus K equals J; hence $P = J - K = J + \pi i \operatorname{Res}_{z=a} f(z)$. If $f(z)$ has several simple poles on the real axis, then K will be $-\pi i$ times the sum of the corresponding residues. Hence the desired formula is

$$\textbf{(14)} \qquad \text{pr. v.} \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum \operatorname{Res} f(z) + \pi i \sum \operatorname{Res} f(z)$$

where the first sum extends over all poles in the upper half-plane and the second over all poles on the real axis, the latter being simple by assumption.

EXAMPLE 4 Poles on the Real Axis

Find the principal value

$$\text{pr. v.} \int_{-\infty}^{\infty} \frac{dx}{(x^2 - 3x + 2)(x^2 + 1)}.$$

Solution. Since

$$x^2 - 3x + 2 = (x - 1)(x - 2),$$

the integrand $f(x)$, considered for complex z, has simple poles at

$$z = 1, \qquad \operatorname*{Res}_{z=1} f(z) = \left[\frac{1}{(z - 2)(z^2 + 1)}\right]_{z=1} = -\frac{1}{2},$$

$$z = 2, \qquad \operatorname*{Res}_{z=2} f(z) = \left[\frac{1}{(z - 1)(z^2 + 1)}\right]_{z=2} = \frac{1}{5},$$

$$z = i, \qquad \operatorname*{Res}_{z=i} f(z) = \left[\frac{1}{(z^2 - 3z + 2)(z + i)}\right]_{z=i} = \frac{1}{6 + 2i} = \frac{3 - i}{20},$$

and at $z = -i$ in the lower half-plane, which is of no interest here. From (14) we get the answer

$$\text{pr. v.} \int_{-\infty}^{\infty} \frac{dx}{(x^2 - 3x + 2)(x^2 + 1)} = 2\pi i \left(\frac{3 - i}{20}\right) + \pi i \left(-\frac{1}{2} + \frac{1}{5}\right) = \frac{\pi}{10} \qquad \blacksquare$$

More integrals of the kind considered in this section are included in the problem set. Try also your CAS, which may sometimes give you false results on complex integrals.

1–8 INTEGRALS INVOLVING COSINE AND SINE

Evaluate the following integrals. (Show the details of your work.)

1. $\int_0^{2\pi} \frac{d\theta}{7 + 6\cos\theta}$

2. $\int_0^{\pi} \frac{d\theta}{2 + \cos\theta}$

3. $\int_0^{2\pi} \frac{d\theta}{37 - 12\cos\theta}$

4. $\int_0^{2\pi} \frac{d\theta}{8 - 2\sin\theta}$

5. $\int_0^{2\pi} \frac{d\theta}{5 - 4\sin\theta}$

6. $\int_0^{2\pi} \frac{\sin^2\theta}{5 - 4\cos\theta}\, d\theta$

7. $\int_0^{2\pi} \frac{\cos\theta}{13 - 12\cos 2\theta}\, d\theta.$

Hint. $\cos 2\theta = \frac{1}{2}\left(z^2 + \frac{1}{z^2}\right)$

8. $\int_0^{2\pi} \frac{1 + 4\cos\theta}{17 - 8\cos\theta}\, d\theta$

9–22 IMPROPER INTEGRALS: INFINITE INTERVAL OF INTEGRATION

Evaluate (showing the details):

9. $\int_{-\infty}^{\infty} \frac{dx}{x^2 + 1}$

10. $\int_{-\infty}^{\infty} \frac{x}{x^4 + 1}\, dx$

11. $\int_{-\infty}^{\infty} \frac{dx}{x^6 + 1}$

12. $\int_{-\infty}^{\infty} \frac{dx}{(x^2 - 2x + 5)^2}$

13. $\int_{-\infty}^{\infty} \frac{dx}{(x^2 + 4)^2}$

14. $\int_{-\infty}^{\infty} \frac{dx}{x^4 + 16}$

15. $\int_{-\infty}^{\infty} \frac{x^3}{1 + x^8}\, dx$

16. $\int_{-\infty}^{\infty} \frac{dx}{(x^2 + 1)(x^2 + 9)}$

17. $\int_{-\infty}^{\infty} \frac{x}{(x^2 - 2x + 2)^2}\, dx$

18. $\int_{-\infty}^{\infty} \frac{x^2 + 1}{x^4 + 1}\, dx$

19. $\int_{-\infty}^{\infty} \frac{\sin x}{x^4 + 1}\, dx$

20. $\int_{-\infty}^{\infty} \frac{\cos x}{x^4 + 1}\, dx$

21. $\int_{-\infty}^{\infty} \frac{\sin 3x}{x^4 + 1}\, dx$

22. $\int_{-\infty}^{\infty} \frac{\cos 4x}{x^4 + 5x^2 + 4}\, dx$

23–27 IMPROPER INTEGRALS: POLES ON THE REAL AXIS

Find the Cauchy principal value (showing details):

23. $\int_{-\infty}^{\infty} \frac{x + 2}{x^3 + x}\, dx$

24. $\int_{-\infty}^{\infty} \frac{x^2}{x^4 - 1}\, dx$

25. $\int_{-\infty}^{\infty} \frac{x + 5}{x^3 - x}\, dx$

26. $\int_{-\infty}^{\infty} \frac{dx}{x^4 + 3x^2 - 4}$

27. $\int_{-\infty}^{\infty} \frac{dx}{x^4 - 1}$

28. **TEAM PROJECT. Comments on Real Integrals.**
(a) Formula (10) follows from (9). Give the details.

(b) Use of auxiliary results. Integrating e^{-z^2} around the boundary C of the rectangle with vertices $-a$, a, $a + ib$, $-a + ib$, letting $a \to \infty$, and using

$$\int_0^{\infty} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2},$$

show that

$$\int_0^{\infty} e^{-x^2} \cos 2bx\, dx = \frac{\sqrt{\pi}}{2}\, e^{-b^2}.$$

(This integral is needed in heat conduction in Sec. 12.6.)

(c) Inspection. Solve Probs. 15 and 21 without calculation.

29. **CAS EXPERIMENT. Check your CAS.** Find out to what extent your CAS can evaluate integrals of the form (1), (4), and (8) correctly. Do this by comparing the results of direct integration (which may come out false) with those of using residues.

30. **CAS EXPERIMENT. Simple Poles on the Real Axis.** Experiment with integrals $\int_{-\infty}^{\infty} f(x)\, dx$. $f(x) = [(x - a_1)(x - a_2) \cdots (x - a_k)]^{-1}$, a_j real and all different, $k > 1$. Conjecture that the principal value of these integrals is 0. Try to prove this for a special k, say, $k = 3$. For general k.

[illegible] STIONS AND PROBLEMS

1. Laurent series generalize Taylor series. Explain the details.
2. Can a function have several Laurent series with the same center? Explain. If your answer is yes, give examples.
3. What is the principal part of a Laurent series? Its significance?
4. What is a pole? An essential singularity? Give examples.
5. What is Picard's theorem? Why did it occur in this chapter?
6. What is the Riemann sphere? The extended complex plane? Its significance?
7. Is e^{1/z^2} analytic or singular at infinity? $\cosh z$? $(z-4)^3$? Explain.
8. What is the residue? Why is it important?
9. State formulas for residues from memory.
10. State some further methods for calculating residues.
11. What is residue integration? To what kind of complex integrals does it apply?
12. By what idea can we apply residue integration to real integrals from $-\infty$ to ∞? Give simple examples.
13. What is a zero of an analytic function? How are zeros classified?
14. What are improper integrals? Cauchy principal values? Give examples.
15. Can the residue at a singular point be 0? At a simple pole?
16. What is a meromorphic function? An entire function? Give examples.

17–28 COMPLEX INTEGRALS

Integrate counterclockwise around C. (Show the details.)

17. $\dfrac{\tan z}{z^4}$, C: $|z| = 1$
18. $\dfrac{\sin 2z}{z^4}$, C: $|z| = 1$
19. $\dfrac{10z}{2z + i}$, C: $|z - 2i| = 3$
20. $\dfrac{iz + 1}{z^2 - iz + 2}$, C: $|z - 1| = 3$
21. $\dfrac{\cosh 5z}{z^2 + 4}$, C: $|z - i| = 2$
22. $\dfrac{4z^3 + 7z}{\cos z}$, C: $|z + 1| = 1$
23. $\cot 8z$, C: $|z| = 0.2$
24. $\dfrac{z^2 \sin z}{4z^2 - 1}$, C: $|z - 1| = 2$
25. $\dfrac{\cos z}{z^n}$, $n = 1, 2, \cdots$, C: $|z| = 1$
26. $\dfrac{z^2 + 1}{z^2 - 2z}$, C: $\dfrac{1}{2}x^2 + y^2 = 1$
27. $\dfrac{15z + 9}{z^3 - 9z}$, C: $|z - 3| = 2$
28. $\dfrac{15z + 9}{z^3 - 9z}$, C: $|z| = 4$

29–35 REAL INTEGRALS

Evaluate by the methods of this chapter (showing the details):

29. $\displaystyle\int_0^{2\pi} \frac{d\theta}{25 - 24\cos\theta}$
30. $\displaystyle\int_0^{\pi} \frac{d\theta}{k + \cos\theta}$, $k > 1$
31. $\displaystyle\int_0^{2\pi} \frac{d\theta}{1 - \frac{1}{2}\sin\theta}$
32. $\displaystyle\int_0^{2\pi} \frac{\sin\theta}{3 + \cos\theta}\, d\theta$
33. $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(1 + x^2)^2}\, dx$
34. $\displaystyle\int_0^{\infty} \frac{dx}{(1 + x^2)^2}$
35. $\displaystyle\int_0^{\infty} \frac{1 + 2x^2}{1 + 4x^4}\, dx$

36. Obtain the answer to Prob. 18 in Sec. 16.4 from the present Prob. 35.