2

The Laplace Transform*

2–1 INTRODUCTION

The Laplace transform method is an operational method that can be used advantageously for solving linear differential equations. By use of Laplace transforms, we can convert many common functions, such as sinusoidal functions, damped sinusoidal functions, and exponential functions, into algebraic functions of a complex variable s. Operations such as differentiation and integration can be replaced by algebraic operations in the complex plane. Thus, a linear differential equation can be transformed into an algebraic equation in a complex variable s. If the algebraic equation in s is solved for the dependent variable, then the solution of the differential equation (the inverse Laplace transform of the dependent variable) may be found by use of a Laplace transform table or by use of the partial-fraction expansion technique, which is presented in Section 2–5 and 2–6.

An advantage of the Laplace transform method is that it allows the use of graphical techniques for predicting the system performance without actually solving system differential equations. Another advantage of the Laplace transform method is that, when we solve the differential equation, both the transient component and steady-state component of the solution can be obtained simultaneously.

Outline of the Chapter. Section 2–1 presents introductory remarks. Section 2–2 briefly reviews complex variables and complex functions. Section 2–3 derives Laplace

*This chapter may be skipped if the student is already familiar with Laplace transforms.

transforms of time functions that are frequently used in control engineering. Section 2–4 presents useful theorems of Laplace transforms, and Section 2–5 treats the inverse Laplace transformation using the partial-fraction expansion of $B(s)/A(s)$, where $A(s)$ and $B(s)$ are polynomials in s. Section 2–6 presents computational methods with MATLAB to obtain the partial-fraction expansion of $B(s)/A(s)$, as well as the zeros and poles of $B(s)/A(s)$. Finally, Section 2–7 deals with solutions of linear time-invariant differential equations by the Laplace transform approach.

2–2 REVIEW OF COMPLEX VARIABLES AND COMPLEX FUNCTIONS

Before we present the Laplace transformation, we shall review the complex variable and complex function. We shall also review Euler's theorem, which relates the sinusoidal functions to exponential functions.

Complex Variable. A complex number has a real part and an imaginary part, both of which are constant. If the real part and/or imaginary part are variables, a complex quantity is called a *complex variable*. In the Laplace transformation we use the notation s as a complex variable; that is,

$$s = \sigma + j\omega$$

where σ is the real part and ω is the imaginary part.

Complex Function. A complex function $G(s)$, a function of s, has a real part and an imaginary part or

$$G(s) = G_x + jG_y$$

where G_x and G_y are real quantities. The magnitude of $G(s)$ is $\sqrt{G_x^2 + G_y^2}$, and the angle θ of $G(s)$ is $\tan^{-1}(G_y/G_x)$. The angle is measured counterclockwise from the positive real axis. The complex conjugate of $G(s)$ is $\bar{G}(s) = G_x - jG_y$.

Complex functions commonly encountered in linear control systems analysis are single-valued functions of s and are uniquely determined for a given value of s.

A complex function $G(s)$ is said to be *analytic* in a region if $G(s)$ and all its derivatives exist in that region. The derivative of an analytic function $G(s)$ is given by

$$\frac{d}{ds}G(s) = \lim_{\Delta s \to 0} \frac{G(s + \Delta s) - G(s)}{\Delta s} = \lim_{\Delta s \to 0} \frac{\Delta G}{\Delta s}$$

Since $\Delta s = \Delta\sigma + j\Delta\omega$, Δs can approach zero along an infinite number of different paths. It can be shown, but is stated without a proof here, that if the derivatives taken along two particular paths, that is, $\Delta s = \Delta\sigma$ and $\Delta s = j\Delta\omega$, are equal, then the derivative is unique for any other path $\Delta s = \Delta\sigma + j\Delta\omega$ and so the derivative exists.

For a particular path $\Delta s = \Delta\sigma$ (which means that the path is parallel to the real axis),

$$\frac{d}{ds}G(s) = \lim_{\Delta\sigma \to 0}\left(\frac{\Delta G_x}{\Delta\sigma} + j\frac{\Delta G_y}{\Delta\sigma}\right) = \frac{\partial G_x}{\partial\sigma} + j\frac{\partial G_y}{\partial\sigma}$$

For another particular path $\Delta s = j\Delta\omega$ (which means that the path is parallel to the imaginary axis).

$$\frac{d}{ds}G(s) = \lim_{j\Delta\omega \to 0}\left(\frac{\Delta G_x}{j\Delta\omega} + j\frac{\Delta G_y}{j\Delta\omega}\right) = -j\frac{\partial G_x}{\partial\omega} + \frac{\Delta G_y}{\partial\omega}$$

If these two values of the derivative are equal,

$$\frac{\partial G_x}{\partial\sigma} + j\frac{\partial G_y}{\partial\sigma} = \frac{\partial G_y}{\partial\omega} - j\frac{\partial G_x}{\partial\omega}$$

or if the following two conditions

$$\frac{\partial G_x}{\partial\sigma} = \frac{\partial G_y}{\partial\omega} \quad \text{and} \quad \frac{\partial G_y}{\partial\sigma} = -\frac{\partial G_x}{\partial\omega}$$

are satisfied, then the derivative $dG(s)/ds$ is uniquely determined. These two conditions are known as the Cauchy–Riemann conditions. If these conditions are satisfied, the function $G(s)$ is analytic.

As an example, consider the following $G(s)$:

$$G(s) = \frac{1}{s+1}$$

Then

$$G(\sigma + j\omega) = \frac{1}{\sigma + j\omega + 1} = G_x + jG_y$$

where

$$G_x = \frac{\sigma + 1}{(\sigma + 1)^2 + \omega^2} \quad \text{and} \quad G_y = \frac{-\omega}{(\sigma + 1)^2 + \omega^2}$$

It can be seen that, except at $s = -1$ (that is, $\sigma = -1$, $\omega = 0$), $G(s)$ satisfies the Cauchy–Riemann conditions:

$$\frac{\partial G_x}{\partial\sigma} = \frac{\partial G_y}{\partial\omega} = \frac{\omega^2 - (\sigma + 1)^2}{[(\sigma + 1)^2 + \omega^2]^2}$$

$$\frac{\partial G_y}{\partial\sigma} = -\frac{\partial G_x}{\partial\omega} = \frac{2\omega(\sigma + 1)}{[(\sigma + 1)^2 + \omega^2]^2}$$

Hence $G(s) = 1/(s + 1)$ is analytic in the entire s plane except at $s = -1$. The derivative $dG(s)/ds$, except at $s = 1$, is found to be

$$\frac{d}{ds}G(s) = \frac{\partial G_x}{\partial\sigma} + j\frac{\partial G_y}{\partial\sigma} = \frac{\partial G_y}{\partial\omega} - j\frac{\partial G_x}{d\omega}$$

$$= -\frac{1}{(\sigma + j\omega + 1)^2} = -\frac{1}{(s + 1)^2}$$

Note that the derivative of an analytic function can be obtained simply by differentiating $G(s)$ with respect to s. In this example,

$$\frac{d}{ds}\left(\frac{1}{s+1}\right) = -\frac{1}{(s + 1)^2}$$

Points in the s plane at which the function $G(s)$ is analytic are called *ordinary* points, while points in the s plane at which the function $G(s)$ is not analytic are called *singular* points. Singular points at which the function $G(s)$ or its derivatives approach infinity are called *poles*. Singular points at which the function $G(s)$ equals zero are called *zeros*.

If $G(s)$ approaches infinity as s approaches $-p$ and if the function

$$G(s)(s + p)^n, \qquad \text{for } n = 1, 2, 3, \ldots$$

has a finite, nonzero value at $s = -p$, then $s = -p$ is called a pole of order n. If $n = 1$, the pole is called a simple pole. If $n = 2, 3, \ldots$, the pole is called a second-order pole, a third-order pole, and so on.

To illustrate, consider the complex function

$$G(s) = \frac{K(s + 2)(s + 10)}{s(s + 1)(s + 5)(s + 15)^2}$$

$G(s)$ has zeros at $s = -2$, $s = -10$, simple poles at $s = 0$, $s = -1$, $s = -5$, and a double pole (multiple pole of order 2) at $s = -15$. Note that $G(s)$ becomes zero at $s = \infty$. Since for large values of s

$$G(s) \doteqdot \frac{K}{s^3}$$

$G(s)$ possesses a triple zero (multiple zero of order 3) at $s = \infty$. If points at infinity are included, $G(s)$ has the same number of poles as zeros. To summarize, $G(s)$ has five zeros ($s = -2$, $s = -10$, $s = \infty$, $s = \infty$, $s = \infty$) and five poles ($s = 0$, $s = -1$, $s = -5$, $s = -15$, $s = -15$).

Euler's Theorem. The power series expansions of $\cos\theta$ and $\sin\theta$ are, respectively,

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$

And so

$$\cos\theta + j\sin\theta = 1 + (j\theta) + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \cdots$$

Since

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

we see that

$$\cos\theta + j\sin\theta = e^{j\theta} \tag{2–1}$$

This is known as *Euler's theorem*.

By using Euler's theorem, we can express sine and cosine in terms of an exponential function. Noting that $e^{-j\theta}$ is the complex conjugate of $e^{j\theta}$ and that

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

we find, after adding or subtracting these two equations, that

$$\cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right) \tag{2-2}$$

$$\sin\theta = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right) \tag{2-3}$$

2-3 LAPLACE TRANSFORMATION

We shall first present a definition of the Laplace transformation and a brief discussion of the condition for the existence of the Laplace transform and then provide examples for illustrating the derivation of Laplace transforms of several common functions.

Let us define

$f(t)$ = a function of time t such that $f(t) = 0$ for $t < 0$

s = a complex variable

$\mathscr{L}$ = an operational symbol indicating that the quantity that it prefixes is to be transformed by the Laplace integral $\int_0^\infty e^{-st}\,dt$

$F(s)$ = Laplace transform of $f(t)$

Then the Laplace transform of $f(t)$ is given by

$$\mathscr{L}[f(t)] = F(s) = \int_0^\infty e^{-st}\,dt[f(t)] = \int_0^\infty f(t)e^{-st}\,dt$$

The reverse process of finding the time function $f(t)$ from the Laplace transform $F(s)$ is called the *inverse Laplace transformation*. The notation for the inverse Laplace transformation is $\mathscr{L}^{-1}$, and the inverse Laplace transform can be found from $F(s)$ by the following inversion integral:

$$\mathscr{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}\,ds, \qquad \text{for } t > 0 \tag{2-4}$$

where c, the abscissa of convergence, is a real constant and is chosen larger than the real parts of all singular points of $F(s)$. Thus, the path of integration is parallel to the $j\omega$ axis and is displaced by the amount c from it. This path of integration is to the right of all singular points.

Evaluating the inversion integral appears complicated. In practice, we seldom use this integral for finding $f(t)$. There are simpler methods for finding $f(t)$. We shall discuss such simpler methods in Sections 2–5 and 2–6.

It is noted that in this book the time function $f(t)$ is always assumed to be zero for negative time; that is,

$$f(t) = 0, \qquad \text{for } t < 0$$

Existence of Laplace Transform. The Laplace transform of a function $f(t)$ exists if the Laplace integral converges. The integral will converge if $f(t)$ is sectionally continuous in every finite interval in the range $t > 0$ and if it is of exponential order as t approaches infinity. A function $f(t)$ is said to be of exponential order if a real, positive constant σ exists such that the function

$$e^{-\sigma t}|f(t)|$$

approaches zero as t approaches infinity. If the limit of the function $e^{-\sigma t}|f(t)|$ approaches zero for σ greater than σ_c and the limit approaches infinity for σ less than σ_c, the value σ_c is called the *abscissa of convergence*.

For the function $f(t) = Ae^{-\alpha t}$

$$\lim_{t\to\infty} e^{-\sigma t}|Ae^{-\alpha t}|$$

approaches zero if $\sigma > -\alpha$. The abscissa of convergence in this case is $\sigma_c = -\alpha$. The integral $\int_0^\infty f(t)e^{-st}\,dt$ converges only if σ, the real part of s, is greater than the abscissa of convergence σ_c. Thus the operator s must be chosen as a constant such that this integral converges.

In terms of the poles of the function $F(s)$, the abscissa of convergence σ_c corresponds to the real part of the pole located farthest to the right in the s plane. For example, for the following function $F(s)$,

$$F(s) = \frac{K(s+3)}{(s+1)(s+2)}$$

the abscissa of convergence σ_c is equal to -1. It can be seen that for such functions as t, $\sin\omega t$, and $t\sin\omega t$ the abscissa of convergence is equal to zero. For functions like e^{-ct}, te^{-ct}, $e^{-ct}\sin\omega t$, and so on, the abscissa of convergence is equal $-c$. For functions that increase faster than the exponential function, however, it is impossible to find suitable values of the abcissa of convergence. Therefore, such functions as e^{t^2} and te^{t^2} do not possess Laplace transforms.

The reader should be cautioned that although e^{t^2}(for $0 \le t \le \infty$) does not possess a Laplace transform, the time function defined by

$$\begin{aligned} f(t) &= e^{t^2}, \qquad && \text{for } 0 \le t \le T < \infty \\ &= 0, && \text{for } t < 0, T < t \end{aligned}$$

does possess a Laplace transform since $f(t) = e^{t^2}$ for only a limited time interval $0 \le t \le T$ and not for $0 \le t \le \infty$. Such a signal can be physically generated. Note that the signals that we can physically generate always have corresponding Laplace transforms.

If a function $f(t)$ has a Laplace transform, then the Laplace transform of $Af(t)$, where A is a constant, is given by

$$\mathcal{L}[Af(t)] = A\mathcal{L}[f(t)]$$

This is obvious from the definition of the Laplace transform. Since Laplace transformation is a linear operation, if functions $f_1(t)$ and $f_2(t)$ have Laplace transforms, $F_1(s)$ and $F_1(s)$, respectively, then the Laplace transform of the function $\alpha f_1(t) + \beta f_2(t)$ is given by

$$\mathcal{L}[\alpha f_1(t) + \beta f_2(t)] = \alpha F_1(s) + \beta F_2(s)$$

In what follows, we derive Laplace transforms of a few commonly encountered functions.

Exponential Function. Consider the exponential function

$$\begin{aligned} f(t) &= 0, && \text{for } t < 0 \\ &= Ae^{-\alpha t}, && \text{for } t \geq 0 \end{aligned}$$

where A and α are constants. The Laplace transform of this exponential function can be obtained as follows:

$$\mathcal{L}[Ae^{-\alpha t}] = \int_0^\infty Ae^{-\alpha t}e^{-st}\,dt = A\int_0^\infty e^{-(\alpha+s)t}\,dt = \frac{A}{s+\alpha}$$

It is seen that the exponential function produces a pole in the complex plane.

In deriving the Laplace transform of $f(t) = Ae^{-\alpha t}$, we required that the real part of s be greater than $-\alpha$ (the abscissa of convergence). A question may immediately arise as to whether or not the Laplace transform thus obtained is valid in the range where $\sigma < -\alpha$ in the s plane. To answer this question, we must resort to the theory of complex variables. In the theory of complex variables, there is a theorem known as the analytic extension theorem. It states that, if two analytic functions are equal for a finite length along any arc in a region in which both are analytic, then they are equal everywhere in the region. The arc of equality is usually the real axis or a portion of it. By using this theorem the form of $F(s)$ determined by an integration in which s is allowed to have any real positive value greater than the abscissa of convergence holds for any complex values of s at which $F(s)$ is analytic. Thus, although we require the real part of s to be greater than the abscissa of convergence to make the integral $\int_0^\infty f(t)e^{-st}\,dt$ absolutely convergent, once the Laplace transform $F(s)$ is obtained, $F(s)$ can be considered valid throughout the entire s plane except at the poles of $F(s)$.

Step Function. Consider the step function

$$\begin{aligned} f(t) &= 0, && \text{for } t < 0 \\ &= A, && \text{for } t > 0 \end{aligned}$$

where A is a constant. Note that it is a special case of the exponential function $Ae^{-\alpha t}$, where $\alpha = 0$. The step function is undefined at $t = 0$. Its Laplace transform is given by

$$\mathcal{L}[A] = \int_0^\infty Ae^{-st}\,dt = \frac{A}{s}$$

In performing this integration, we assumed that the real part of s was greater than zero (the abscissa of convergence) and therefore that $\lim_{t\to\infty} e^{-st}$ was zero. As stated earlier, the Laplace transform thus obtained is valid in the entire s plane except at the pole $s = 0$.

The step function whose height is unity is called *unit-step* function. The unit-step function that occurs at $t = t_0$ is frequently written as $1(t - t_0)$. The step function of height A that occurs at $t = 0$ can then be written as $f(t) = A1(t)$. The Laplace transform of the unit-step function, which is defined by

$$\begin{aligned} 1(t) &= 0, && \text{for } t < 0 \\ &= 1, && \text{for } t > 0 \end{aligned}$$

is $1/s$, or

$$\mathcal{L}[1(t)] = \frac{1}{s}$$

Physically, a step function occurring at $t = 0$ corresponds to a constant signal suddenly applied to the system at time t equals zero.

Ramp Function. Consider the ramp function

$$\begin{aligned} f(t) &= 0, & &\text{for } t < 0 \\ &= At, & &\text{for } t \geq 0 \end{aligned}$$

where A is a constant. The Laplace transform of this ramp function is obtained as

$$\begin{aligned} \mathcal{L}[At] &= \int_0^\infty Ate^{-st}\,dt = At\frac{e^{-st}}{-s}\bigg|_0^\infty - \int_0^\infty \frac{Ae^{-st}}{-s}\,dt \\ &= \frac{A}{s}\int_0^\infty e^{-st}\,dt = \frac{A}{s^2} \end{aligned}$$

Sinusoidal Function. The Laplace transform of the sinusoidal function

$$\begin{aligned} f(t) &= 0, & &\text{for } t < 0 \\ &= A\sin\omega t, & &\text{for } t \geq 0 \end{aligned}$$

where A and ω are constants, is obtained as follows. Referring to Equation (2–3), $\sin\omega t$ can be written

$$\sin\omega t = \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)$$

Hence

$$\begin{aligned} \mathcal{L}[A\sin\omega t] &= \frac{A}{2j}\int_0^\infty \left(e^{j\omega t} - e^{-j\omega t}\right)e^{-st}\,dt \\ &= \frac{A}{2j}\frac{1}{s - j\omega} - \frac{A}{2j}\frac{1}{s + j\omega} = \frac{A\omega}{s^2 + \omega^2} \end{aligned}$$

Similarly, the Laplace transform of $A\cos\omega t$ can be derived as follows:

$$\mathcal{L}[A\cos\omega t] = \frac{As}{s^2 + \omega^2}$$

Comments. The Laplace transform of any Laplace transformable function $f(t)$ can be found by multiplying $f(t)$ by e^{-st} and then integrating the product from $t = 0$ to $t = \infty$. Once we know the method of obtaining the Laplace transform, however, it is not necessary to derive the Laplace transform of $f(t)$ each time. Laplace transform tables can conveniently be used to find the transform of a given function $f(t)$. Table 2–1 shows Laplace transforms of time functions that will frequently appear in linear control systems analysis.

Table 2–1 Laplace Transform Pairs

	$f(t)$	$F(s)$
1	Unit impulse $\delta(t)$	1
2	Unit step $1(t)$	$\frac{1}{s}$
3	t	$\frac{1}{s^2}$
4	$\frac{t^{n-1}}{(n-1)!} \quad (n = 1, 2, 3, \dots)$	$\frac{1}{s^n}$
5	$t^n \quad (n = 1, 2, 3, \dots)$	$\frac{n!}{s^{n+1}}$
6	e^{-at}	$\frac{1}{s+a}$
7	te^{-at}	$\frac{1}{(s+a)^2}$
8	$\frac{1}{(n-1)!} t^{n-1} e^{-at} \quad (n = 1, 2, 3, \dots)$	$\frac{1}{(s+a)^n}$
9	$t^n e^{-at} \quad (n = 1, 2, 3, \dots)$	$\frac{n!}{(s+a)^{n+1}}$
10	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
11	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
12	$\sinh \omega t$	$\frac{\omega}{s^2-\omega^2}$
13	$\cosh \omega t$	$\frac{s}{s^2-\omega^2}$
14	$\frac{1}{a}(1 - e^{-at})$	$\frac{1}{s(s+a)}$
15	$\frac{1}{b-a}(e^{-at} - e^{-bt})$	$\frac{1}{(s+a)(s+b)}$
16	$\frac{1}{b-a}(be^{-bt} - ae^{-at})$	$\frac{s}{(s+a)(s+b)}$
17	$\frac{1}{ab}\left[1 + \frac{1}{a-b}(be^{-at} - ae^{-bt})\right]$	$\frac{1}{s(s+a)(s+b)}$

(*continues on next page*)

Table 2–1 *(continued)*

18	$\frac{1}{a^2}(1 - e^{-at} - ate^{-at})$	$\frac{1}{s(s+a)^2}$
19	$\frac{1}{a^2}(at - 1 + e^{-at})$	$\frac{1}{s^2(s+a)}$
20	$e^{-at}\sin\omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
21	$e^{-at}\cos\omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
22	$\frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin\omega_n\sqrt{1-\zeta^2}\,t \quad (0<\zeta<1)$	$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$
23	$-\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}\,t - \phi)$ $\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$ $(0<\zeta<1, \quad 0<\phi<\pi/2)$	$\frac{s}{s^2+2\zeta\omega_n s+\omega_n^2}$
24	$1-\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}\,t + \phi)$ $\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$ $(0<\zeta<1, \quad 0<\phi<\pi/2)$	$\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$
25	$1-\cos\omega t$	$\frac{\omega^2}{s(s^2+\omega^2)}$
26	$\omega t - \sin\omega t$	$\frac{\omega^3}{s^2(s^2+\omega^2)}$
27	$\sin\omega t - \omega t\cos\omega t$	$\frac{2\omega^3}{(s^2+\omega^2)^2}$
28	$\frac{1}{2\omega}t\sin\omega t$	$\frac{s}{(s^2+\omega^2)^2}$
29	$t\cos\omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
30	$\frac{1}{\omega_2^2-\omega_1^2}(\cos\omega_1 t - \cos\omega_2 t) \qquad (\omega_1^2 \neq \omega_2^2)$	$\frac{s}{(s^2+\omega_1^2)(s^2+\omega_2^2)}$
31	$\frac{1}{2\omega}(\sin\omega t + \omega t\cos\omega t)$	$\frac{s^2}{(s^2+\omega^2)^2}$

Table 2–2 Properties of Laplace Transforms

1	$\mathscr{L}[Af(t)] = AF(s)$
2	$\mathscr{L}[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$
3	$\mathscr{L}_\pm\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0\pm)$
4	$\mathscr{L}_\pm\left[\frac{d^2}{dt^2}f(t)\right] = s^2F(s) - sf(0\pm) - \dot{f}(0\pm)$
5	$\mathscr{L}_\pm\left[\frac{d^n}{dt^n}f(t)\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k}\overset{(k-1)}{f}(0\pm)$ where $\overset{(k-1)}{f}(t) = \frac{d^{k-1}}{dt^{k-1}}f(t)$
6	$\mathscr{L}_\pm\left[\int f(t)\,dt\right] = \frac{F(s)}{s} + \frac{1}{s}\left[\int f(t)\,dt\right]_{t=0\pm}$
7	$\mathscr{L}_\pm\left[\int\cdots\int f(t)(dt)^n\right] = \frac{F(s)}{s^n} + \sum_{k=1}^{n}\frac{1}{s^{n-k+1}}\left[\int\cdots\int f(t)(dt)^k\right]_{t=0\pm}$
8	$\mathscr{L}\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$
9	$\int_0^\infty f(t)\,dt = \lim_{s\to 0} F(s) \qquad \text{if } \int_0^\infty f(t)\,dt \text{ exists}$
10	$\mathscr{L}[e^{-\alpha t}f(t)] = F(s + a)$
11	$\mathscr{L}[f(t-\alpha)1(t-\alpha)] = e^{-\alpha s}F(s) \qquad \alpha \geq 0$
12	$\mathscr{L}[tf(t)] = -\frac{dF(s)}{ds}$
13	$\mathscr{L}[t^2f(t)] = \frac{d^2}{ds^2}F(s)$
14	$\mathscr{L}[t^nf(t)] = (-1)^n\frac{d^n}{ds^n}F(s) \qquad (n = 1, 2, 3, \dots)$
15	$\mathscr{L}\left[\frac{1}{t}f(t)\right] = \int_s^\infty F(s)\,ds \qquad \text{if } \lim_{t\to 0}\frac{1}{t}f(t) \text{ exists}$
16	$\mathscr{L}\left[f\left(\frac{1}{a}\right)\right] = aF(as)$
17	$\mathscr{L}\left[\int_0^t f_1(t-\tau)f_2(\tau)\,d\tau\right] = F_1(s)F_2(s)$
18	$\mathscr{L}[f(t)g(t)] = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty}F(p)G(s-p)\,dp$