Then, using $\dot{x}_3 = \ddot{y}_1$ and $\dot{x}_4 = \ddot{y}_2$, we get the state-space representation as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -K_1/M_1 & K_1/M_1 & 0 & 0 \\ K_2/M_1 & -K_2/M_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/M_1 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/M_2 \end{bmatrix} u_2 \quad \text{(state equation)}$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + 0 \cdot u_1 + 0 \cdot u_2 \quad \text{(output equation)}$$

(4-32)

where the state equation is a set of four first-order differential equations.

4-1-2 Rotational Motion

The rotational motion of a body can be defined as motion about a fixed axis. The extension of Newton's law of motion for rotational motion states that the *algebraic sum of moments or torque about a fixed axis is equal to the product of the inertia and the angular acceleration about the axis.* Or

$$\sum \text{torques} = J\alpha \tag{4-33}$$

where J denotes the inertia and α is the angular acceleration. The other variables generally used to describe the motion of rotation are **torque** T, **angular velocity** ω, and **angular displacement** θ. The elements involved with the rotational motion are as follows:

- **Inertia.** *Inertia, J, is considered a property of an element that stores the kinetic energy of rotational motion.* The inertia of a given element depends on the geometric composition about the axis of rotation and its density. For instance, the inertia of a circular disk or shaft, of radius r and mass M, about its geometric axis is given by

$$J = \frac{1}{2}Mr^2 \tag{4-34}$$

When a torque is applied to a body with inertia J, as shown in Fig. 4-14, the torque equation is written

$$T(t) = J\alpha(t) = J\frac{d\omega(t)}{dt} = J\frac{d^2\theta(t)}{dt^2} \tag{4-35}$$

where $\theta(t)$ is the angular displacement; $\omega(t)$, the angular velocity; and $\alpha(t)$, the angular acceleration.

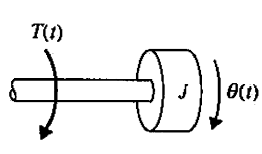


Figure 4-14 Torque-inertia system.

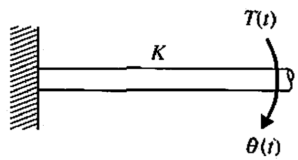


Figure 4-15 Torque torsional spring system.

- **Torsional spring.** As with the linear spring for translational motion, a **torsional spring constant** K, in torque-per-unit angular displacement, can be devised to represent the compliance of a rod or a shaft when it is subject to an applied torque. Fig. 4-15 illustrates a simple torque-spring system that can be represented by the equation

$$T(t) = K\theta(t) \tag{4-36}$$

If the torsional spring is preloaded by a preload torque of TP, Eq. (4-36) is modified to

$$T(t) - TP = K\theta(t) \tag{4-37}$$

- **Friction for rotational motion.** The three types of friction described for translational motion can be carried over to the motion of rotation. Therefore, Eqs. (4-6), (4-7), and (4-8) can be replaced, respectively, by their counterparts:
  - **Viscous friction.**

$$T(t) = B\frac{d\theta(t)}{dt} \tag{4-38}$$

  - **Static friction.**

$$T(t) = \pm(F_s)|_{\dot{\theta}=0} \tag{4-39}$$

  - **Coulomb friction.**

$$T(t) = F_c \frac{\frac{d\theta(t)}{dt}}{\left|\frac{d\theta(t)}{dt}\right|} \tag{4-40}$$

Table 4-2 shows the SI and other measurement units for inertia and the variables in rotational mechanical systems.

► **EXAMPLE 4-1-4** The rotational system shown in Fig. 4-16(a) consists of a disk mounted on a shaft that is fixed at one end. The moment of inertia of the disk about the axis of rotation is J. The edge of the disk is riding on the surface, and the viscous friction coefficient between the two surfaces is B. The inertia of the shaft is negligible, but the torsional spring constant is K.

Assume that a torque is applied to the disk, as shown; then the torque or moment equation about the axis of the shaft is written from the free-body diagram of Fig. 4-16(b):

$$T(t) = J\frac{d^2\theta(t)}{dt^2} + B\frac{d\theta(t)}{dt} + K\theta(t) \tag{4-41}$$

Notice that this system is analogous to the translational system in Fig. 4-5. The state equations may be written by defining the state variables as $x_1(t) = \theta(t)$ and $x_2(t) = dx_1(t)/dt$.

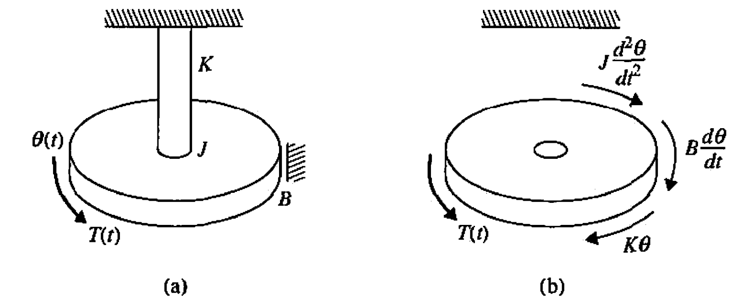


Figure 4-16 Rotational system for Example 4-1-4.

TABLE 4-2 Basic Rotational Mechanical System Properties and Their Units

Parameter	Symbol Used	SI Units	Other Units	Conversion Factors
Inertia	J	kg-m^2	slug-ft^2 lb-ft-sec^2 oz-in.-sec^2	1 g-cm = 1.417×10^{-5} oz-in.-sec^2 1 lb-ft-sec^2 = 192 oz-in.-sec^2 = 32.2 lb-ft^2 1 oz-in.-sec^2 = 386 oz-in^2 1 g-cm-sec^2 = 980 g-cm^2
Angular Displacement	T	Radian	Radian	$1 \text{ rad} = \frac{180}{\pi} = 57.3 \text{ deg}$
Angular Velocity	O	radian/sec	radian/sec	$1 \text{ rpm} = \frac{2\pi}{60}$ = 0.1047 rad/sec 1 rpm = 6 deg/sec
Angular Acceleration	A	radian/sec^2	radian/sec^2	
Torque	T	(N-m) dyne-cm	lb-ft oz-in.	1 g-cm = 0.0139 oz-in. 1 lb-ft = 192 oz-in. 1 oz-in. = 0.00521 lb-ft
Spring Constant	K	N-m/rad	ft-lb/rad	
Viscous Friction Coefficient	B	N-m/rad/sec	ft-lb/rad/sec	
Energy	Q	J (joules)	Btu Calorie	1 J = 1 N-m 1 Btu = 1055 J 1 cal = 4.184 J

EXAMPLE 4-1-5 Fig. 4-17(a) shows the diagram of a motor coupled to an inertial load through a shaft with a spring constant K. A non-rigid coupling between two mechanical components in a control system often causes torsional resonances that can be transmitted to all parts of the system. The system variables and parameters are defined as follows:

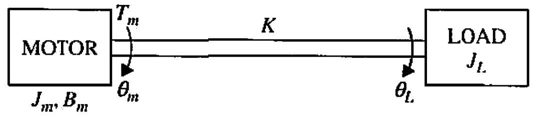
$T_m(t)$ = motor torque

B_m = motor viscous-friction coefficient

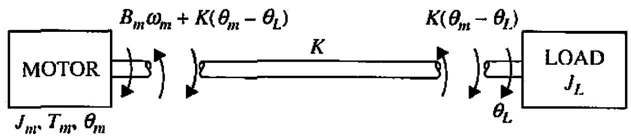
K = spring constant of the shaft

$\theta_m(t)$ = motor displacement

$\omega_m(t)$ = motor velocity



(a)



(b)

Figure 4-17 (a) Motor–load system. (b) Free-body diagram.

J_m = motor inertia

$\theta_L(t)$ = load displacement

$\omega_L(t)$ = load velocity

J_L = load inertia

The free-body diagrams of the system are shown in Fig. 4-17(b). The torque equations of the system are

$$\frac{d^2\theta_m(t)}{dt^2} = -\frac{B_m}{J_m}\frac{d\theta_m(t)}{dt} - \frac{K}{J_m}[\theta_m(t) - \theta_L(t)] + \frac{1}{J_m}T_m(t) \tag{4-42}$$

$$K[\theta_m(t) - \theta_L(t)] = J_L\frac{d^2\theta_L(t)}{dt^2} \tag{4-43}$$

In this case, the system contains three energy-storage elements in J_m, J_L, and K. Thus, there should be three state variables. Care should be taken in constructing the state diagram and assigning the state variables so that a minimum number of the latter are incorporated. Eqs. (4-42) and (4-43) are rearranged as

$$\frac{d^2\theta_m(t)}{dt^2} = -\frac{B_m}{J_m}\frac{d\theta_m(t)}{dt} - \frac{K}{J_m}[\theta_m(t) - \theta_L(t)] + \frac{1}{J_m}T_m(t) \tag{4-44}$$

$$\frac{d^2\theta_L(t)}{dt^2} = \frac{K}{J_L}[\theta_m(t) - \theta_L(t)] \tag{4-45}$$

The state variables in this case are defined as $x_1(t) = \theta_m(t) - \theta_L(t)$, $x_2(t) = d\theta_L(t)/dt$, and $x_3(t) = d\theta_m(t)/dt$. The state equations are

$$\begin{aligned}\frac{dx_1(t)}{dt} &= x_3(t) - x_2(t)\\ \frac{dx_2(t)}{dt} &= \frac{K}{J_L}x_1(t)\\ \frac{dx_3(t)}{dt} &= -\frac{K}{J_m}x_1(t) - \frac{B_m}{J_m}x_3(t) + \frac{1}{J_m}T_m(t)\end{aligned} \tag{4-46}$$

The SFG representation is shown in Fig. 4-18.

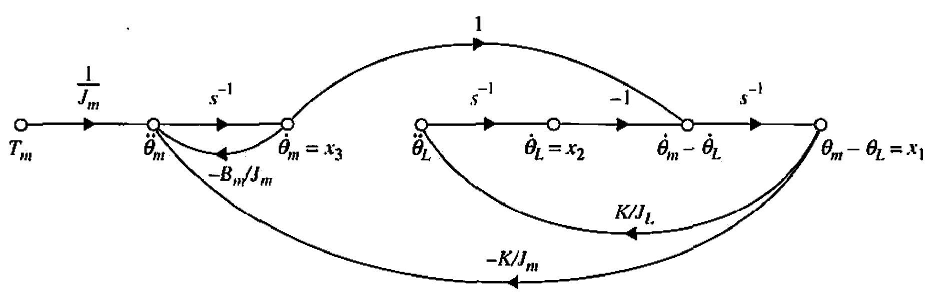


Figure 4-18 Rotational system of Eq. (4-46) signal-flow graph representation.

and using Eq. (2.123) for M_3,

$$-f_{v_3}sX_1(s) - f_{v_4}sX_2(s) + \left[M_3s^2 + (f_{v_3} + f_{v_4})s\right]X_3(s) = 0 \tag{2.126}$$

Equations (2.124) through (2.126) are the equations of motion. We can solve them for any displacement, $X_1(s)$, $X_2(s)$, or $X_3(s)$, or transfer function.

Skill-Assessment Exercise 2.8

PROBLEM: Find the transfer function, $G(s) = X_2(s)/F(s)$, for the translational mechanical system shown in Figure 2.21.

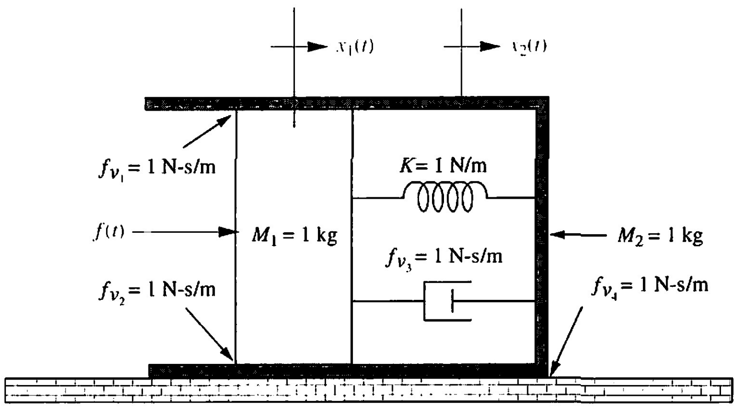


FIGURE 2.21 Translational mechanical system for Skill-Assessment Exercise 2.8

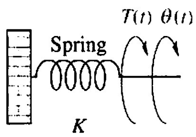
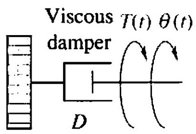
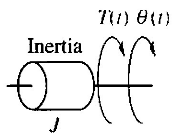
ANSWER: $G(s) = \dfrac{3s + 1}{s(s^3 + 7s^2 + 5s + 1)}$

The complete solution is at www.wiley.com/college/nise.

2.6 Rotational Mechanical System Transfer Functions

Having covered electrical and translational mechanical systems, we now move on to consider rotational mechanical systems. Rotational mechanical systems are handled the same way as translational mechanical systems, except that torque replaces force and angular displacement replaces translational displacement. The mechanical components for rotational systems are the same as those for translational systems, except that the components undergo rotation instead of translation. Table 2.5 shows the components along with the relationships between torque and angular velocity, as well as angular displacement. Notice that the symbols for the

TABLE 2.5 Torque-angular velocity, torque-angular displacement, and impedance rotational relationships for springs, viscous dampers, and inertia

Component	Torque-angular velocity	Torque-angular displacement	Impedence $Z_M(s) = T(s)/\theta(s)$
	$T(t) = K\int_0^t \omega(\tau)d\tau$	$T(t) = K\theta(t)$	K
	$T(t) = D\omega(t)$	$T(t) = D\frac{d\theta(t)}{dt}$	Ds
	$T(t) = J\frac{d\omega(t)}{dt}$	$T(t) = J\frac{d^2\theta(t)}{dt^2}$	Js^2

Note: The following set of symbols and units is used throughout this book: $T(t)$ – N-m (newton-meters), $\theta(t)$ – rad(radians), $\omega(t)$ – rad/s(radians/second), K – N-m/rad(newton- meters/radian), D – N-m-s/rad (newton- meters-seconds/radian). J – kg-m^2(kilograms-meters^2 – newton-meters-seconds^2/radian).

components look the same as translational symbols, but they are undergoing rotation and not translation.

Also notice that the term associated with the mass is replaced by inertia. The values of K, D, and J are called *spring constant*, *coefficient of viscous friction*, and *moment of inertia*, respectively. The impedances of the mechanical components are also summarized in the last column of Table 2.5. The values can be found by taking the Laplace transform, assuming zero initial conditions, of the torque-angular displacement column of Table 2.5.

The concept of degrees of freedom carries over to rotational systems, except that we test a point of motion by *rotating* it while holding still all other points of motion. The number of points of motion that can be rotated while all others are held still equals the number of equations of motion required to describe the system.

Writing the equations of motion for rotational systems is similar to writing them for translational systems; the only difference is that the free-body diagram consists of torques rather than forces. We obtain these torques using superposition. First, we rotate a body while holding all other points still and place on its free-body diagram all torques due to the body's own motion. Then, holding the body still, we rotate adjacent points of motion one at a time and add the torques due to the adjacent motion to the free-body diagram. The process is repeated for each point of motion. For each free-body diagram, these torques are summed and set equal to zero to form the equations of motion.

Two examples will demonstrate the solution of rotational systems. The first one uses free-body diagrams; the second uses the concept of impedances to write the equations of motion by inspection.

Example 2.19

Transfer Function—Two Equations of Motion

PROBLEM: Find the transfer function, $\theta_2(s)/T(s)$, for the rotational system shown in Figure 2.22(a). The rod is supported by bearings at either end and is undergoing torsion. A torque is applied at the left, and the displacement is measured at the right.

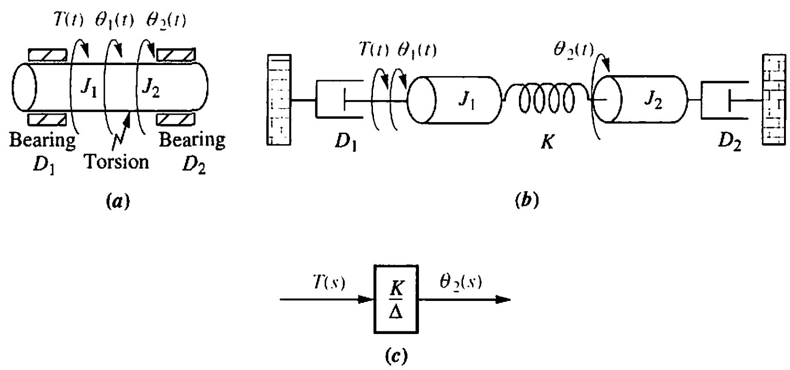


FIGURE 2.22 **a.** Physical system; **b.** schematic; **c.** block diagram

SOLUTION: First, obtain the schematic from the physical system. Even though torsion occurs throughout the rod in Figure 2.22(a),[9] we approximate the system by assuming that the torsion acts like a spring concentrated at one particular point in the rod, with an inertia J_1 to the left and an inertia J_2 to the right.[10] We also assume that the damping inside the flexible shaft is negligible. The schematic is shown in Figure 2.22(b). There are two degrees of freedom, since each inertia can be rotated while the other is held still. Hence, it will take two simultaneous equations to solve the system.

Next, draw a free-body diagram of J_1, using superposition. Figure 2.23(a) shows the torques on J_1 if J_2 is held still and J_1 rotated. Figure 2.23(b) shows the torques on J_1 if J_1 is held still and J_2 rotated. Finally, the sum of Figures 2.23(a) and 2.23(b) is shown in Figure 2.23(c), the final free-body diagram for J_1. The same process is repeated in Figure 2.24 for J_2.

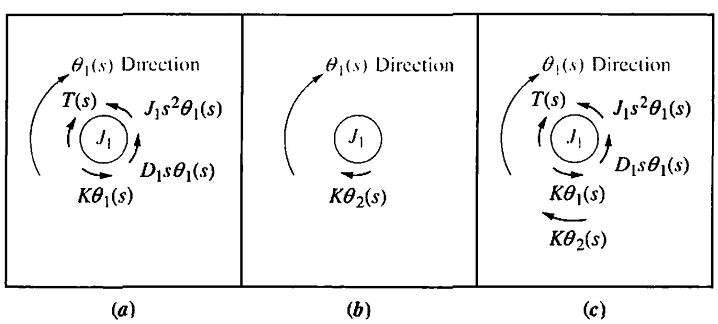


FIGURE 2.23 **a.** Torques on J_1 due only to the motion of J_1; **b.** torques on J_1 due only to the motion of J_2; **c.** final free-body diagram for J_1

[9] In this case the parameter is referred to as a *distributed* parameter.

[10] The parameter is now referred to as a *lumped* parameter.

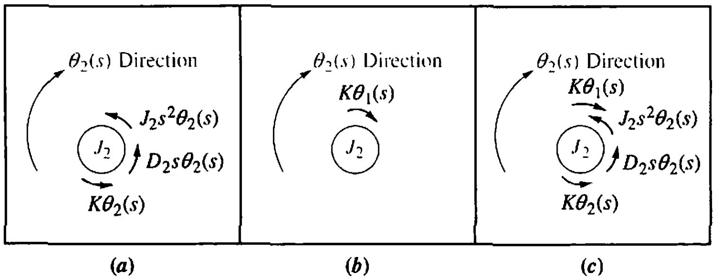


FIGURE 2.24 **a.** Torques on J_2 due only to the motion of J_2; **b.** torques on J_2 due only to the motion of J_1; **c.** final free-body diagram for J_2

Summing torques respectively from Figures 2.23(*c*) and 2.24(*c*) we obtain the equations of motion,

$$(J_1s^2 + D_1s + K)\theta_1(s) \qquad - K\theta_2(s) = T(s) \tag{2.127a}$$

$$-K\theta_1(s) + (J_2s^2 + D_2s + K)\theta_2(s) = 0 \tag{2.127b}$$

from which the required transfer function is found to be

$$\frac{\theta_2(s)}{T(s)} = \frac{K}{\Delta} \tag{2.128}$$

as shown in Figure 2.22(*c*), where

$$\Delta = \begin{vmatrix} (J_1s^2 + D_1s + K) & -K \\ -K & (J_2s^2 + D_2s + K) \end{vmatrix}$$

Notice that Eq. (2.127) have that now well-known form

$$\begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{connected} \\ \text{to the motion} \\ \text{at } \theta_1 \end{bmatrix} \theta_1(s) - \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ \theta_1 \text{ and } \theta_2 \end{bmatrix} \theta_2(s) = \begin{bmatrix} \text{Sum of} \\ \text{applied torques} \\ \text{at } \theta_1 \end{bmatrix} \tag{2.129a}$$

$$-\begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ \theta_1 \text{ and } \theta_2 \end{bmatrix} \theta_1(s) + \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{connected} \\ \text{to the motion} \\ \text{at } \theta_2 \end{bmatrix} \theta_2(s) = \begin{bmatrix} \text{Sum of} \\ \text{applied torques} \\ \text{at } \theta_2 \end{bmatrix} \tag{2.129b}$$

TryIt 2.9

Use the following MATLAB and Symbolic Math Toolbox statements to help you get Eq. (2.128).

```
syms s J1 D1 K T J2 D2...
  theta1 theta2
A=[(J1*s^2+D1*s+K) -K
    -K (J2*s^2+D2*s+K)];
B=[theta1
    theta2];
C=[T
    0];
B=inv(A)*C;
theta2=B(2);
'theta2'
pretty(theta2)
```

Example 2.20

Equations of Motion By Inspection

PROBLEM: Write, but do not solve, the Laplace transform of the equations of motion for the system shown in Figure 2.25.

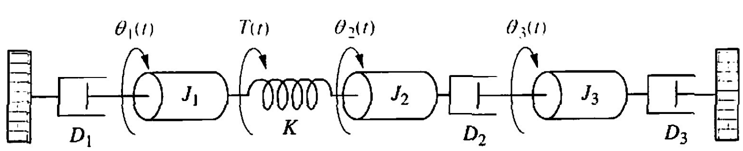


FIGURE 2.25 Three-degrees-of-freedom rotational system