Skill-Assessment Exercise 2.4

PROBLEM: Find the differential equation corresponding to the transfer function,

$$G(s) = \frac{2s + 1}{s^2 + 6s + 2}$$

ANSWER: $\dfrac{d^2c}{dt^2} + 6\dfrac{dc}{dt} + 2c = 2\dfrac{dr}{dt} + r$

The complete solution is at www.wiley.com/college/nise.

Skill-Assessment Exercise 2.5

PROBLEM: Find the ramp response for a system whose transfer function is

$$G(s) = \frac{s}{(s + 4)(s + 8)}$$

ANSWER: $c(t) = \dfrac{1}{32} - \dfrac{1}{16}e^{-4t} + \dfrac{1}{32}e^{-8t}$

WileyPLUS

WPCS

Control Solutions

The complete solution is at www.wiley.com/college/nise.

In general, a physical system that can be represented by a linear, time-invariant differential equation can be modeled as a transfer function. The rest of this chapter will be devoted to the task of modeling individual subsystems. We will learn how to represent electrical networks, translational mechanical systems, rotational mechanical systems, and electromechanical systems as transfer functions. As the need arises, the reader can consult the Bibliography at the end of the chapter for discussions of other types of systems, such as pneumatic, hydraulic, and heat-transfer systems (*Cannon, 1967*).

# ◖ 2.4 Electrical Network Transfer Functions

In this section, we formally apply the transfer function to the mathematical modeling of electric circuits including passive networks and operational amplifier circuits. Subsequent sections cover mechanical and electromechanical systems.

Equivalent circuits for the electric networks that we work with first consist of three passive linear components: resistors, capacitors, and inductors.[2] Table 2.3 summarizes the components and the relationships between voltage and current and between voltage and charge under zero initial conditions.

We now combine electrical components into circuits, decide on the input and output, and find the transfer function. Our guiding principles are Kirchhoff's laws. We sum voltages around loops or sum currents at nodes, depending on which technique involves the least effort in algebraic manipulation, and then equate the result to zero. From these relationships we can write the differential equations for the circuit. Then we can take the Laplace transforms of the differential equations and finally solve for the transfer function.

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[2] *Passive* means that there is no internal source of energy.

TABLE 2.3   Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s) = V(s)/I(s)$	Admittance $Y(s) = I(s)/V(s)$
$\dashv\vdash$ Capacitor	$v(t) = \dfrac{1}{C}\displaystyle\int_0^1 i(\tau)d\tau$	$i(t) = C\dfrac{dv(t)}{dt}$	$v(t) = \dfrac{1}{C}q(t)$	$\dfrac{1}{Cs}$	Cs
$\text{--}\mathcal{W}\text{--}$ Resistor	$v(t) = Ri(t)$	$i(t) = \dfrac{1}{R}v(t)$	$v(t) = R\dfrac{dq(t)}{dt}$	R	$\dfrac{1}{R} = G$
$\text{--}\text{0000}\text{--}$ Inductor	$v(t) = L\dfrac{di(t)}{dt}$	$i(t) = \dfrac{1}{L}\displaystyle\int_0^1 v(\tau)d\tau$	$v(t) = L\dfrac{d^2q(t)}{dt^2}$	Ls	$\dfrac{1}{Ls}$

Note: The following set of symbols and units is used throughout this book: $v(t) \sim$ V (volts), $i(t) -$ A (amps), $q(t) -$ Q (coulombs), $C -$ F (farads), $R - \Omega$ (ohms), $G - \Omega$ (mhos), $L - H$ (henries).

Simple Circuits via Mesh Analysis

Transfer functions can be obtained using Kirchhoff's voltage law and summing voltages around loops or meshes.[3] We call this method *loop* or *mesh analysis* and demonstrate it in the following example.

Example 2.6

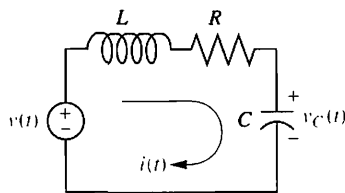### Transfer Function—Single Loop via the Differential Equation

FIGURE 2.3   *RLC* network

PROBLEM: Find the transfer function relating the capacitor voltage, $V_C(s)$, to the input voltage, $V(s)$ in Figure 2.3.

SOLUTION: In any problem, the designer must first decide what the input and output should be. In this network, several variables could have been chosen to be the output—for example, the inductor voltage, the capacitor voltage, the resistor voltage, or the current. The problem statement, however, is clear in this case: We are to treat the capacitor voltage as the output and the applied voltage as the input.

Summing the voltages around the loop, assuming zero initial conditions, yields the integro-differential equation for this network as

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = v(t) \tag{2.61}$$

Changing variables from current to charge using $i(t) = dq(t)/dt$ yields

$$L\frac{d^2q(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{1}{C}q(t) = v(t) \tag{2.62}$$

From the voltage-charge relationship for a capacitor in Table 2.3,

$$q(t) = Cv_C(t) \tag{2.63}$$

Substituting Eq. (2.63) into Eq. (2.62) yields

$$LC\frac{d^2v_C(t)}{dt^2} + RC\frac{dv_C(t)}{dt} + v_C(t) = v(t) \tag{2.64}$$

---

[3] A particular loop that resembles the spaces in a screen or fence is called a *mesh*.

Taking the Laplace transform assuming zero initial conditions, rearranging terms, and simplifying yields

$$(LCs^2 + RCs + 1)V_C(s) = V(s) \qquad (2.65)$$

Solving for the transfer function, $V_C(s)/V(s)$, we obtain

$$\frac{V_C(s)}{V(s)} = \frac{1/LC}{s^2 + \dfrac{R}{L}s + \dfrac{1}{LC}} \qquad (2.66)$$
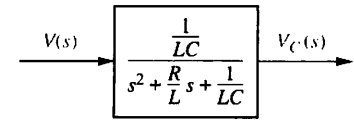
as shown in Figure 2.4.

FIGURE 2.4 Block diagram of series RLC electrical network

Let us now develop a technique for simplifying the solution for future problems. First, take the Laplace transform of the equations in the voltage-current column of Table 2.3 assuming zero initial conditions.

For the capacitor,

$$\boxed{V(s) = \frac{1}{Cs}I(s)} \qquad (2.67)$$

For the resistor,

$$\boxed{V(s) = RI(s)} \qquad (2.68)$$

For the inductor,

$$\boxed{V(s) = LsI(s)} \qquad (2.69)$$

Now define the following transfer function:

$$\boxed{\frac{V(s)}{I(s)} = Z(s)} \qquad (2.70)$$

Notice that this function is similar to the definition of resistance, that is, the ratio of voltage to current. But, unlike resistance, this function is applicable to capacitors and inductors and carries information on the dynamic behavior of the component, since it represents an equivalent differential equation. We call this particular transfer function *impedance*. The impedance for each of the electrical elements is shown in Table 2.3.

Let us now demonstrate how the concept of impedance simplifies the solution for the transfer function. The Laplace transform of Eq. (2.61), assuming zero initial conditions, is

$$\left(Ls + R + \frac{1}{Cs}\right)I(s) = V(s) \qquad (2.71)$$

Notice that Eq. (2.71), which is in the form

$$[\text{Sum of impedances}]I(s) = [\text{Sum of applied voltages}] \qquad (2.72)$$

suggests the series circuit shown in Figure 2.5. Also notice that the circuit of Figure 2.5 could have been obtained immediately from the circuit of Figure 2.3 simply by replacing each element with its impedance. We call this altered circuit the *transformed circuit*. Finally, notice that the transformed circuit leads immediately to Eq. (2.71) if we add impedances in series as we add resistors in series. Thus, rather than writing the differential equation first and then taking the
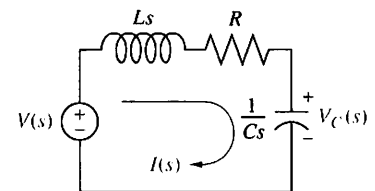

FIGURE 2.5 Laplace-transformed network

Laplace transform, we can draw the transformed circuit and obtain the Laplace transform of the differential equation simply by applying Kirchhoff's voltage law to the transformed circuit. We summarize the steps as follows:

1. Redraw the original network showing all time variables, such as $v(t)$, $i(t)$, and $v_C(t)$, as Laplace transforms $V(s)$, $I(s)$, and $V_C(s)$, respectively.

2. Replace the component values with their impedance values. This replacement is similar to the case of dc circuits, where we represent resistors with their resistance values.

We now redo Example 2.6 using the transform methods just described and bypass the writing of the differential equation.

(Example 2.7)

Transfer Function—Single Loop via Transform Methods

PROBLEM: Repeat Example 2.6 using mesh analysis and transform methods without writing a differential equation.

SOLUTION: Using Figure 2.5 and writing a mesh equation using the impedances as we would use resistor values in a purely resistive circuit, we obtain

$$\left(Ls + R + \frac{1}{Cs}\right)I(s) = V(s) \tag{2.73}$$

Solving for $I(s)/V(s)$,

$$\frac{I(s)}{V(s)} = \frac{1}{Ls + R + \dfrac{1}{Cs}} \tag{2.74}$$

But the voltage across the capacitor, $V_C(s)$, is the product of the current and the impedance of the capacitor. Thus,

$$V_C(s) = I(s)\frac{1}{Cs} \tag{2.75}$$

Solving Eq. (2.75) for $I(s)$, substituting $I(s)$ into Eq. (2.74), and simplifying yields the same result as Eq. (2.66).

Simple Circuits via Nodal Analysis

Transfer functions also can be obtained using Kirchhoff's current law and summing currents flowing from nodes. We call this method *nodal analysis*. We now demonstrate this principle by redoing Example 2.6 using Kirchhoff's current law and the transform methods just described to bypass writing the differential equation.

(Example 2.8)

Transfer Function—Single Node via Transform Methods

PROBLEM: Repeat Example 2.6 using nodal analysis and without writing a differential equation.

SOLUTION: The transfer function can be obtained by summing currents flowing out of the node whose voltage is $V_C(s)$ in Figure 2.5. We assume that currents leaving the node are positive and currents entering the node are negative. The currents consist of the current through the capacitor and the current flowing through the series resistor and inductor. From Eq. (2.70), each $I(s) = V(s)/Z(s)$. Hence,

$$\frac{V_C(s)}{1/Cs} + \frac{V_C(s) - V(s)}{R + Ls} = 0 \tag{2.76}$$

where $V_C(s)/(1/Cs)$ is the current flowing out of the node through the capacitor, and $[V_C(s) - V(s)]/(R + Ls)$ is the current flowing out of the node through the series resistor and inductor. Solving Eq. (2.76) for the transfer function, $V_C(s)/V(s)$, we arrive at the same result as Eq. (2.66).

Simple Circuits via Voltage Division

Example 2.6 can be solved directly by using voltage division on the transformed network. We now demonstrate this technique.

Example 2.9

Transfer Function—Single Loop via Voltage Division

PROBLEM: Repeat Example 2.6 using voltage division and the transformed circuit.

SOLUTION: The voltage across the capacitor is some proportion of the input voltage, namely the impedance of the capacitor divided by the sum of the impedances. Thus,

$$V_C(s) = \frac{1/Cs}{\left(Ls + R + \dfrac{1}{Cs}\right)} V(s) \tag{2.77}$$

Solving for the transfer function, $V_C(s)/V(s)$, yields the same result as Eq. (2.66).

Review Examples 2.6 through 2.9. Which method do you think is easiest for this circuit?

The previous example involves a simple, single-loop electrical network. Many electrical networks consist of multiple loops and nodes, and for these circuits we must write and solve simultaneous differential equations in order to find the transfer function, or solve for the output.

Complex Circuits via Mesh Analysis

To solve complex electrical networks—those with multiple loops and nodes—using mesh analysis, we can perform the following steps:

1. Replace passive element values with their impedances.

2. Replace all sources and time variables with their Laplace transform.

3. Assume a transform current and a current direction in each mesh.

4. Write Kirchhoff's voltage law around each mesh.

5. Solve the simultaneous equations for the output.

6. Form the transfer function.

Let us look at an example.

## Example 2.10

Transfer Function—Multiple Loops

PROBLEM: Given the network of Figure 2.6(a), find the transfer function, $I_2(s)/V(s)$.

SOLUTION: The first step in the solution is to convert the network into Laplace transforms for impedances and circuit variables, assuming zero initial conditions. The result is shown in Figure 2.6(b). The circuit with which we are dealing requires two simultaneous equations to solve for the transfer function. These equations can be found by summing voltages around each mesh through which the assumed currents, $I_1(s)$ and $I_2(s)$, flow. Around Mesh 1, where $I_1(s)$ flows,
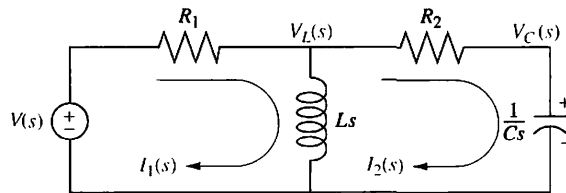
$$R_1 I_1(s) + Ls I_1(s) - Ls I_2(s) = V(s) \qquad (2.78)$$

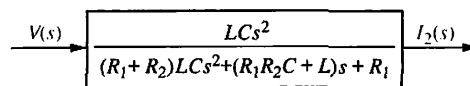Around Mesh 2, where $I_2(s)$ flows,

$$Ls I_2(s) + R_2 I_2(s) + \frac{1}{Cs} I_2(s) - Ls I_1(s) = 0 \qquad (2.79)$$

(a)

(b)
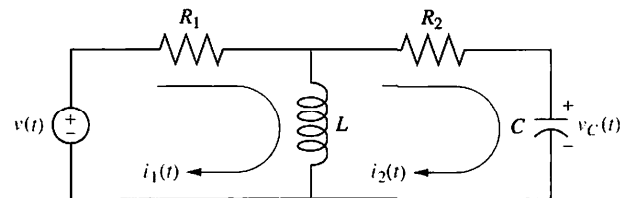
FIGURE 2.6   a. Two-loop electrical network; **b.** transformed two-loop electrical network; **c.** block diagram

(c)

Combining terms, Eqs. (2.78) and (2.79) become simultaneous equations in $I_1(s)$ and $I_2(s)$:

$$(R_1 + Ls)I_1(s) \qquad\qquad - LsI_2(s) = V(s) \qquad\qquad (2.80a)$$

$$- LsI_1(s) + \left(Ls + R_2 + \frac{1}{Cs}\right)I_2(s) = 0 \qquad\qquad (2.80b)$$

We can use Cramer's rule (or any other method for solving simultaneous equations) to solve Eq. (2.80) for $I_2(s)$.[4] Hence,

$$I_2(s) = \frac{\begin{vmatrix} (R_1 + Ls) & V(s) \\ -Ls & 0 \end{vmatrix}}{\Delta} = \frac{LsV(s)}{\Delta} \qquad\qquad (2.81)$$

where

$$\Delta = \begin{vmatrix} (R_1 + Ls) & -Ls \\ -Ls & \left(Ls + R_2 + \frac{1}{Cs}\right) \end{vmatrix}$$

Forming the transfer function, $G(s)$, yields

$$G(s) = \frac{I_2(s)}{V(s)} = \frac{Ls}{\Delta} = \frac{LCs^2}{(R_1 + R_2)LCs^2 + (R_1R_2C + L)s + R_1} \qquad\qquad (2.82)$$

as shown in Figure 2.6(c).

We have succeeded in modeling a physical network as a transfer function: The network of Figure 2.6(a) is now modeled as the transfer function of Figure 2.6(c). Before leaving the example, we notice a pattern first illustrated by Eq. (2.72). The form that Eq. (2.80) take is

$$\begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{around Mesh 1} \end{bmatrix} I_1(s) - \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{common to the} \\ \text{two meshes} \end{bmatrix} I_2(s) = \begin{bmatrix} \text{Sum of applied} \\ \text{voltages around} \\ \text{Mesh 1} \end{bmatrix} \qquad (2.83a)$$

$$-\begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{common to the} \\ \text{two meshes} \end{bmatrix} I_1(s) + \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{around Mesh 2} \end{bmatrix} I_2(s) = \begin{bmatrix} \text{Sum of applied} \\ \text{voltages around} \\ \text{Mesh 2} \end{bmatrix} \qquad (2.83b)$$

Recognizing the form will help us write such equations rapidly; for example, mechanical equations of motion (covered in Sections 2.5 and 2.6) have the same form.

Symbolic Math
SM

Students who are performing the MATLAB exercises and want to explore the added capability of MATLAB's Symbolic Math Toolbox should now run ch2sp4 in Appendix F at www.wiley.com/college/nise, where Example 2.10 is solved. You will learn how to use the Symbolic Math Toolbox to solve simultaneous equations using Cramer's rule. Specifically, the Symbolic Math Toolbox will be used to solve for the transfer function in Eq. (2.82) using Eq. (2.80).

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[4] See Appendix G (Section G.4) at www.wiley.com/college/nise for Cramer's rule.

Complex Circuits via Nodal Analysis

Often, the easiest way to find the transfer function is to use nodal analysis rather than mesh analysis. The number of simultaneous differential equations that must be written is equal to the number of nodes whose voltage is unknown. In the previous example we wrote simultaneous mesh equations using Kirchhoff's voltage law. For multiple nodes we use Kirchhoff's current law and sum currents flowing from each node. Again, as a convention, currents flowing from the node are assumed to be positive, and currents flowing into the node are assumed to be negative.

Before progressing to an example, let us first define *admittance*, $Y(s)$, as the reciprocal of impedance, or

$$Y(s) = \frac{1}{Z(s)} = \frac{I(s)}{V(s)} \qquad (2.84)$$

When writing nodal equations, it can be more convenient to represent circuit elements by their admittance. Admittances for the basic electrical components are shown in Table 2.3. Let us look at an example.

(Example 2.11)

Transfer Function—Multiple Nodes

PROBLEM: Find the transfer function, $V_C(s)/V(s)$, for the circuit in Figure 2.6(b). Use nodal analysis.

SOLUTION: For this problem, we sum currents at the nodes rather than sum voltages around the meshes. From Figure 2.6(b) the sum of currents flowing from the nodes marked $V_L(s)$ and $V_C(s)$ are, respectively,

$$\frac{V_L(s) - V(s)}{R_1} + \frac{V_L(s)}{Ls} + \frac{V_L(s) - V_C(s)}{R_2} = 0 \qquad (2.85a)$$

$$CsV_C(s) + \frac{V_C(s) - V_L(s)}{R_2} = 0 \qquad (2.85b)$$

Rearranging and expressing the resistances as conductances,[5] $G_1 = 1/R_1$ and $G_2 = 1/R_2$, we obtain,

$$\left(G_1 + G_2 + \frac{1}{Ls}\right)V_L(s) \qquad - G_2V_C(s) = V(s)G_1 \qquad (2.86a)$$

$$-G_2V_L(s) + (G_2 + Cs)V_C(s) = 0 \qquad (2.86b)$$

Solving for the transfer function, $V_C(s)/V(s)$, yields

$$\frac{V_C(s)}{V(s)} = \frac{\dfrac{G_1G_2}{C}s}{(G_1 + G_2)s^2 + \dfrac{G_1G_2L + C}{LC}s + \dfrac{G_2}{LC}} \qquad (2.87)$$
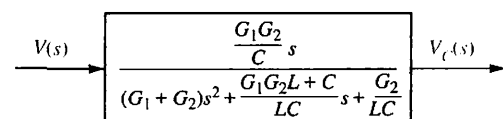
as shown in Figure 2.7.

$V(s)$ → $\dfrac{\dfrac{G_1G_2}{C}s}{(G_1 + G_2)s^2 + \dfrac{G_1G_2L + C}{LC}s + \dfrac{G_2}{LC}}$ → $V_c(s)$

FIGURE 2.7   Block diagram of the network of Figure 2.6

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[5] In general, admittance is complex. The real part is called conductance and the imaginary part is called susceptance. But when we take the reciprocal of resistance to obtain the admittance, a purely real quantity results. The reciprocal of resistance is called conductance.

Another way to write node equations is to replace voltage sources by current sources. A voltage source presents a constant voltage to any load; conversely, a current source delivers a constant current to any load. Practically, a current source can be constructed from a voltage source by placing a large resistance in series with the voltage source. Thus, variations in the load do not appreciably change the current, because the current is determined approximately by the large series resistor and the voltage source. Theoretically, we rely on *Norton's theorem*, which states that a voltage source, $V(s)$, in series with an impedance, $Z_s(s)$, can be replaced by a current source, $I(s) = V(s)/Z_s(s)$, in parallel with $Z_s(s)$.

In order to handle multiple-node electrical networks, we can perform the following steps:

1. Replace passive element values with their admittances.
2. Replace all sources and time variables with their Laplace transform.
3. Replace transformed voltage sources with transformed current sources.
4. Write Kirchhoff's current law at each node.
5. Solve the simultaneous equations for the output.
6. Form the transfer function.

Let us look at an example.

Example 2.12

Transfer Function—Multiple Nodes with Current Sources

PROBLEM: For the network of Figure 2.6, find the transfer function, $V_C(s)/V(s)$, using nodal analysis and a transformed circuit with current sources.

SOLUTION: Convert all impedances to admittances and all voltage sources in series with an impedance to current sources in parallel with an admittance using Norton's theorem.

Redrawing Figure 2.6(b) to reflect the changes, we obtain Figure 2.8, where $G_1 = 1/R_1$, $G_2 = 1/R_2$, and the node voltages—the voltages across the inductor and the capacitor—have been identified as $V_L(s)$ and $V_C(s)$, respectively. Using the general relationship, $I(s) = Y(s)V(s)$, and summing currents at the node $V_L(s)$,

FIGURE 2.8 Transformed network ready for nodal analysis

$$G_1 V_L(s) + \frac{1}{Ls} V_L(s) + G_2[V_L(s) - V_C(s)] = V(s)G_1 \qquad (2.88)$$

Summing the currents at the node $V_C(s)$ yields

$$Cs V_C(s) + G_2[V_C(s) - V_L(s)] = 0 \qquad (2.89)$$

Combining terms, Eqs. (2.88) and (2.89) become simultaneous equations in $V_C(s)$ and $V_L(s)$, which are identical to Eq. (2.86) and lead to the same solution as Eq. (2.87).

An advantage of drawing this circuit lies in the form of Eq. (2.86) and its direct relationship to Figure 2.8, namely
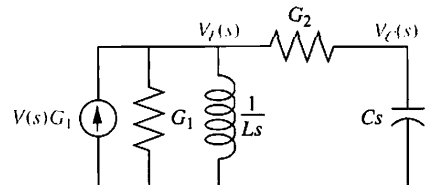
$$\left[\begin{array}{c}\text{Sum of admittances}\\\text{connected to Node 1}\end{array}\right]V_L(s) - \left[\begin{array}{c}\text{Sum of admittances}\\\text{common to the two}\\\text{nodes}\end{array}\right]V_C(s) = \left[\begin{array}{c}\text{Sum of applied}\\\text{currents at Node 1}\end{array}\right]$$

(2.90a)

$$-\left[\begin{array}{c}\text{Sum of admittances}\\\text{common to the two}\\\text{nodes}\end{array}\right]V_L(s) + \left[\begin{array}{c}\text{Sum of admittances}\\\text{connected to Node 2}\end{array}\right]V_C(s) = \left[\begin{array}{c}\text{Sum of applied}\\\text{currents at Node 2}\end{array}\right]$$

(2.90b)

A Problem-Solving Technique

In all of the previous examples, we have seen a repeating pattern in the equations that we can use to our advantage. If we recognize this pattern, we need not write the equations component by component; we can sum impedances around a mesh in the case of mesh equations or sum admittances at a node in the case of node equations. Let us now look at a three-loop electrical network and write the mesh equations by inspection to demonstrate the process.

(Example 2.13)

Mesh Equations via Inspection

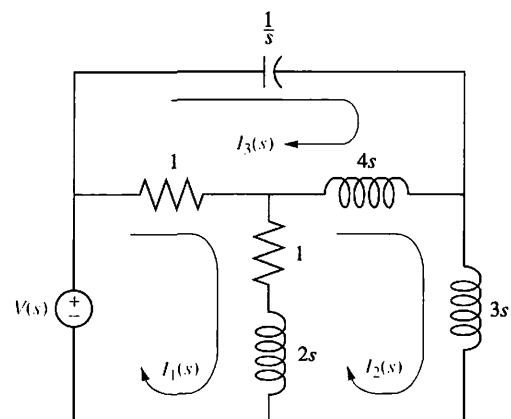**PROBLEM:** Write, but do not solve, the mesh equations for the network shown in Figure 2.9.

FIGURE 2.9 Three-loop electrical network

SOLUTION: Each of the previous problems has illustrated that the mesh equations and nodal equations have a predictable form. We use that knowledge to solve this three-loop problem. The equation for Mesh 1 will have the following form: