3
Block Diagram Representation

3.1 Introduction :

If the given system is complicated, it is very difficult to analyse it as a whole. With the help of transfer function approach, we can find transfer function of each and every element of the complicated system. And by showing connection between the elements, complete system can be splitted into different blocks and can be analysed conveniently.

Basically block diagram is a pictorial representation of the given system. It is a very simple way of representing the given complicated practical system. In block diagram, the interconnection of system components to form a system can be conveniently shown by the blocks arranged in proper sequence. It explains the cause and effect relationship existing between input and output of the system, through the blocks.

To draw the block diagram of a practical system, each element of practical system is represented by a block. The block is called as **functional block**. It means, block explains mathematical operation on the input by the element to produce the corresponding output. The actual mathematical function is indicated by inserting corresponding transfer function of the element inside the block. For a closed loop systems, the function of comparing the different signals is indicated by the **summing point** while a point from which signal is taken for the feedback purpose is indicated by **take off point** in block diagrams. All these summing points, blocks and take off points are then must be connected exactly as per actual elements connected in practical system. The connection between the blocks is shown by lines called as branches of the block diagram. An arrow is associated with each and every branch which indicates the direction of flow of signal along the branch. The signal can travel along the direction of an arrow only. It cannot pass against the direction of an arrow. Hence block diagram is a unilateral property of the system.

In short any block diagram has following five basic elements associated with it :

1) Blocks.

2) Transfer functions of elements shown inside the blocks.

3) Summing points.

4) Take off points.

5) Arrows.

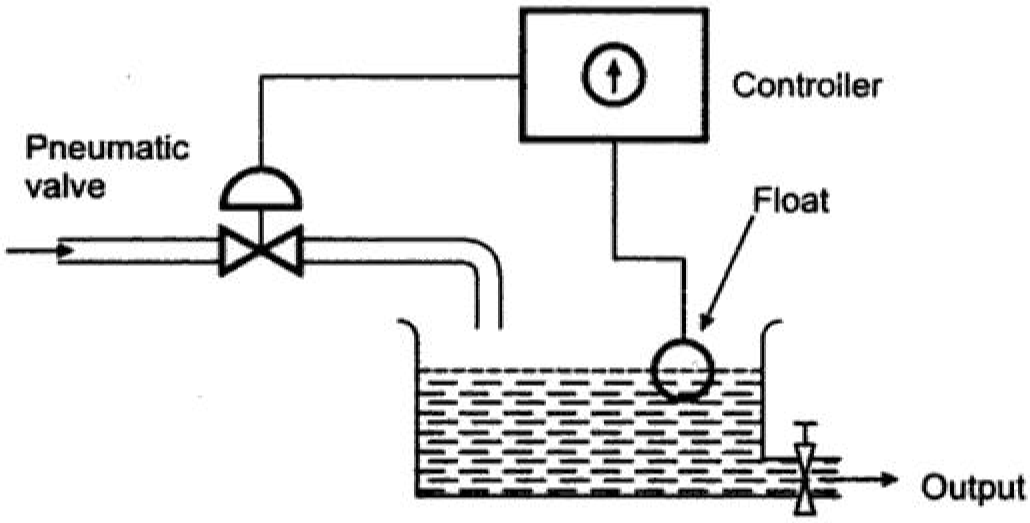


Fig. 3.1

For example consider the liquid level system as shown in Fig. 3.1. So to represent this by block diagram, identify the elements which are
i) Controller (ii) Pneumatic valve (iii) Tank (iv) Float.

Hence indicating them by blocks, the block diagram can be developed as in Fig.3.2.

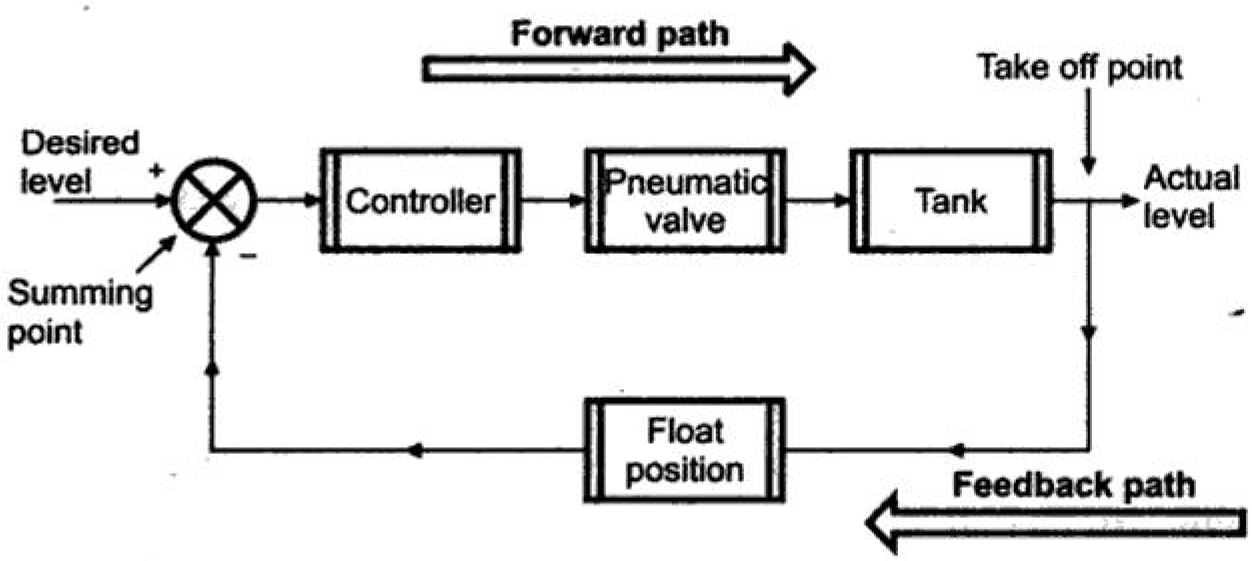


Fig. 3.2 Liquid level control

Consider another example of bottle filling mechanism. When bottle gets filled by the contents upto the required level it should get replaced by an empty bottle. This system can be made closed loop and hence can be shown as in Fig. 3.3 (See Fig. on next page)

In the system shown, conveyor belt is driven by the controller as well as valve position is also controlled by the controller.

When empty bottle comes at the specific position, weight sensor senses the weight and gives signal to controller. Controller stops conveyor movement and opens the valve so bottle starts getting filled. When required level is achieved, again weight sensor sensing the proper weight sends a signal to controller which sends signals to start movement of belt and also closing the valve position with proper time delay till next empty bottle comes at the proper position.

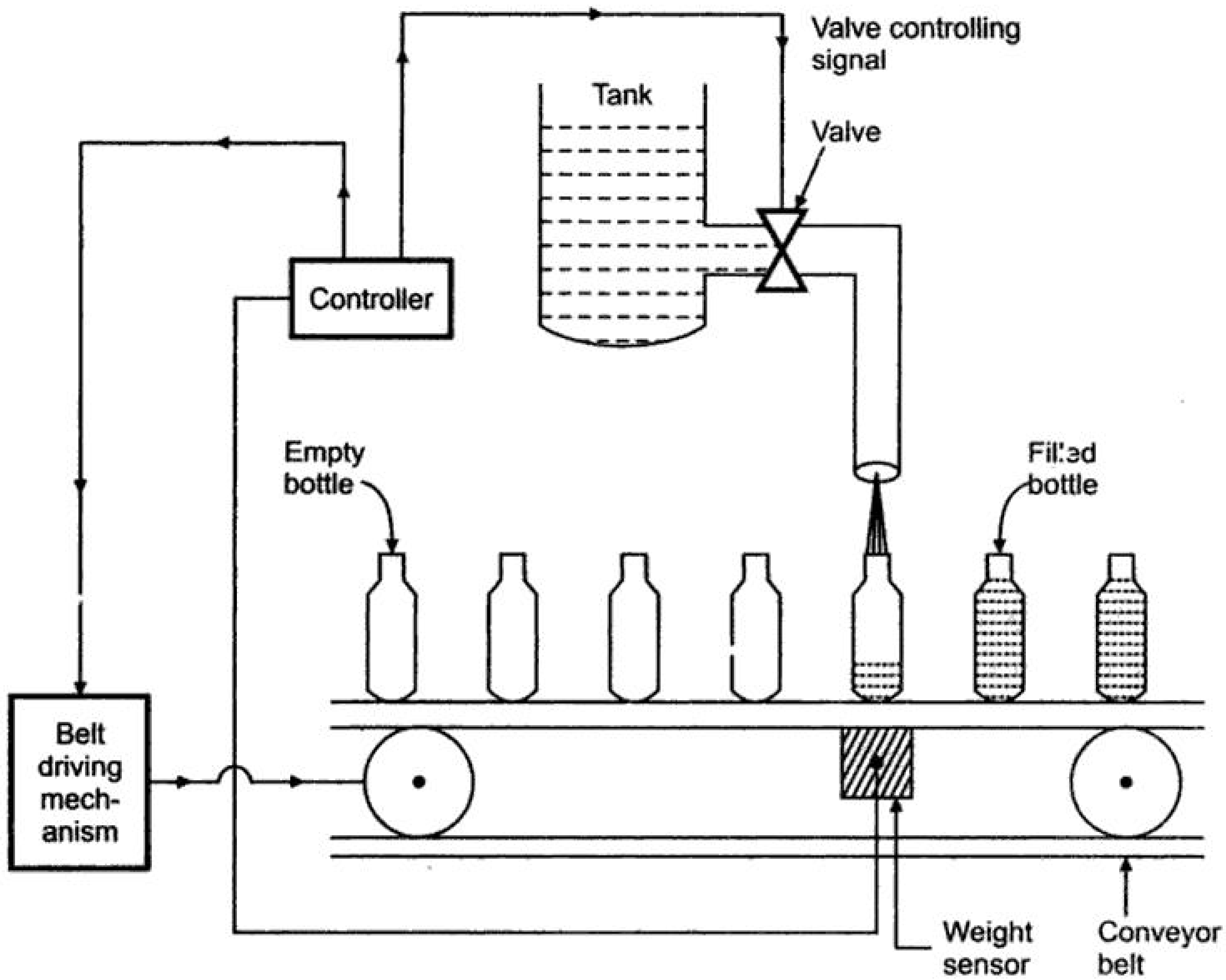


Fig. 3.3

This system can be represented as a block diagram as shown in Fig. 3.4

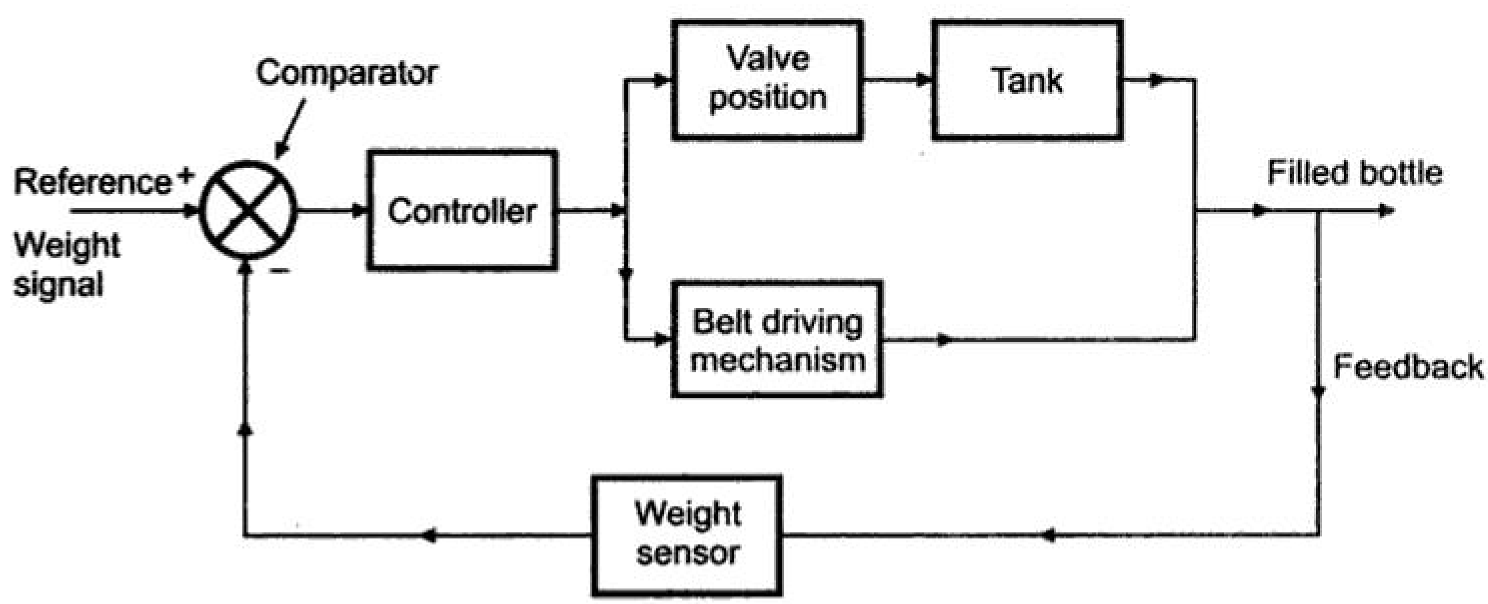


Fig. 3.4 Automatic bottle filling mechanism

3.1.1 Advantages of Block Diagram :

1) Very simple to construct the block diagram for complicated systems.

2) The function of individual element can be visualised from block diagram.
3) Individual as well as overall performance of the system can be studied by using transfer functions shown in the block diagram.
4) Overall closed loop T.F. can be easily calculated by using block diagram reduction rules.

3.1.2 Disadvantages :

1) Block diagram does not include any information about the physical construction of the system.
2) Source of energy is generally not shown in the block diagram. So number of different block diagrams can be drawn depending upon the point of view of analysis. So block diagram for given system is not unique.

3.2 Simple or Canonical Form of Closed Loop System :

A block diagram in which, forward path contains only one block, feedback path contains only one block, one summing point and one take off point represents **simple** or **canonical** form of a closed loop system. This can be achieved by using block diagram reduction rules without disturbing output of the system. This form is very useful as its closed loop transfer function can be easily calculated by using standard result. This result is derived in this section .

The simple form can be shown as in Fig. 3.5.

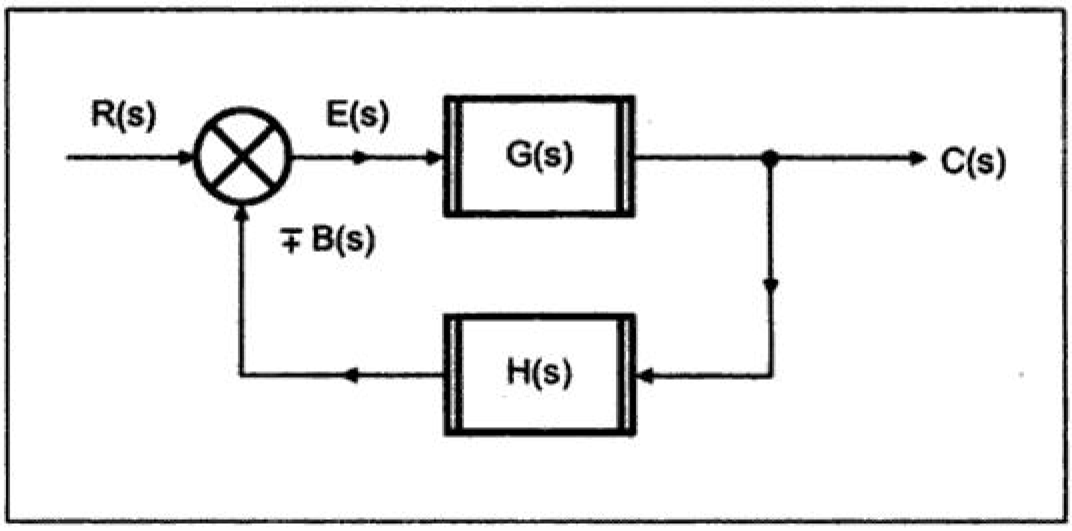


Fig. 3.5

where , R(s) → Laplace of reference input r(t)
C(s) → Laplace of controlled output c(t)
E(s) → Laplace of error signal e(t)
B(s) → Laplace of feedback signal b(t)
G(s) → Equivalent forward path transfer function .
H(s) → Equivalent feedback path transfer function .

G(s) and H(s) can be obtained by reducing complicated block diagram by using block diagram reduction rules.

3.2.1 Derivation of T.F. of Simple Closed Loop System :

Referring to Fig. 3.5, we can write following equations as,

$$E(s) = R(s) \pm B(s) \qquad \text{... (1)}$$

$$B(s) = C(s)\,H(s) \qquad \text{... (2)}$$

$$C(s) = E(s)\,G(s) \qquad \text{... (3)}$$

$$B(s) = C(s)\,H(s) \text{ and substituting in equation (1)}$$

$$E(s) = R(s) \pm C(s)\,H(s)$$

$$E(s) = \frac{C(s)}{G(s)}$$

$$\frac{C(s)}{G(s)} = R(s) \pm C(s)\,H(s)$$

$$C(s) = R(s)\,G(s) \pm C(s)\,G(s)\,H(s)$$

$$\therefore \quad C(s)\,[1 \pm G(s)\,H(s)] = R(s)\,G(s)$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm G(s)\,H(s)}$$

+ sign → negative feedback

– sign → positive feedback.

This can be represented as in Fig. 3.6

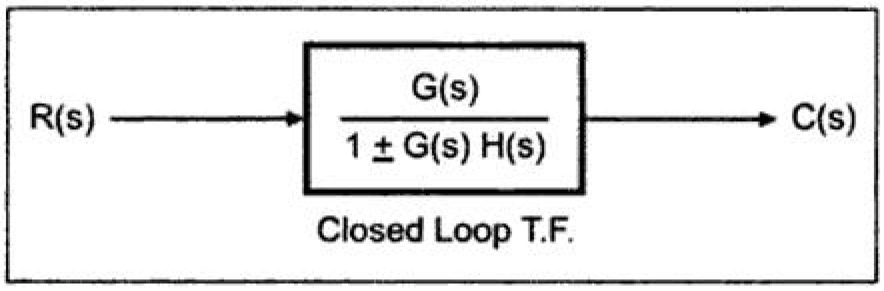


Fig. 3.6

This can be used as a standard result to eliminate such simple loop in a complicated system reduction procedure.

3.3 Rules for Block Diagram Reduction :

Any complicated system if brought into its simple form as shown in Fig. 3.5, its T.F. can be calculated by using the result derived eariler. To bring it into simple form it is necessary to reduce the block diagram but using proper logic such that output of that system and the value of any feedback signal should not get disturbed. This can be achieved by using following mathematical rules while block diagram reduction.

Rule 1 : Associative law : Consider two summing points as shown in Fig. 3.7.

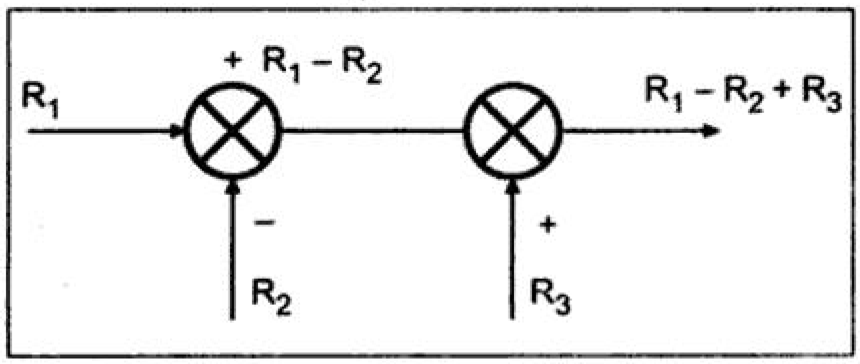


Fig. 3.7

Now change the position of two summing points. Output remains same.

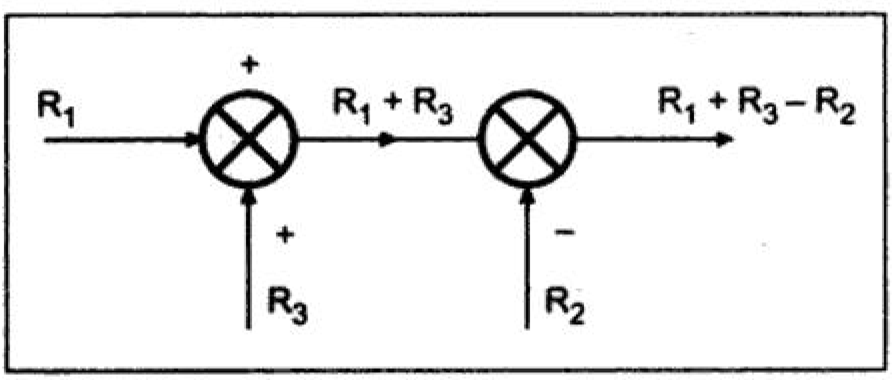


Fig. 3.8

So associative law holds good for summing points which are directly connected to each other (i.e. there is no intermediate block between two summing points).

Consider summing points with a block in between as shown in Fig. 3.9.

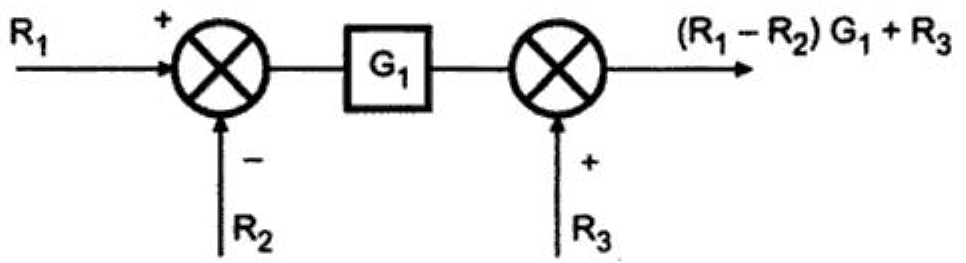


Fig. 3.9

Now interchange two summing points.

So the output does not remain same. So associative law is applicable to summing points which are directly connected to each other.

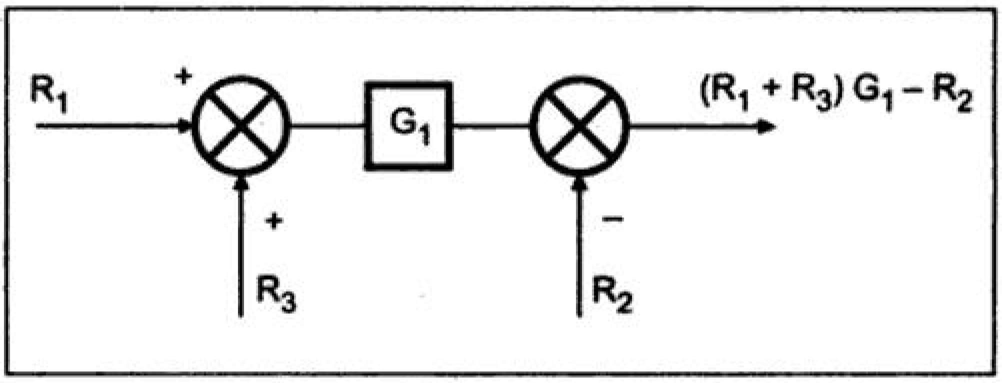


Fig. 3.10

Rule 2 : For blocks in series :

The transfer functions of the blocks which are connected in series get multiplied with each other.

Consider system as shown in Fig. 3.11

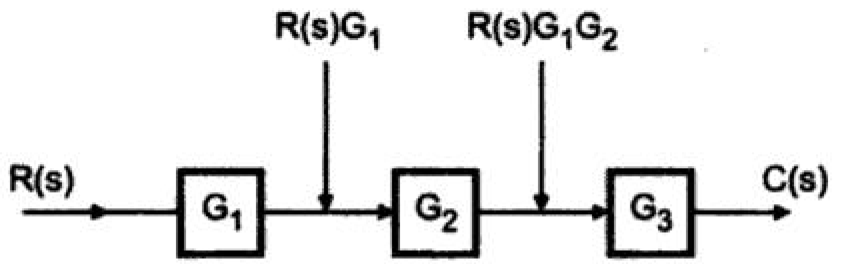


Fig. 3.11

$$C(s) = R(s)\,[G_1 G_2 G_3]$$

So instead of three different blocks, only one block with T.F. $[G_1 G_2 G_3]$ can be shown in system (Fig. 3.12)

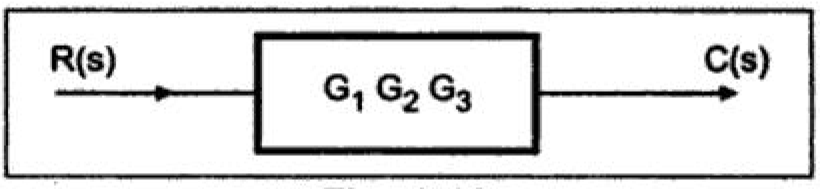


Fig. 3.12

Output in both cases is same.

It is important to note that if there is take off or summing point in between the blocks, the blocks cannot be said to be in series.

Consider the combination of the blocks as shown the Fig. 3.13

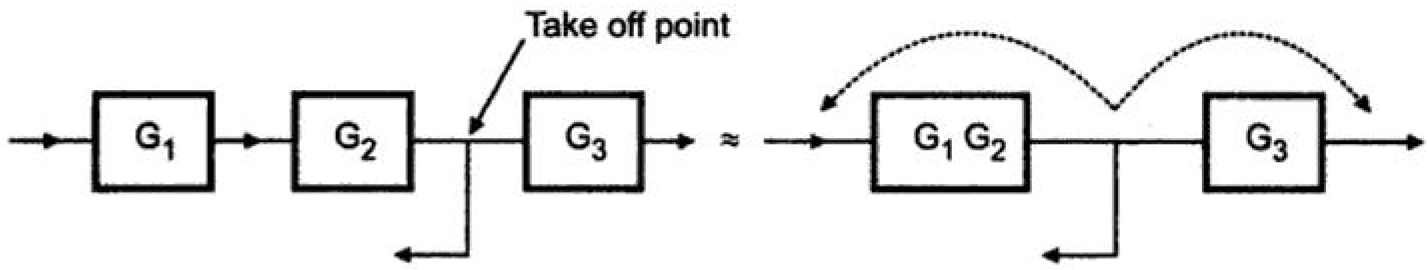


Fig. 3.13

In this combination G_1G_2 are in series and can be combined as G_1G_2 but G_3 is now not in series with G_1G_2 as there is take off point in between. To call G_3 to be in series with G_1G_2 it is necessary to shift the take off point before G_1G_2 or after G_3. The rules for such shifting are discussed later.

Rule 3 : For blocks in parallel. :

The transfer functions of the blocks which are connected in parallel get added algebraically (considering the sign).

Consider system as shown in Fig. 3.14.

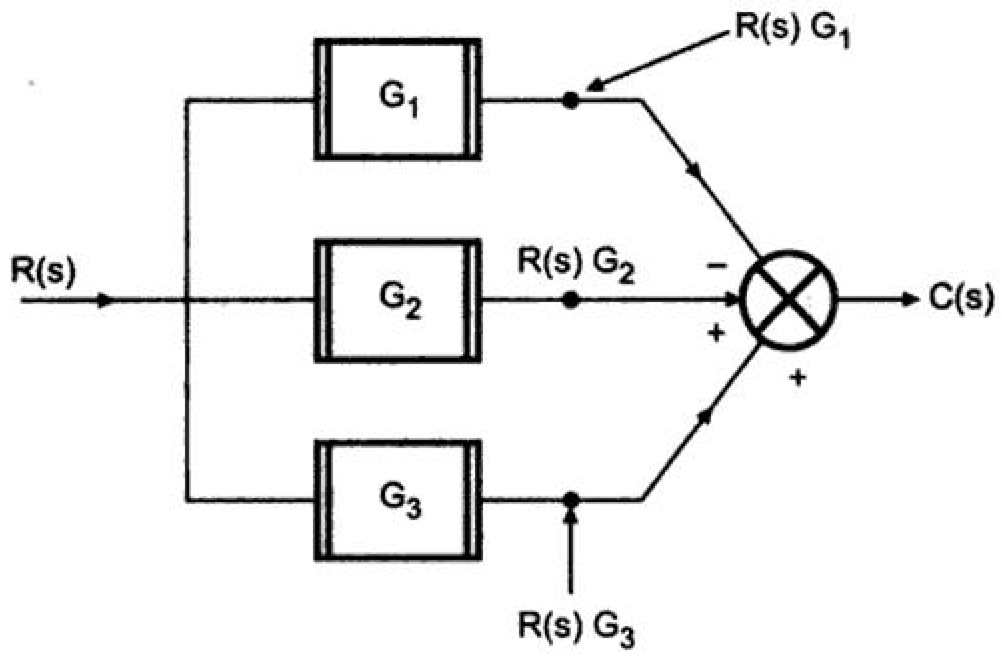


Fig. 3.14

$$C(s) = -R(s)\,G_1 + R(s)\,G_2 + R(s)\,G_3$$
$$= R(s)\,[G_2 + G_3 - G_1]$$

Now replace three block with only one block with T.F. $G_2 + G_3 - G_1$ (Fig. 3.15)

$$C(s) = R(s)\,[G_2 + G_3 - G_1]$$

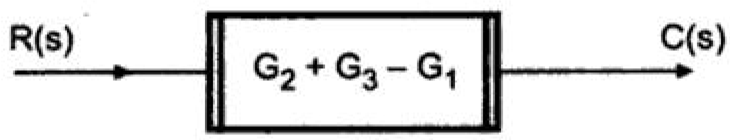


Fig. 3.15

Output is same. So blocks which are in parallel get added algebraically.

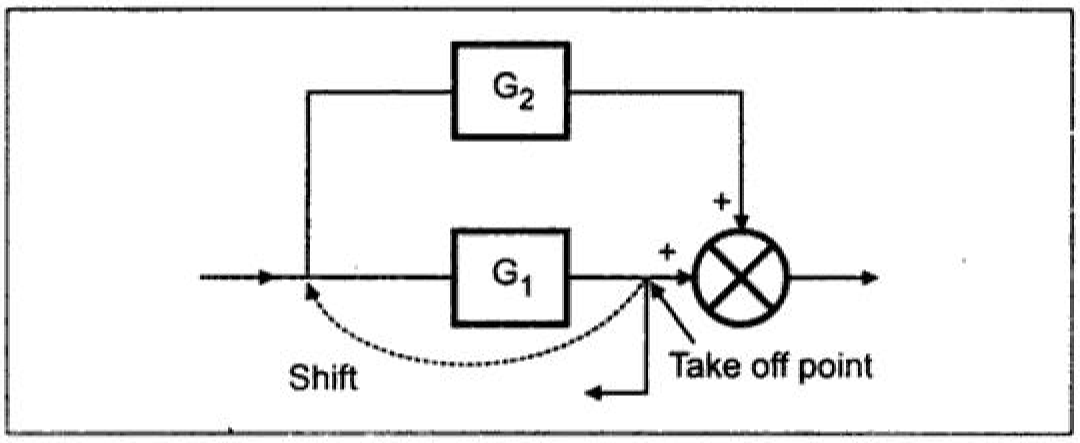
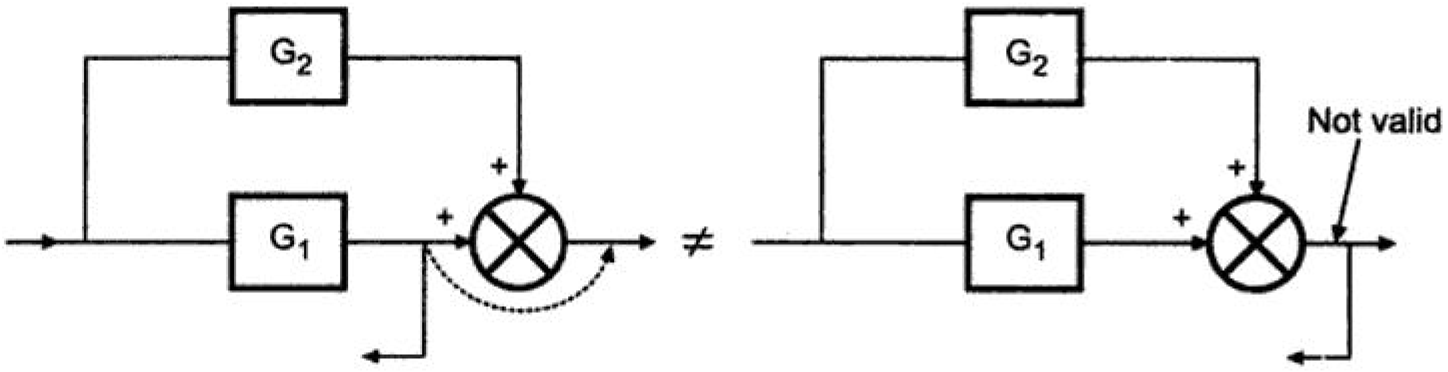


Fig. 3.16

The students may make mistakes while identifying blocks in parallel in following cases. If there exists a takeoff point as shown in the Fig. 3.16 along with blocks G_1, G_2 which appear to be in parallel.

But unless and untill this takeoff point is shifted before the block, blocks can not be said to be in parallel. Shifting of takeoff point is discussed next. Secondly the shifting a take off point after a summing point needs some adjustment to keep out put same. In above case the take off point can not be shown after summing point without any alteration. This type of shifting is discussed as critical rules later as such shifting makes the block diagram complicated and should be avoided as far as possible.



Avoid such shifting as far as possible
Fig. 3.17(a)

Without any alteration such shifting is invalid
Fig. 3.17(b)

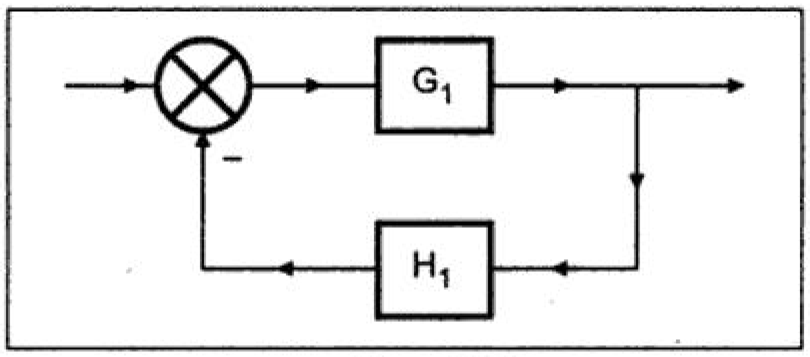


Fig. 3.18

Similarly consider a configuraion as shown in the Fig. 3.18.

This combination is not the parallel combination of G_1 and H_1. **For a parallel combination the direction of signals through the blocks in parallel must be same.**

In this case direction of signal through G_1 and H_1 is opposite. Such a combination is called as **minor feedback** loop and reduction rule for this is discussed later.

Rule 4 : Shifting a summing point behind the block :

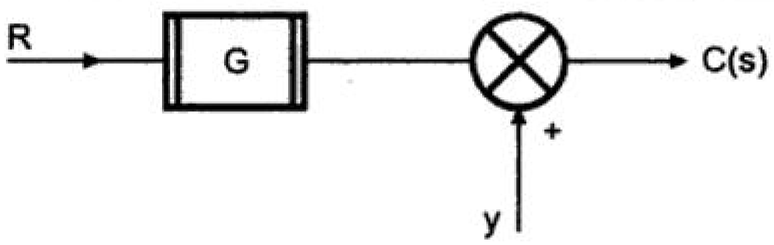


Fig. 3.19

$$C(s) = RG + y$$

Now we have to shift summing point behind the block.

Now output must remain same.

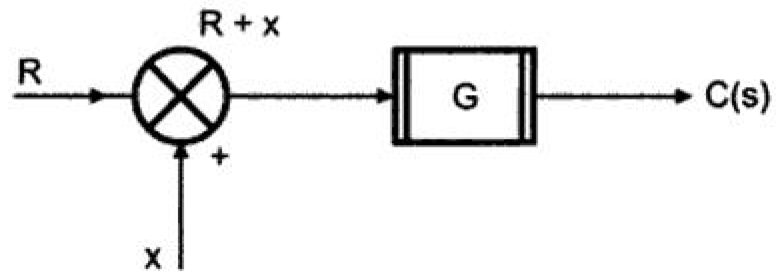


Fig. 3.20

$\therefore \quad (R + x)G = C(s)$

$RG + xG = RG + y$

$\therefore \quad xG = y$

$\therefore \quad x = \frac{y}{G}$ so signal y must be multiplied with $\frac{1}{G}$ to keep output same.

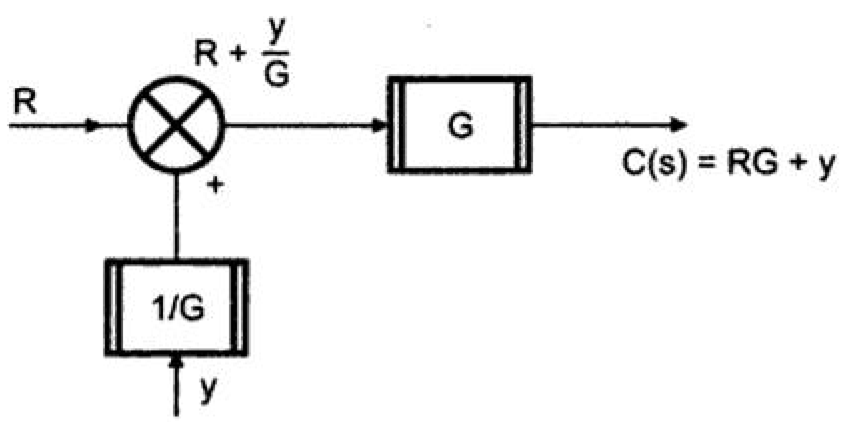


Fig. 3.21

Thus while shifting a summing point behind the block i.e. before the block, add a block having T.F. as reciprocal of the T.F. of the block before which summing point is to be shifted, in series with all the signals at that summing point.

Rule 5 : Shifting a summing point beyond the block.

Consider the combination shown in the Fig. 3.22.

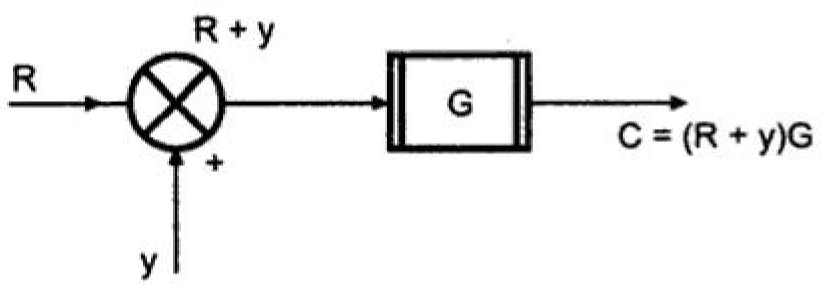


Fig. 3.22

Now to shift summing point after block keeping output same, consider the shifted summing point without any change.

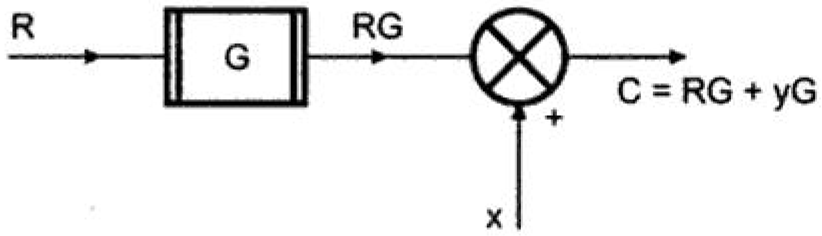


Fig. 3.23

$$\therefore \quad RG + x = RG + yG$$

$$\therefore \quad x = yG$$

i.e. signal y must get multiplied with T.F. of block beyond which summing point is to be shifted.

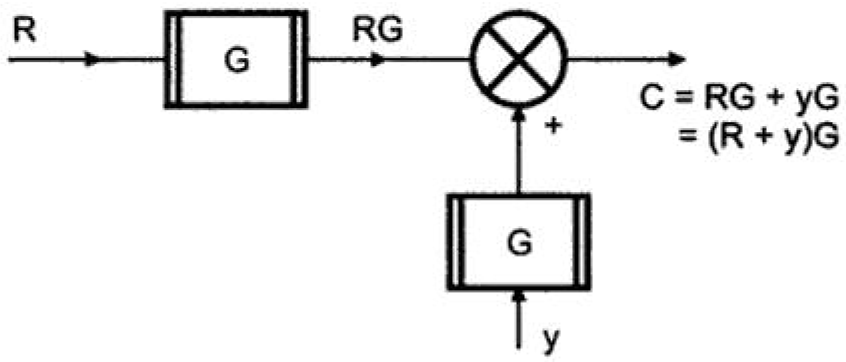


Fig. 3.24

Thus while shifting a summing point after a block, add a block having T.F. same as that of block after which summing point is to be shifted, in series with all the signals at that summing point.

Rule 6 : Shifting a take off point behind the blocks :

Consider the combination shown in the Fig. 3.25.

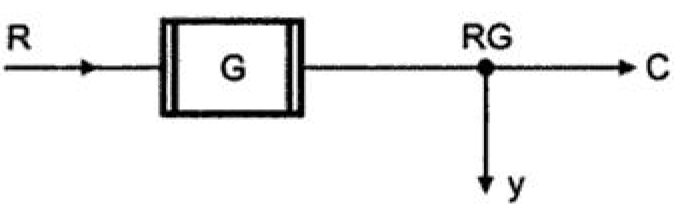


Fig. 3.25

$$C = RG$$

$$y = RG$$

To shift take off point behind block value of signal taking off must remain same.

Though shifting of take off point without any change does not affect output directly, the value of feedback signal which is changed affects the output indirectly which must be kept same. But without any change it is just R as shown in Fig. 3.26.

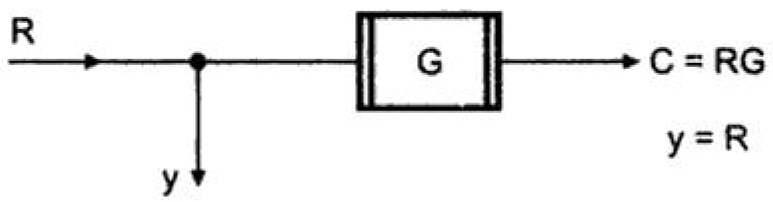


Fig. 3.26

But it must be equal to RG. So a block with T.F. G must be introduced i.e. signal taking off after the block must be multiplied with T.F. of that block while shifting behind the block.

This while shifting a take off point behind the block, add a block having T.F. same as that of the block behind which take off point is to be shifted, in series with all the signals taking off from that take off point.

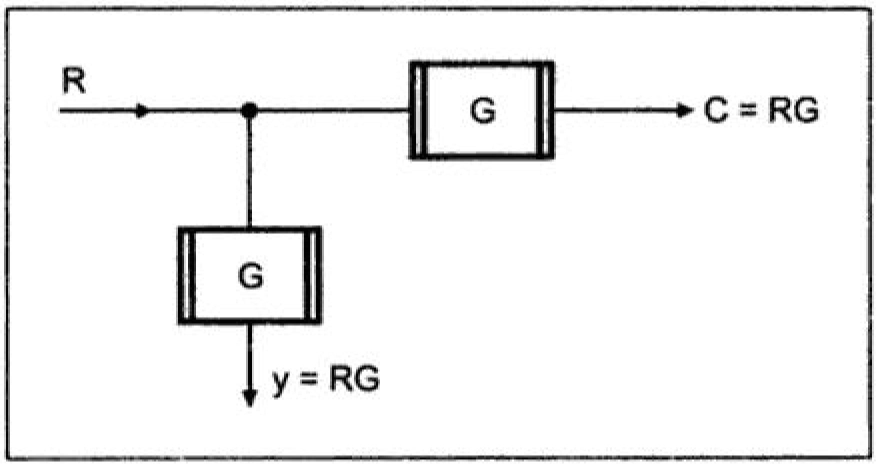
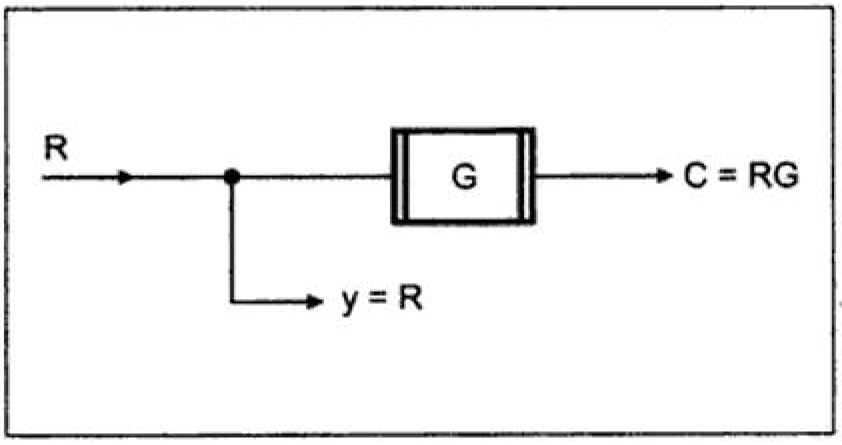


Fig. 3.27

Rule 7 : Shifting a take-off point beyond the block :

Consider the combination shown in the Fig. 3.28.



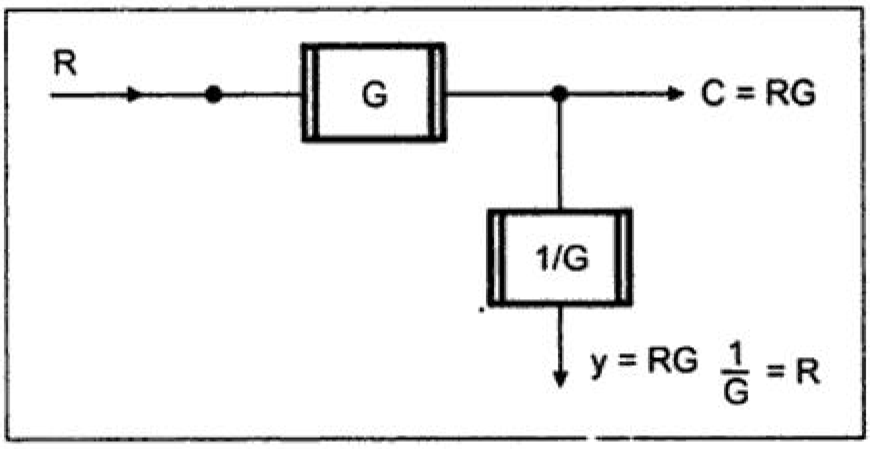


Fig. 3.28 Fig. 3.29

To shift take off point beyond the block, value of 'y' must remain same. To keep value of 'y' constant it must be multiplied by '1/G'. While shifting a take off point beyond the block, add a block in series with all the signals which are taking off from that point, having T.F. as reciprocal of the T.F. of the block beyond which take off point is to be shifted.

Rule 8 : Removing minor feedback loop :

This includes the removal of internal simple forms of the loops by using standard result derived earlier in section 3.2.

After eliminating such a minor loop if summing point carries only one signal input and one signal output, it should be removed from the block diagram to avoid further confusion.

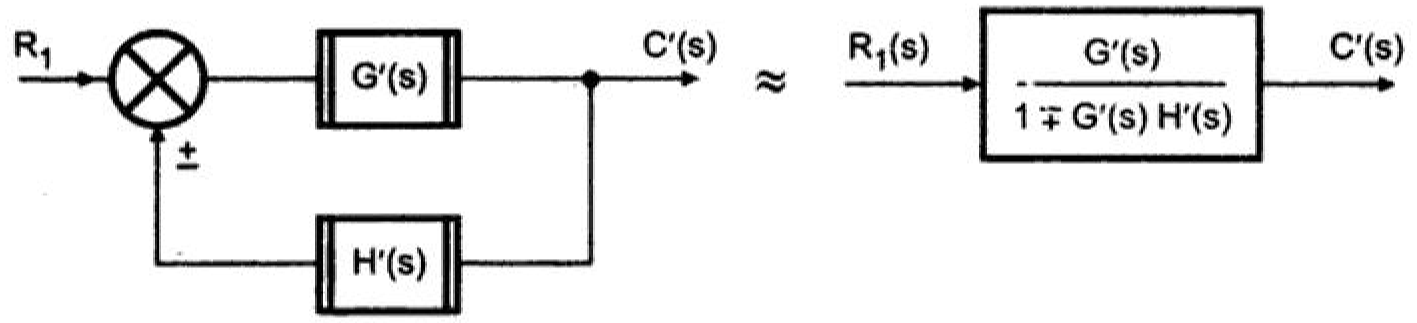


Fig. 3.30

Rule 9 : For multiple input system use superposition theorem :

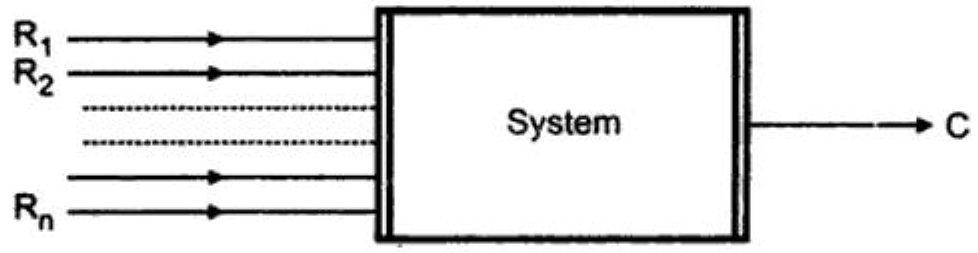


Fig. 3.31

Consider only one input at a time treating all other as zero.

Consider R_1, $R_2 = R_3 = \ldots\ldots\ldots\ldots R_n = 0$ and find output C_1,

Then consider R_2, $R_1 = R_3 = \ldots\ldots\ldots\ldots R_n = 0$ and find output C_2

At the end when all inputs are covered take algebraic sum of all the outputs.

Total output $C = C_1 + C_2 + \ldots\ldots\ldots C_n$

Same logic can be extended to find the outputs if system is multiple input multiple output type. Separate ratio of each output with each input is to be calculated, assuming all other input and outputs zero. Then such components of outputs can be added to get resultant outputs of the system. In very few cases, it is not possible to reduce the block diagram to its simple form by use of above discussed nine rules. In such case there is a requirement to shift a summing point before or after a takeoff point to solve the problem. These rules are discussed below but reader should avoid to use these rules unless and until it is the requirement of the problem. Use of these rules in simple problems may complicate the block diagram. The use of these rules in actual problem solving is illustrated in solved problem no. 21.

3.3.1 Critical Rules :

Rule 10 : Shifting take off point after a summing point. Consider a situation as shown in Fig. 3.32.

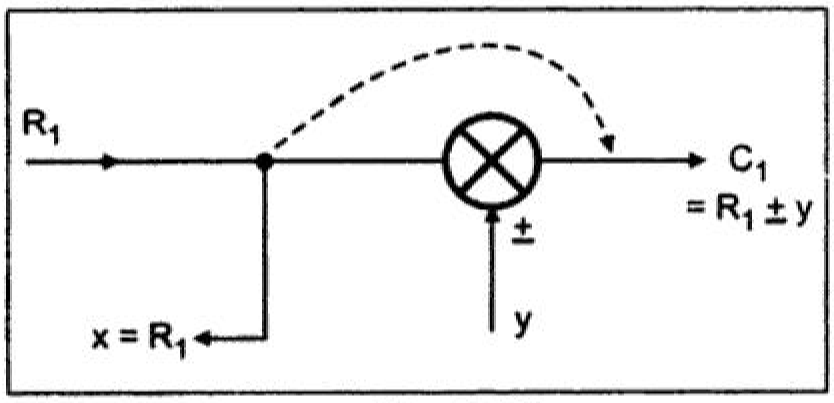


Fig. 3.32

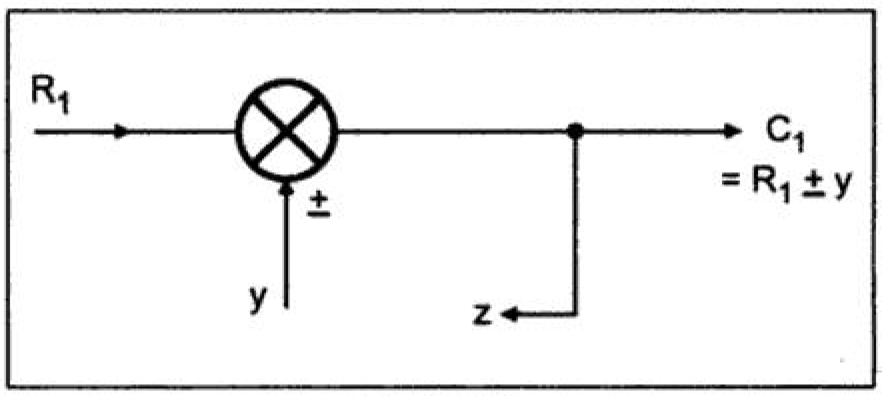


Fig. 3.33

Now after shifting the take off point , let signal taking off be 'z' as shown in Fig. 3.33.

Now $$z = R_1 \pm y$$

But we want feedback signal as $x = R_1$ only.

So signal 'y' must be inverted and added to C_1 to keep feedback signal value same. And to add the signal, summing point must be introduced in series with take off signal. So modified configuration becomes as shown in Fig. 3.34.

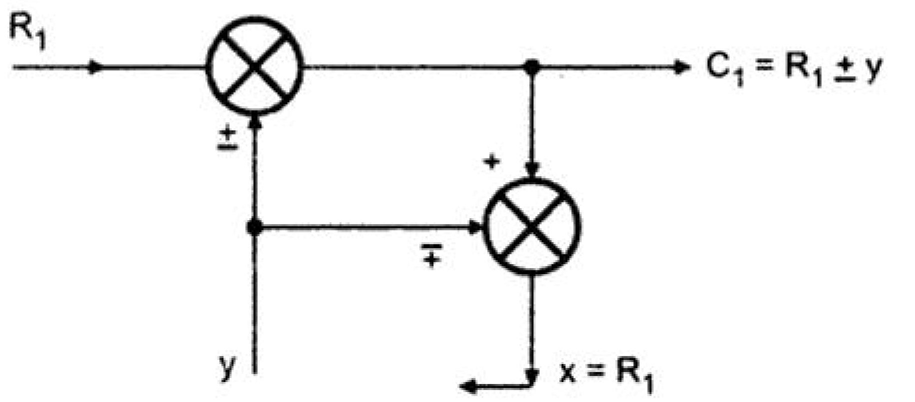


Fig. 3.34

Rule 11 : Shifting take off point before a summing point :

Consider a situation as shown in Fig. 3.35.

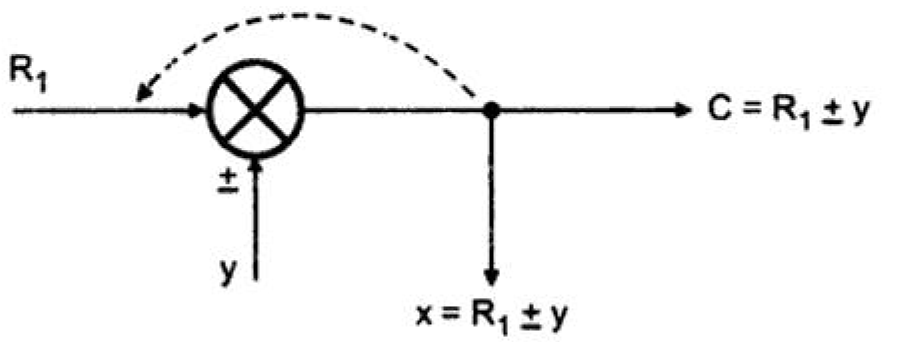


Fig. 3.35

. Now after shifting the take off point, let signal taking off be 'z' as shown in Fig. 3.36.

Now $z = R_1$ only because nothing is changed.

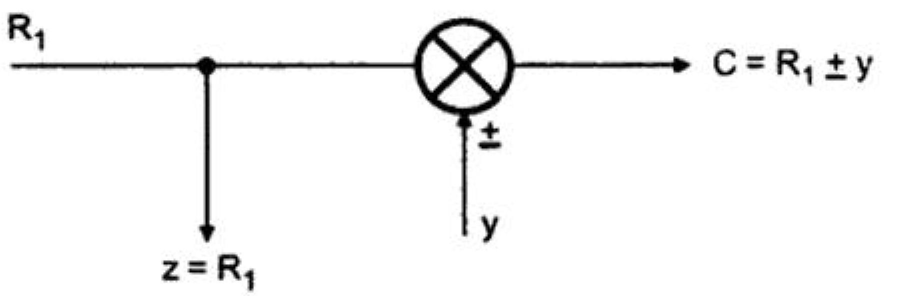


Fig. 3.36

But we want feedback signal x again which is $R_1 \pm y$. Hence to z, signal 'y' must be added with same sign as it is present at summing point which can be achieved by using summing point in series with take off signal as shown in Fig. 3.37.

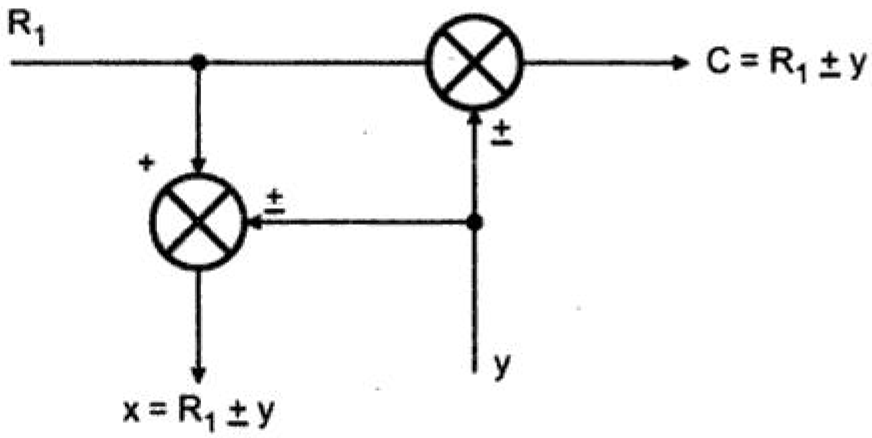


Fig. 3.37

Thus it can be noticed that shifting of take off point before or after a summing point adds an additional summing point in the block diagram and this complicates the block diagram. No doubt, in same rare cases, it is not possible to reduce the block diagram without such shifting of take off point before or after a summing point. Apart from such cases, students should not use such shifting which will complicate the simple block diagrams.

3.3.2 Procedure to solve block diagram reduction problems :

Step 1 : Reduce the blocks connected in series.

Step 2 : Reduce the blocks connected in parallel.

Step 3 : Reduce the minor internal feedback loops.

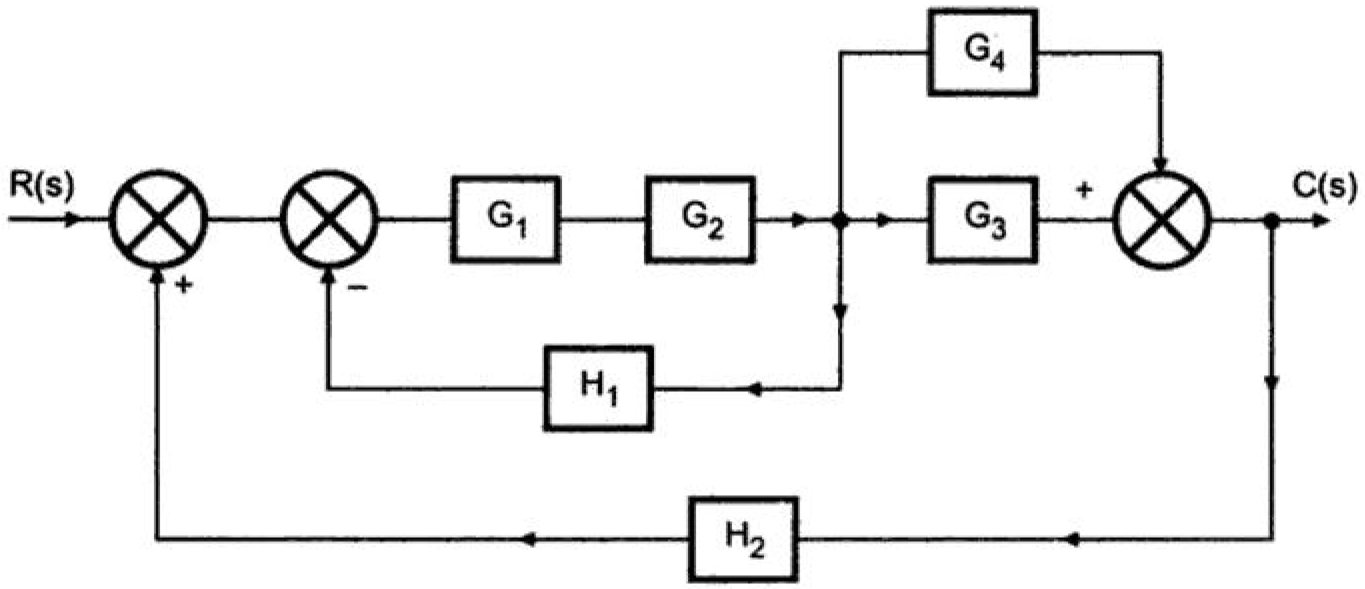
Step 4 : As far as possible **try to shift take off point towards right** and **summing points to the left**. Unless and until it is the requirement of problem do not use rule 10 and 11.

Step 5 : Repeat step 1 to 4 till simple form is obtained.

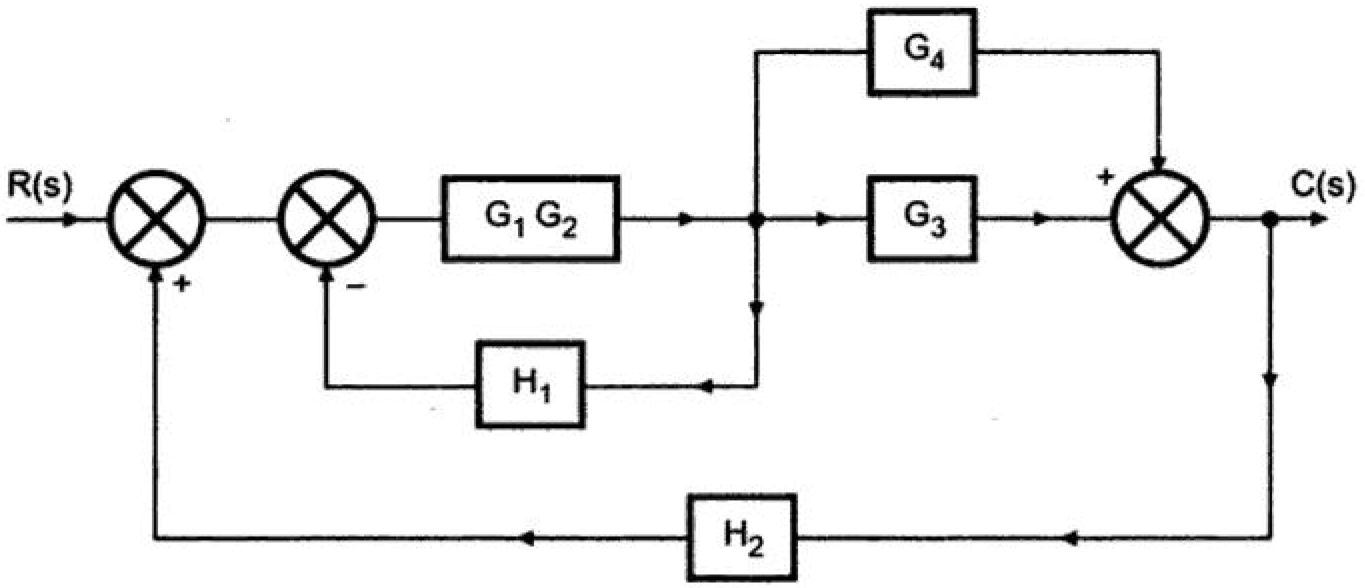
Step 6 : Using standard T.F. of simple closed loop system obtain the closed loop T.F. $\frac{C(s)}{R(s)}$ of the overall system.

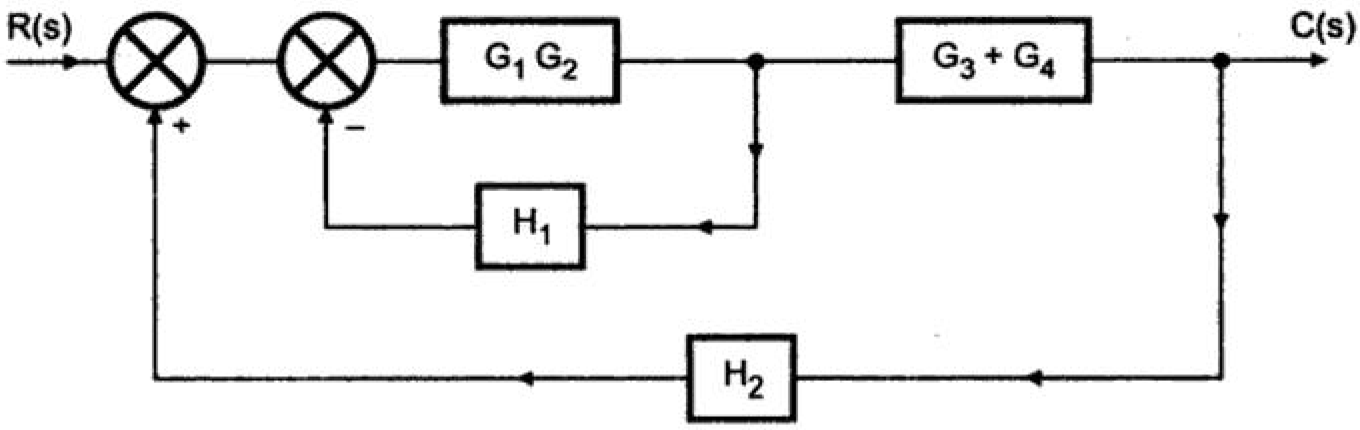
Solved Problems on Block Diagram Reduction

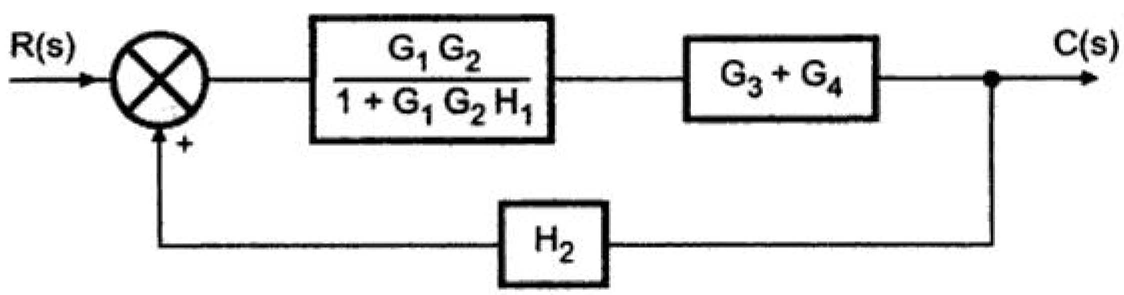
Ex. 3.1 *Reduce the given block diagram to its canonical (simple) form and hence obtain the equivalent transfer function* $\frac{C(s)}{R(s)}$.

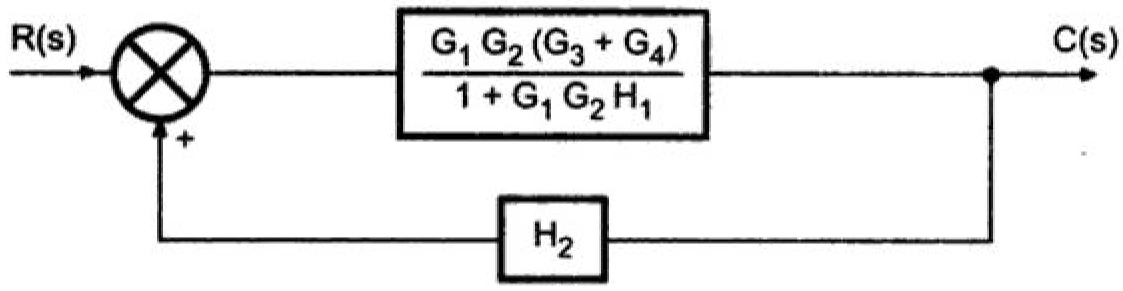

R(s)
C(s)
G4
G1
G2
G3
H1
H2
+
–
+

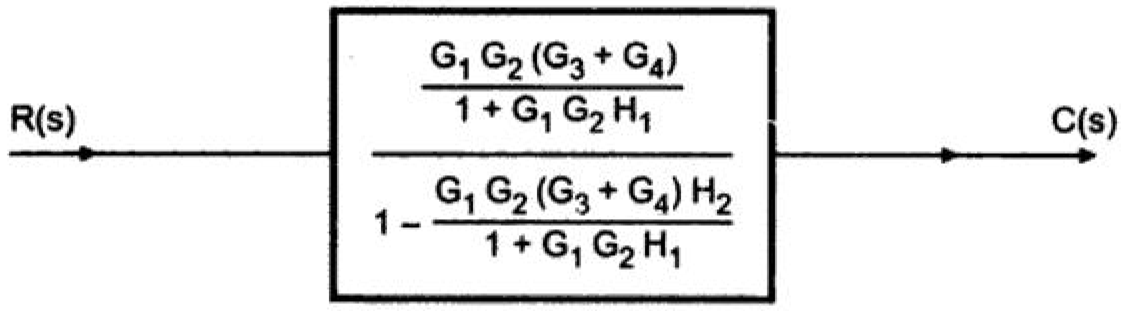
Sol. :


R(s)
C(s)
G4
G1 G2
G3
H1
H2
+
–
+


R(s)
C(s)
G1 G2
G3 + G4
H1
H2
+
–

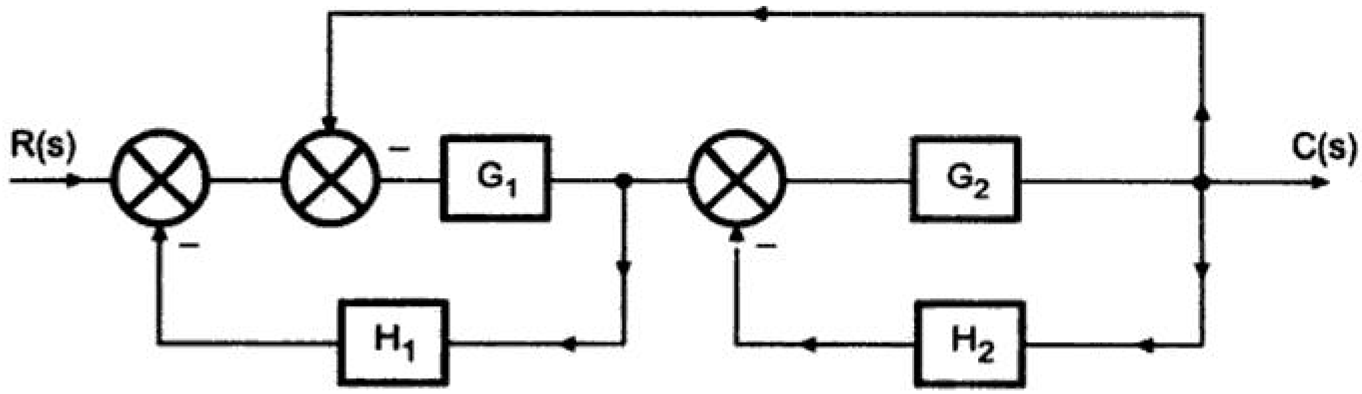

R(s)
C(s)
G1 G2
1 + G1 G2 H1
G3 + G4
H2
+



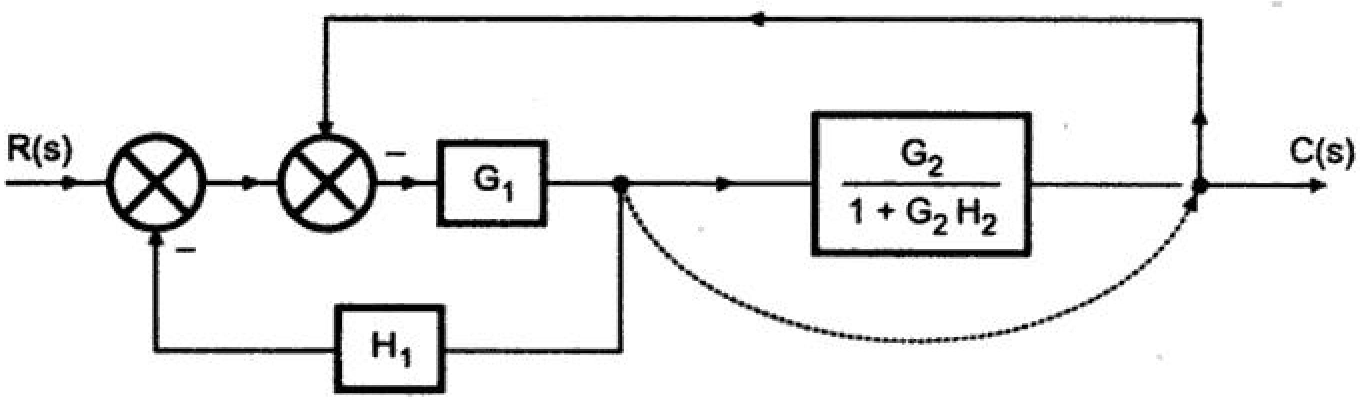


$$\therefore \quad \frac{C(s)}{R(s)} = \frac{G_1 \; G_2 (G_3 + G_4)}{1 + G_1 G_2 H_1 - G_1 G_2 (G_3 + G_4) H_2}$$

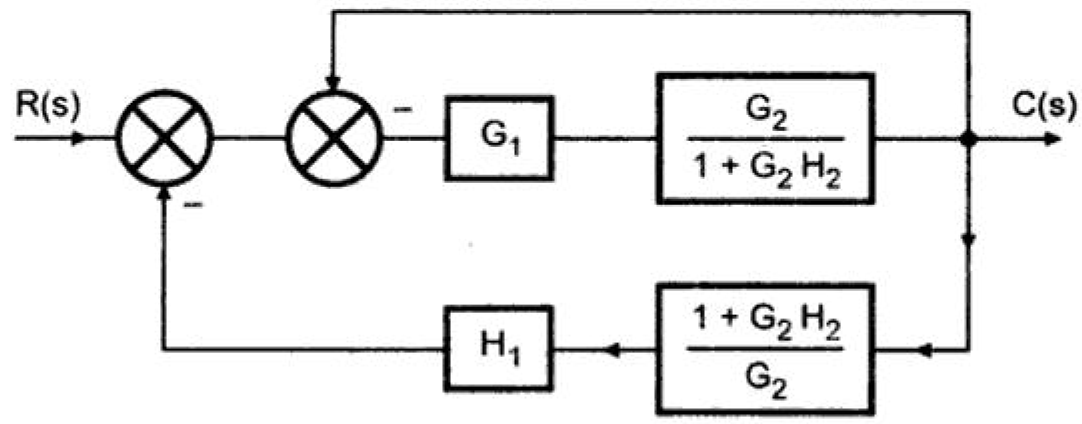
Ex. 3.2

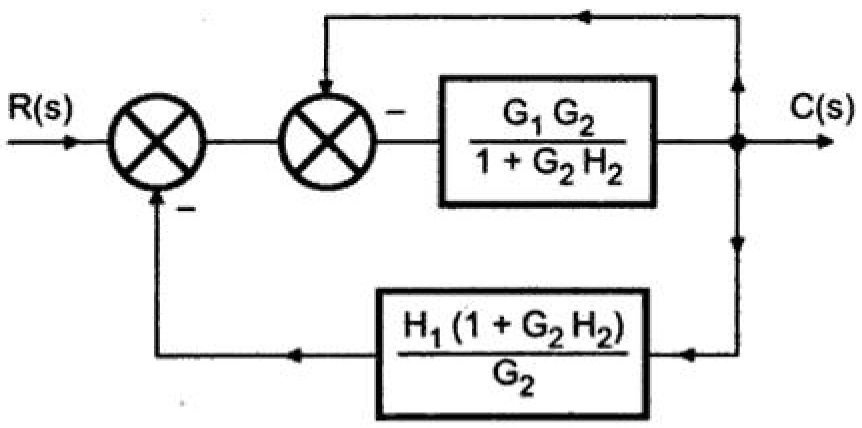


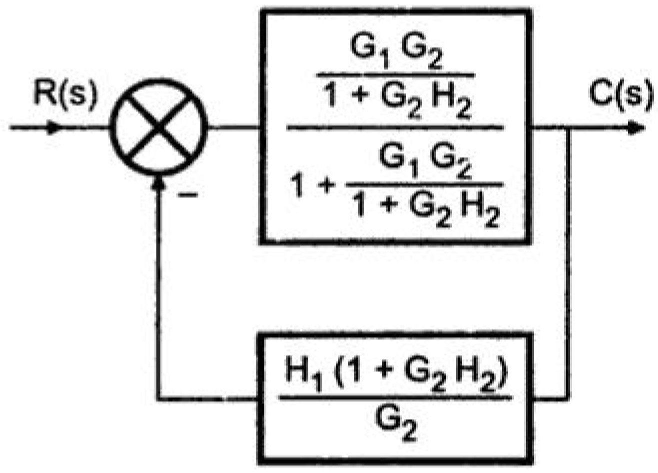
Sol. : No blocks are connected in series or parallel. Blocks having transfer functions G_2 and H_2 form minor feedback loop so eliminating that loop we get,

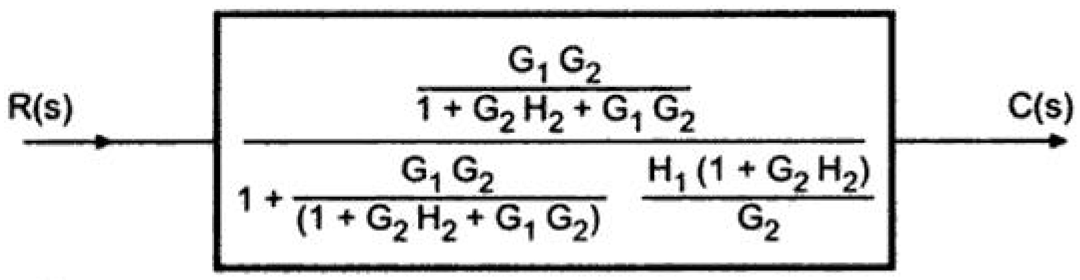


Always try to shift take off point towards right i.e. output and summing point towards left i.e. input.



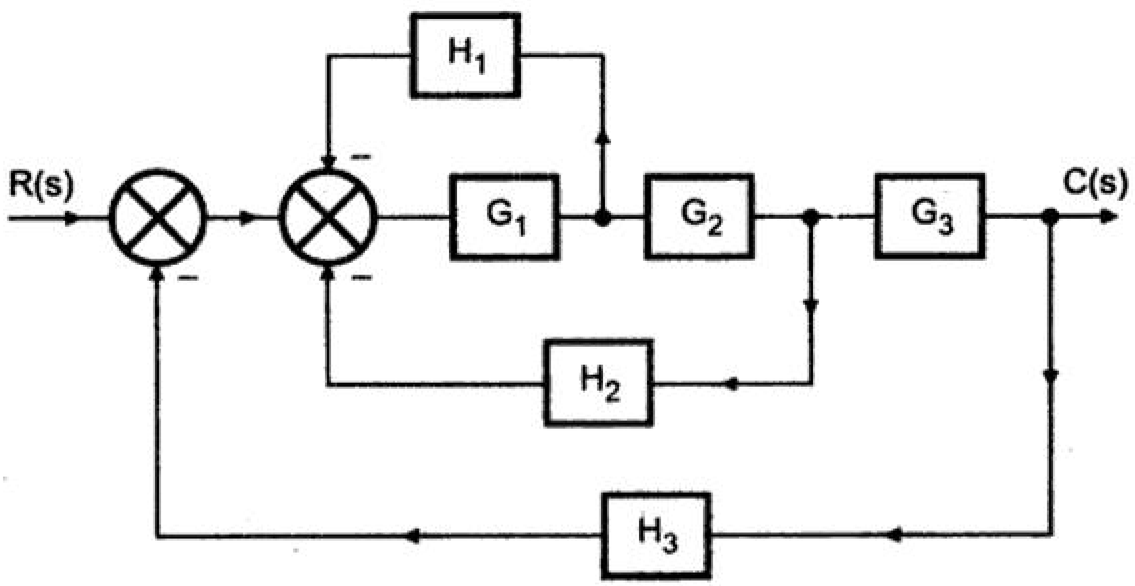




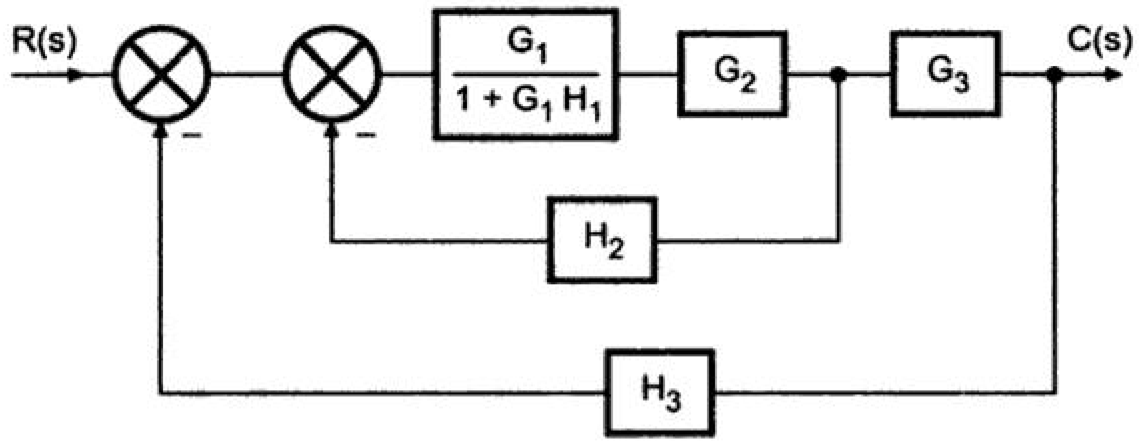


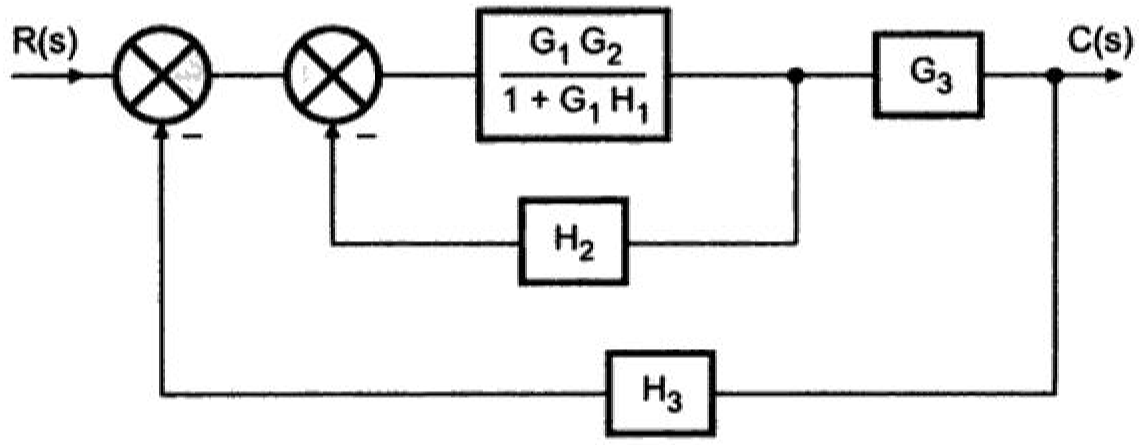
Simplifying $$\frac{C(s)}{R(s)} = \frac{G_1 G_2}{1 + G_1 G_2 + G_2 H_2 + G_1 H_1 + G_1 G_2 H_1 H_2}$$

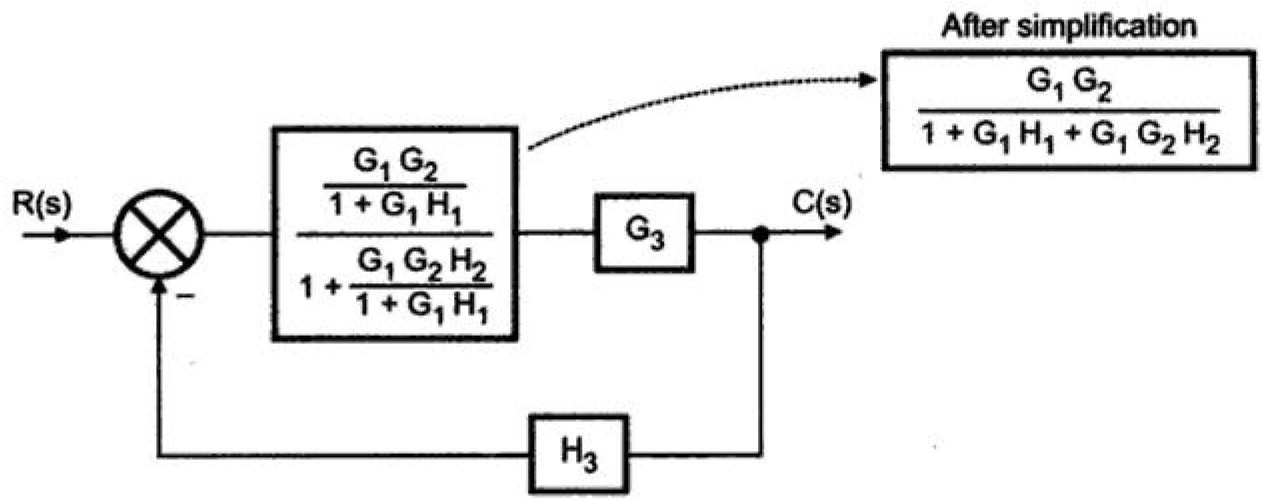
Ex. 3.3

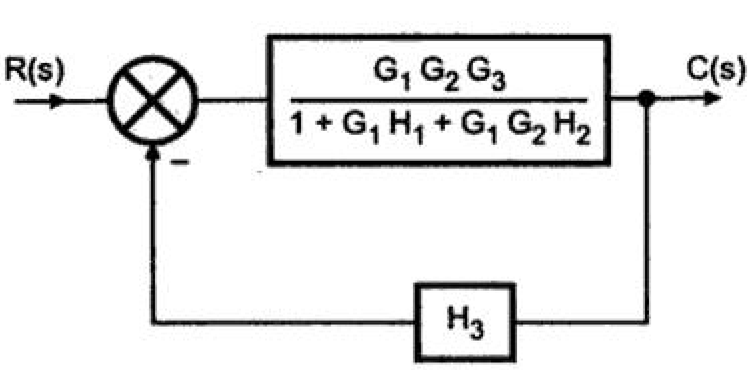


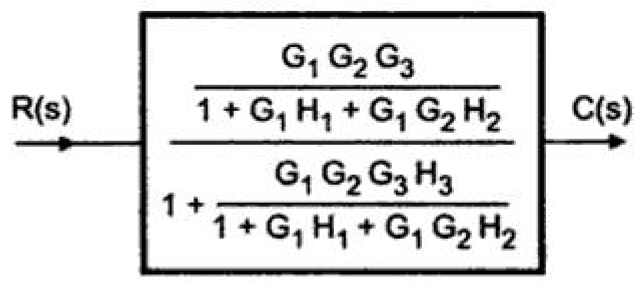
Sol. : No blocks are connected in series or parallel so reducing minor feedback loop formed by blocks with transfer function G_1 and H_1.







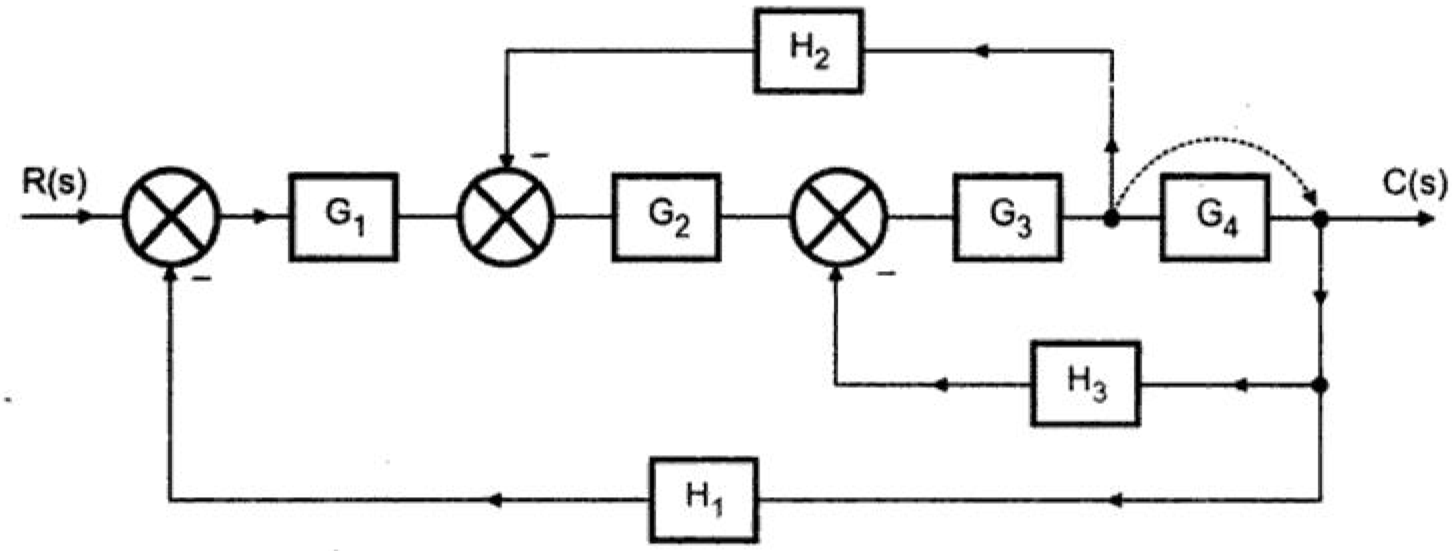




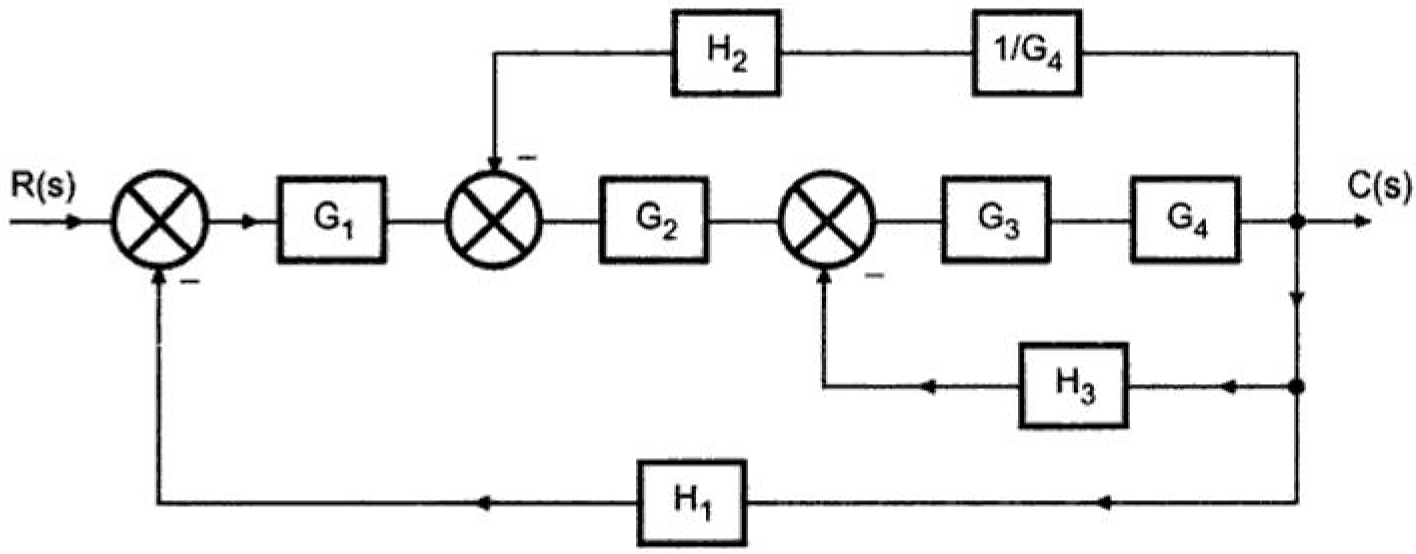
After simplification,

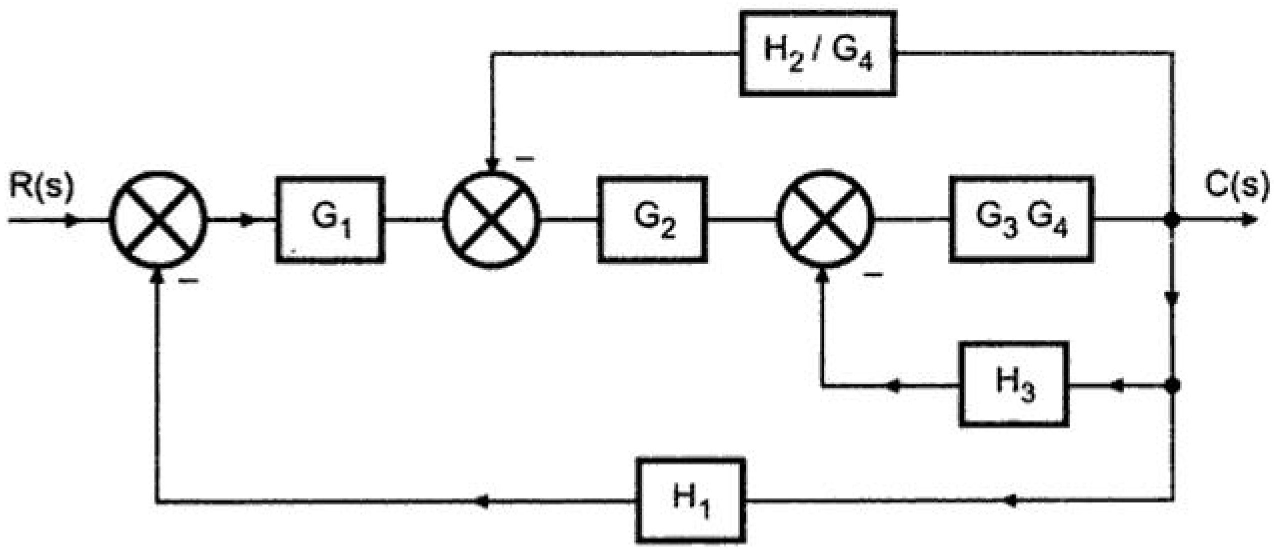
$$\therefore \qquad \frac{C(s)}{R(s)} = \frac{G_1 G_2 G_3}{1 + G_1 H_1 + G_1 G_2 H_2 + G_1 G_2 G_3 H_3}$$

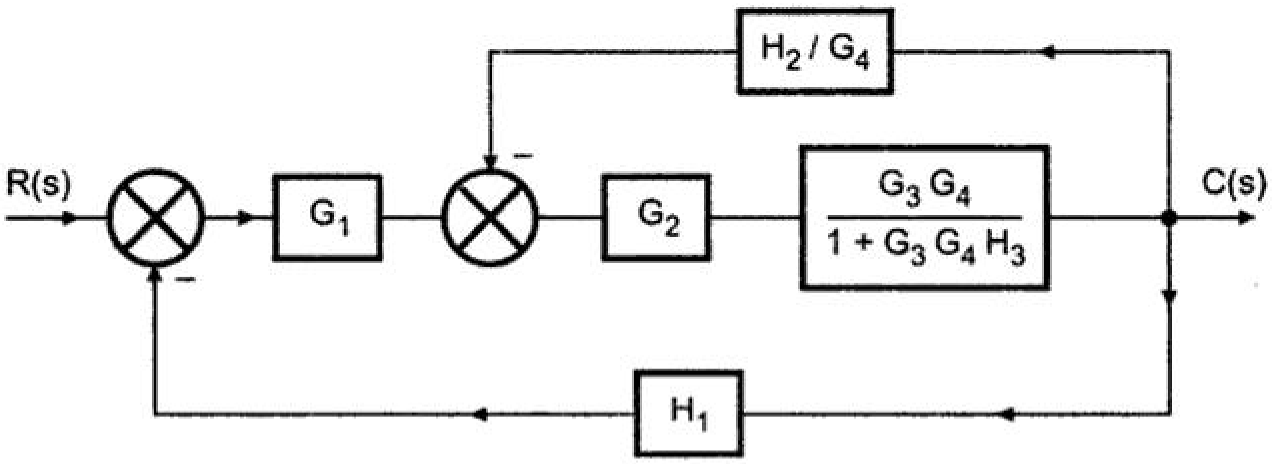
Ex. 3.4

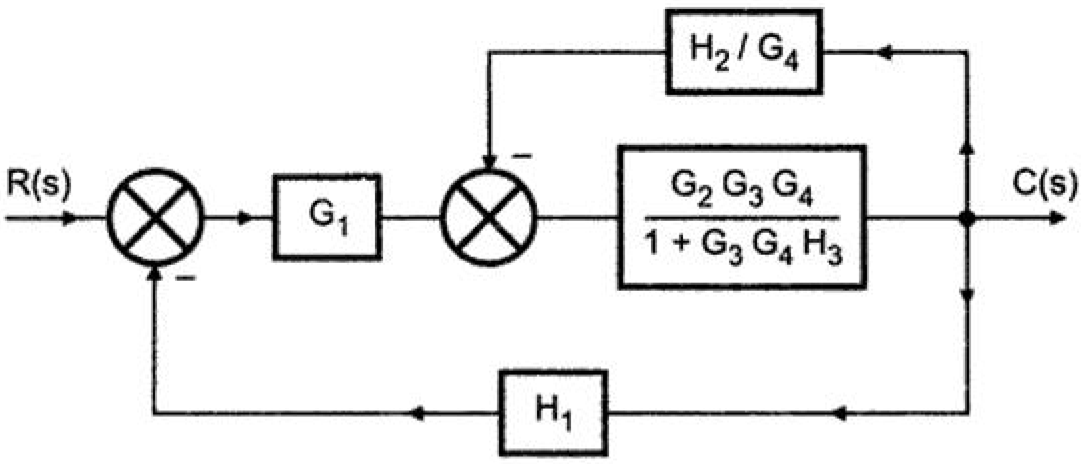


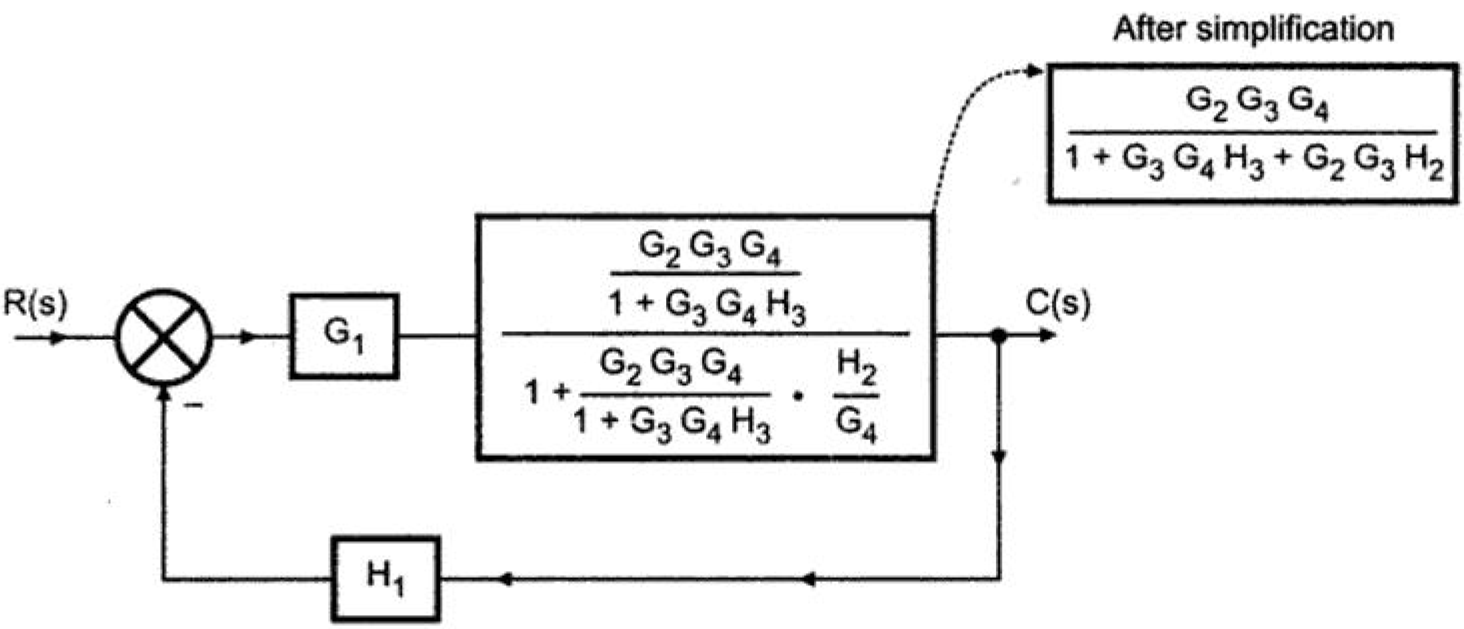
Sol. : No blocks are in series or parallel, similarly there is no minor feedback loop existing. Hence shifting takeoff point towards right as shown we get,

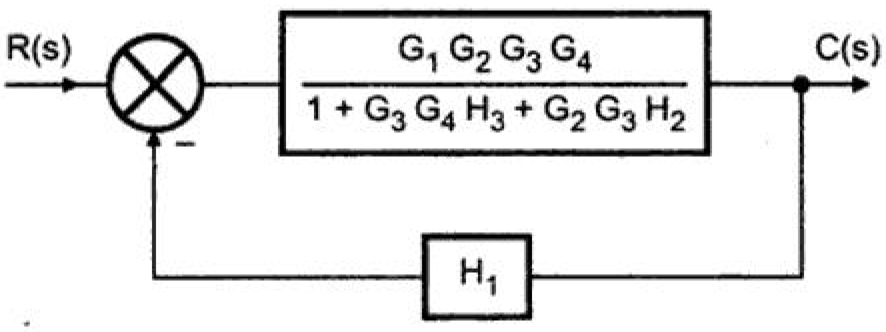


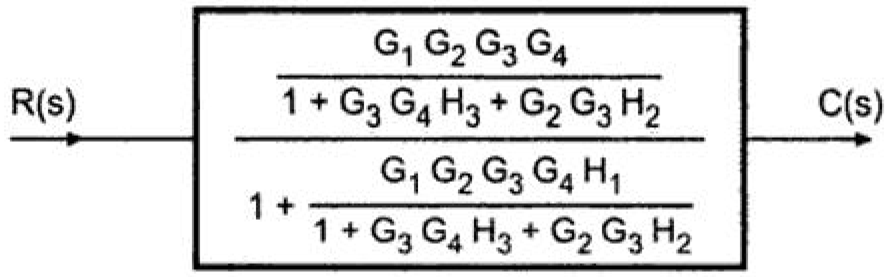



H2 / G4
R(s)
G1
G2
G3 G4 / 1 + G3 G4 H3
C(s)
H1


H2 / G4
R(s)
G1
G2 G3 G4 / 1 + G3 G4 H3
C(s)
H1

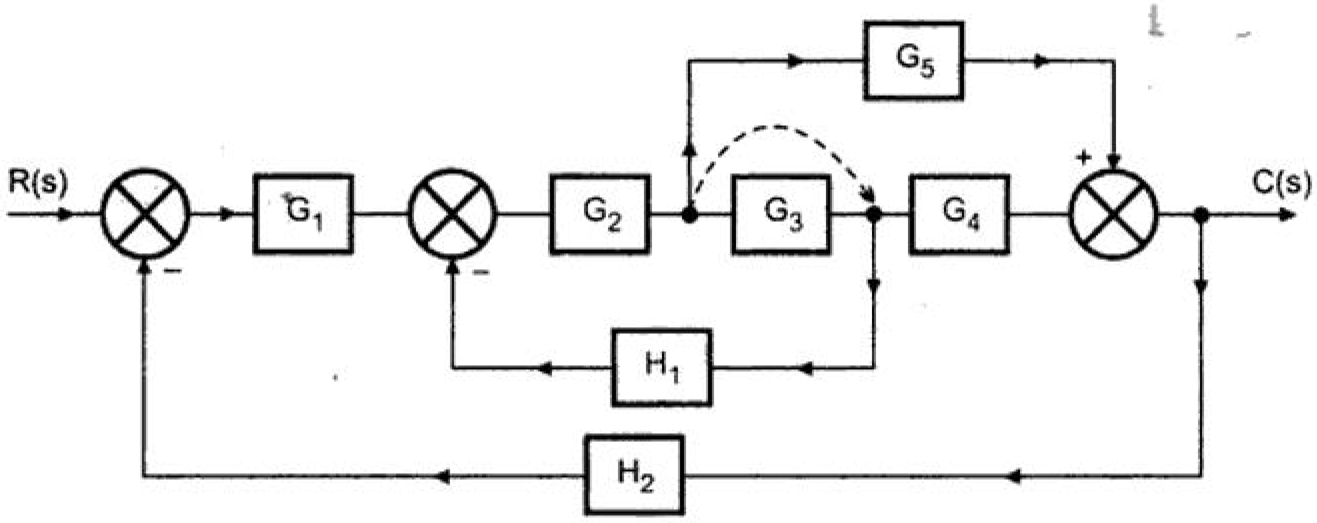

After simplification
G2 G3 G4 / 1 + G3 G4 H3 + G2 G3 H2
R(s)
G1
C(s)
H1


R(s)
G1 G2 G3 G4 / 1 + G3 G4 H3 + G2 G3 H2
C(s)
H1

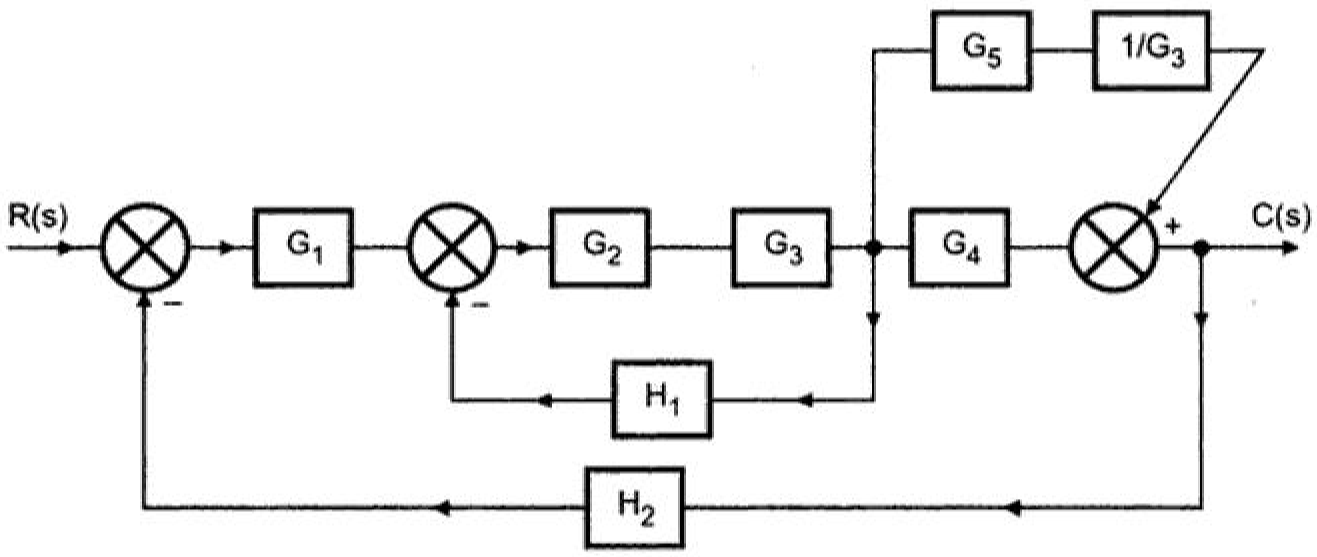


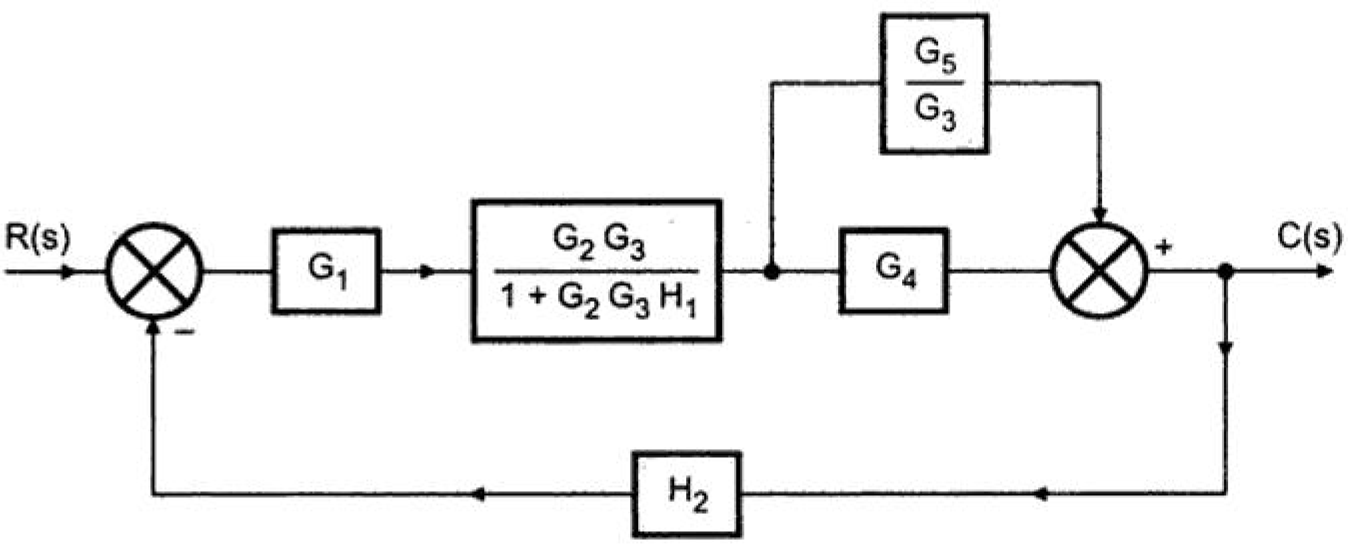
After simplification, $\dfrac{C(s)}{R(s)} = \dfrac{G_1 G_2 G_3 G_4}{1 + G_3 G_4 H_3 + G_2 G_3 H_2 + G_1 G_2 G_3 G_4 H_1}$

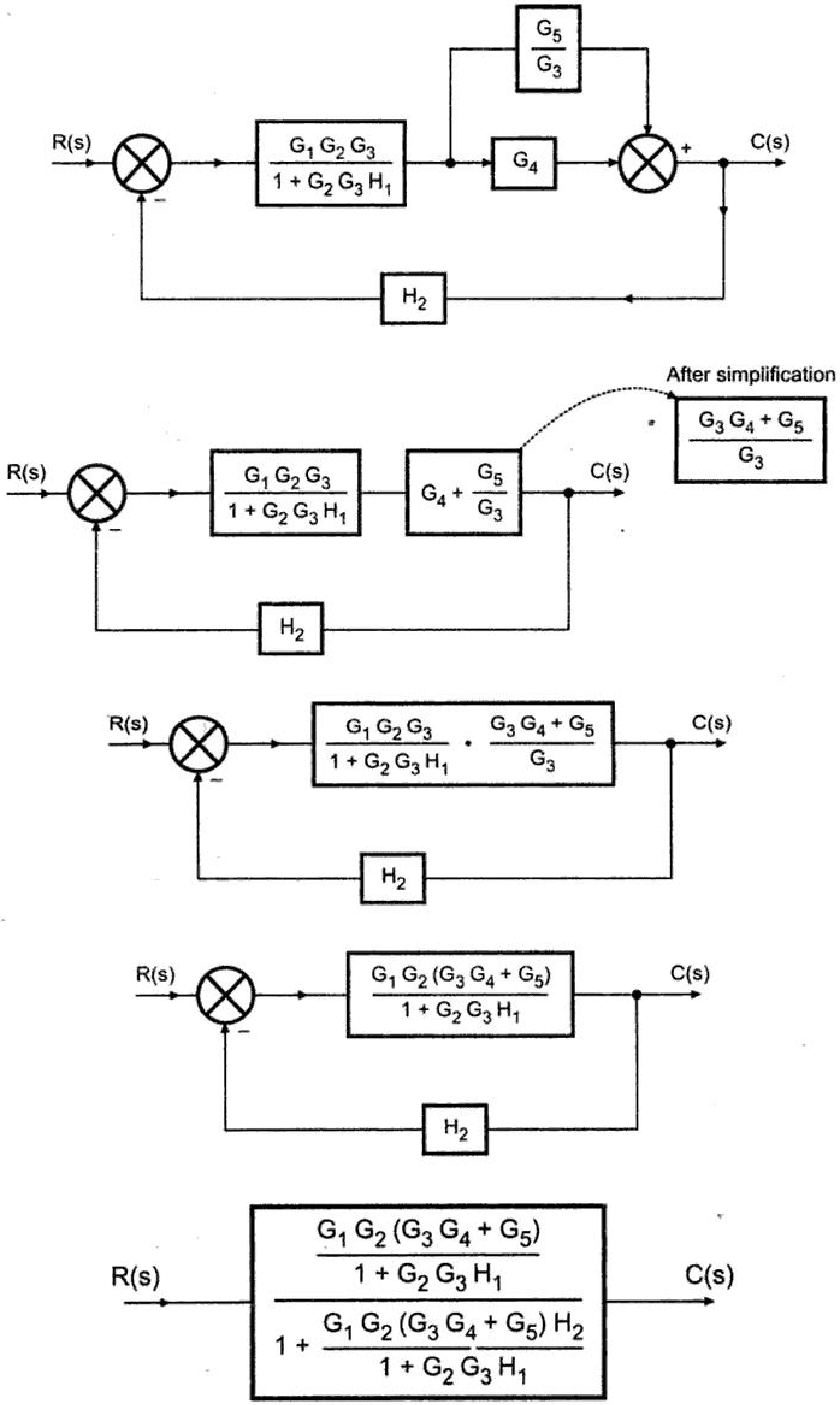
Ex. 3.5



Sol. : No blocks are in series or parallel, similarly there is no minor feedback loop so shifting takeoff point towards right as shown by dotted line we get,






G5 / G3
R(s)
G1 G2 G3 / 1 + G2 G3 H1
G4
+
C(s)
−
H2
After simplification
G3 G4 + G5 / G3
R(s)
G1 G2 G3 / 1 + G2 G3 H1
G4 + G5 / G3
C(s)
−
H2
R(s)
G1 G2 G3 / 1 + G2 G3 H1 · G3 G4 + G5 / G3
C(s)
−
H2
R(s)
G1 G2 (G3 G4 + G5) / 1 + G2 G3 H1
C(s)
−
H2
R(s)
G1 G2 (G3 G4 + G5) / 1 + G2 G3 H1
1 + G1 G2 (G3 G4 + G5) H2 / 1 + G2 G3 H1
C(s)

3–9 SIGNAL FLOW GRAPHS

The block diagram is useful for graphically representing control system dynamics and is used extensively in the analysis and design of control systems. An alternate approach for graphically representing control system dynamics is the signal flow graph approach, due to S. J. Mason. It is noted that the signal flow graph approach and the block diagram approach yield the same information and one is in no sense superior to the other.

Signal Flow Graphs. A signal flow graph is a diagram that represents a set of simultaneous linear algebraic equations. When applying the signal flow graph method to analyses of control systems, we must first transform linear differential equations into algebraic equations in s.

A signal flow graph consists of a network in which nodes are connected by directed branches. Each node represents a system variable, and each branch connected between two nodes acts as a signal multiplier. Note that the signal flows in only one direction. The direction of signal flow is indicated by an arrow placed on the branch, and the multiplication factor is indicated along the branch. The signal flow graph depicts the flow of signals from one point of a system to another and gives the relationships among the signals.

As mentioned earlier, a signal flow graph contains essentially the same information as a block diagram. If a signal flow graph is used to represent a control system, then a gain formula, called Mason's gain formula, may be used to obtain the relationships among system variables without carrying out reduction of the graph.

Definitions. Before we discuss signal flow graphs, we must define certain terms.

Node. A node is a point representing a variable or signal.

Transmittance. The transmittance is a real gain or complex gain between two nodes. Such gains can be expressed in terms of the transfer function between two nodes.

Branch. A branch is a directed line segment joining two nodes. The gain of a branch is a transmittance.

Input node or source. An input node or source is a node that has only outgoing branches. This corresponds to an independent variable.

Output node or sink. An output node or sink is a node that has only incoming branches. This corresponds to a dependent variable.

Mixed node. A mixed node is a node that has both incoming and outgoing branches.

Path. A path is a traversal of connected branches in the direction of the branch arrows. If no node is crossed more than once, the path is open. If the path ends at the same node from which it began and does not cross any other node more than once, it is closed. If a path crosses some node more than once but ends at a different node from which it began, it is neither open nor closed.

Loop. A loop is a closed path.

Loop gain. The loop gain is the product of the branch transmittances of a loop.

Nontouching loops. Loops are nontouching if they do not possess any common nodes.

Forward path. A forward path is a path from an input node (source) to an output node (sink) that does not cross any nodes more than once.

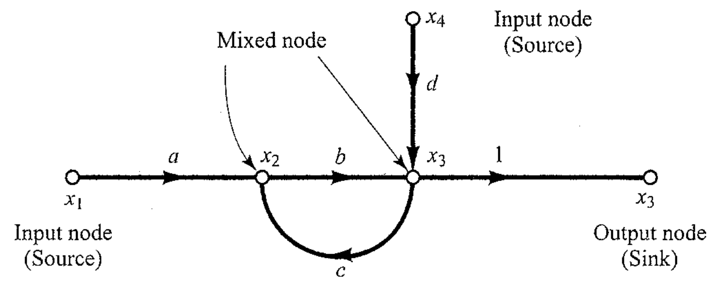


Figure 3–35
Signal flow graph.

Forward path gain. A forward path gain is the product of the branch transmittances of a forward path.

Figure 3–35 shows nodes and branches, together with transmittances.

Properties of Signal Flow Graphs. A few important properties of signal flow graphs are as follows:

1. A branch indicates the functional dependence of one signal on another. A signal passes through only in the direction specified by the arrow of the branch.
2. A node adds the signals of all incoming branches and transmits this sum to all outgoing branches.
3. A mixed node, which has both incoming and outgoing branches, may be treated as an output node (sink) by adding an outgoing branch of unity transmittance. (See Figure 3–35. Notice that a branch with unity transmittance is directed from x_3 to another node, also denoted by x_3.) Note, however, that we cannot change a mixed node to a source by this method.
4. For a given system, a signal flow graph is not unique. Many different signal flow graphs can be drawn for a given system by writing the system equations differently.

Signal Flow Graph Algebra. A signal flow graph of a linear system can be drawn using the foregoing definitions. In doing so, we usually bring the input nodes (sources) to the left and the output nodes (sinks) to the right. The independent and dependent variables of the equations become the input nodes (sources) and output nodes (sinks), respectively. The branch transmittances can be obtained from the coefficients of the equations.

To determine the input-output relationship, we may use Mason's formula, which will be given later, or we may reduce the signal flow graph to a graph containing only input and output nodes. To accomplish this, we use the following rules:

1. The value of a node with one incoming branch, as shown in Figure 3–36(a), is $x_2 = ax_1$.
2. The total transmittance of cascaded branches is equal to the product of all the branch transmittances. Cascaded branches can thus be combined into a single branch by multiplying the transmittances, as shown in Figure 3–36(b).
3. Parallel branches may be combined by adding the transmittances, as shown in Figure 3–36(c).
4. A mixed node may be eliminated, as shown in Figure 3–36(d).
5. A loop may be eliminated, as shown in Figure 3–36(e). Note that

$$x_3 = bx_2, \qquad x_2 = ax_1 + cx_3$$

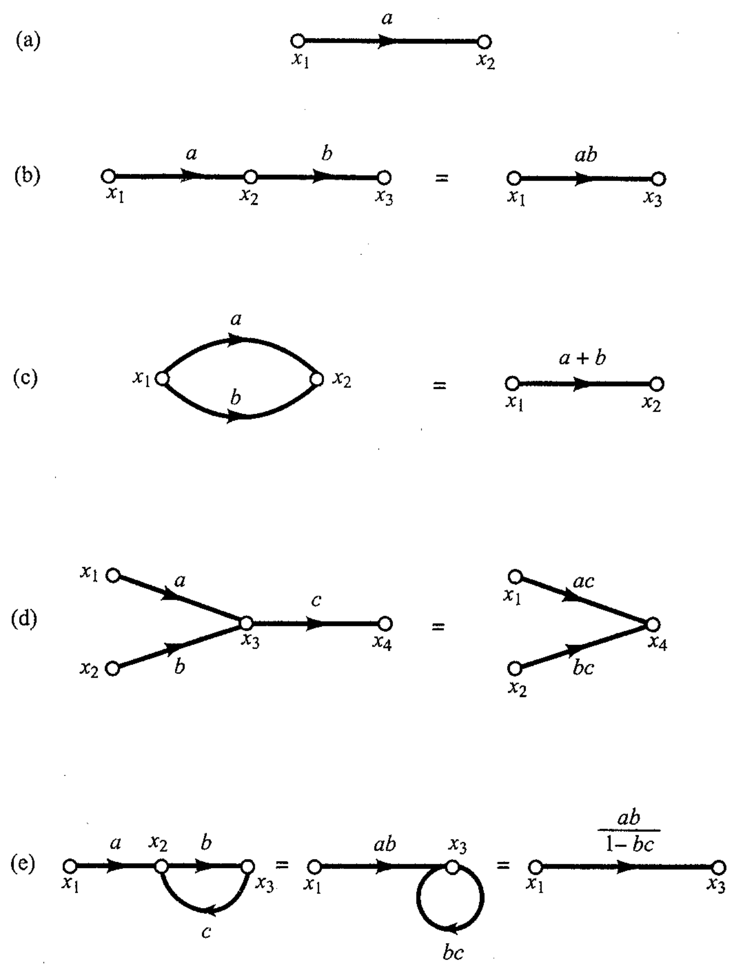


Figure 3–36 Signal flow graphs and simplifications.

Hence

$$x_3 = abx_1 + bcx_3 \tag{3–77}$$

or

$$x_3 = \frac{ab}{1 - bc}x_1 \tag{3–78}$$

Equation (3–77) corresponds to a diagram having a self-loop of transmittance bc. Elimination of the self-loop yields Equation (3–78), which clearly shows that the overall transmittance is $ab/(1 - bc)$.

Signal Flow Graph Representation of Linear Systems. Signal flow graphs are widely applied to linear-system analysis. Here the graph can be drawn from the system equations or, with practice, can be drawn by inspection of the physical system. Routine reduction by use of the foregoing rules gives the relation between an input and output variable.

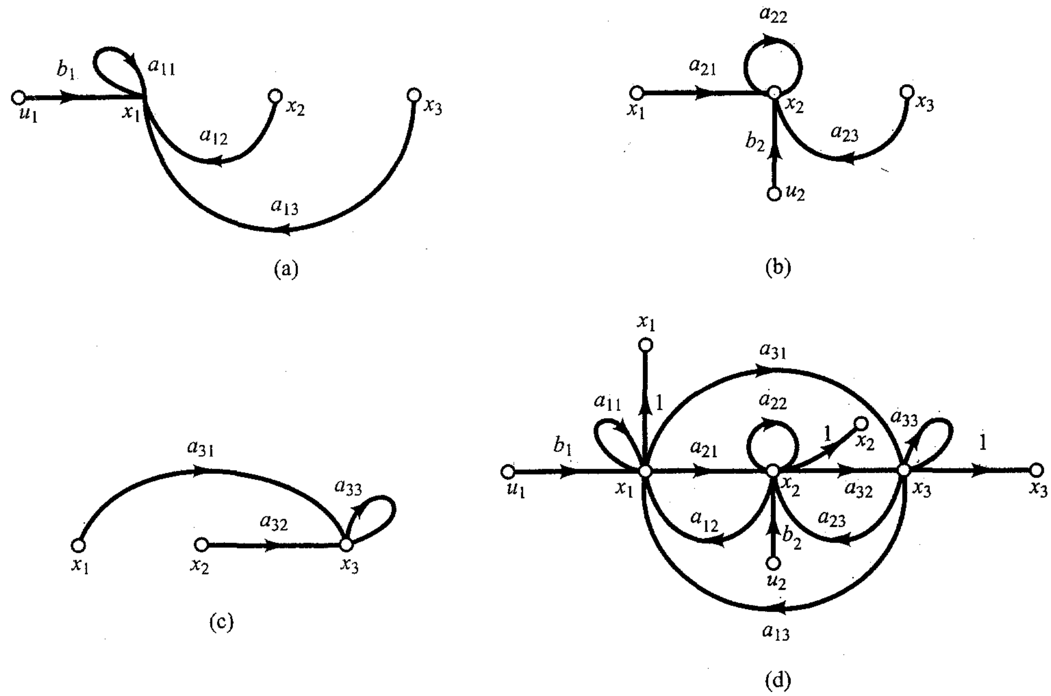


Figure 3–37 Signal flow graphs representing (a) Equation (3–79), (b) Equation (3–80), and (c) Equation (3–81); (d) complete signal flow graph for the system described by Equations (3–79)–(3–81).

Consider a system defined by the following set of equations:

$$x_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + b_1u_1 \tag{3–79}$$

$$x_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + b_2u_2 \tag{3–80}$$

$$x_3 = a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \tag{3–81}$$

where u_1 and u_2 are input variables and x_1, x_2, and x_3 are output variables. A signal flow graph for this system, a graphical representation of these three simultaneous equations, indicating the interdependence of the variables, can be obtained as follows: First locate the nodes x_1, x_2, and x_3 as shown in Figure 3–37(a). Note that a_{ij} is the transmittance between x_j and x_i. Equation (3–79) states that x_1 is equal to the sum of the four signals $a_{11}x_1$, $a_{12}x_2$, $a_{13}x_3$, and b_1u_1. The signal flow graph representing Equation (3–79) is shown in Figure 3–37(a). Equation (3–80) states that x_2 is equal to the sum of $a_{21}x_1$, $a_{22}x_2$, $a_{23}x_3$, and b_2u_2. The corresponding signal flow graph is shown in Figure 3–37(b). The signal flow graph representing Equation (3–81) is shown in Figure 3–37(c).

The signal flow graph representing Equations (3–79), (3–80), and (3–81) is then obtained by combining Figures 3–37(a), (b), and (c). Finally, the complete signal flow graph for the given simultaneous equations is shown in Figure 3–37(d).

In dealing with a signal flow graph, the input nodes (sources) may be considered one at a time. The output signal is then equal to the sum of the individual contributions of each input.

The overall gain from an input to an output may be obtained directly from the signal flow graph by inspection, by use of Mason's formula, or by a reduction of the graph to a simpler form.

Signal Flow Graphs of Control Systems. Some signal flow graphs of simple control systems are shown in Figure 3–38. For such simple graphs, the closed-loop transfer function $C(s)/R(s)$ [or $C(s)/N(s)$] can be obtained easily by inspection. For more complicated signal flow graphs, Mason's gain formula is quite useful.

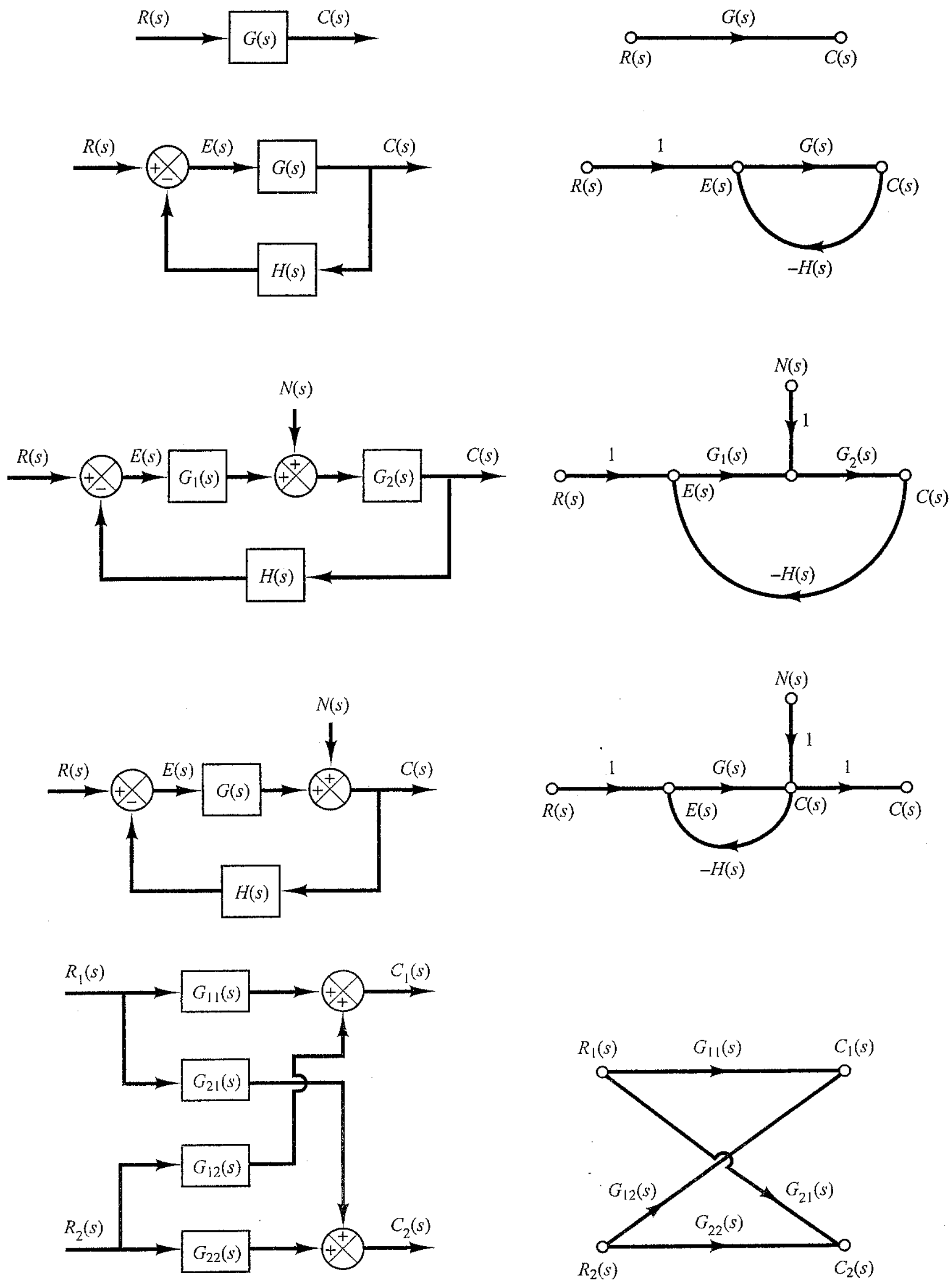


Figure 3–38
Block diagrams and corresponding signal flow graphs.

Mason's Gain Formula for Signal Flow Graphs. In many practical cases, we wish to determine the relationship between an input variable and an output variable of the signal flow graph. The transmittance between an input node and an output node is the overall gain, or overall transmittance, between these two nodes.

Mason's gain formula, which is applicable to the overall gain, is given by

$$P = \frac{1}{\Delta} \sum_k P_k \Delta_k$$

where

P_k = path gain or transmittance of kth forward path

Δ = determinant of graph

= 1 − (sum of all individual loop gains) + (sum of gain products of all possible combinations of two nontouching loops) − (sum of gain products of all possible combinations of three nontouching loops) + ⋯

$$= 1 - \sum_a L_a + \sum_{b,c} L_b L_c - \sum_{d,e,f} L_d L_e L_f + \cdots$$

$\sum_a L_a$ = sum of all individual loop gains

$\sum_{b,c} L_b L_c$ = sum of gain products of all possible combinations of two nontouching loops

$\sum_{d,e,f} L_d L_e L_f$ = sum of gain products of all possible combinations of three nontouching loops

Δ_k = cofactor of the kth forward path determinant of the graph with the loops touching the kth forward path removed, that is, the cofactor Δ_k is obtained from Δ by removing the loops that touch path P_k

(Note that the summations are taken over all possible paths from input to output.)

In the following, we shall illustrate the use of Mason's gain formula by means of two examples.

EXAMPLE 3–13 Consider the system shown in Figure 3–39. A signal flow graph for this system is shown in Figure 3–40. Let us obtain the closed-loop transfer function $C(s)/R(s)$ by use of Mason's gain formula.

In this system there is only one forward path between the input $R(s)$ and the output $C(s)$. The forward path gain is

$$P_1 = G_1 G_2 G_3$$

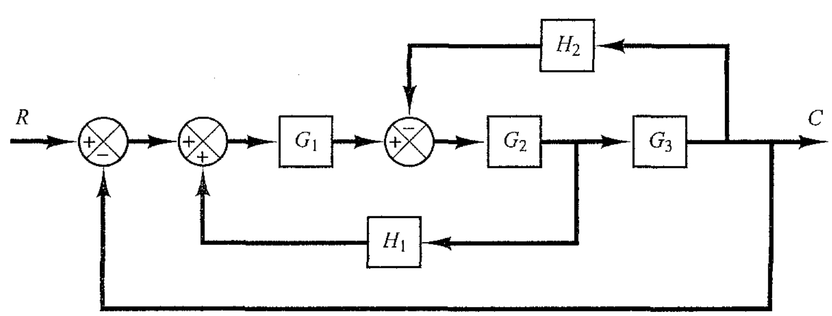


Figure 3–39 Multiple-loop system.

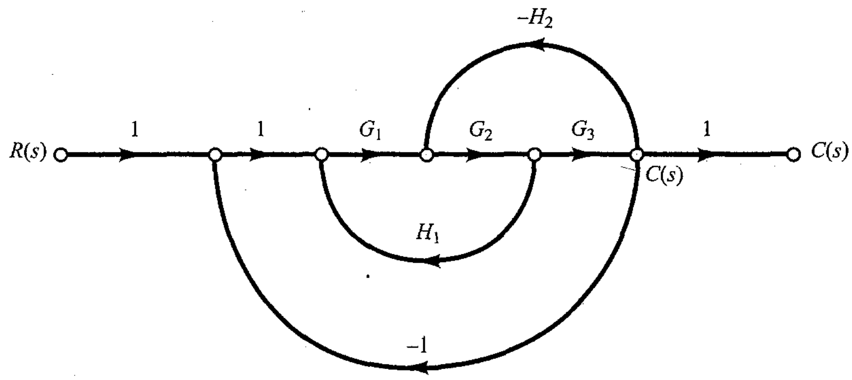


Figure 3–40
Signal flow graph for the system in Figure 3–39.

From Figure 3–40, we see that there are three individual loops. The gains of these loops are

$$L_1 = G_1G_2H_1$$
$$L_2 = -G_2G_3H_2$$
$$L_3 = -G_1G_2G_3$$

Note that since all three loops have a common branch, there are no nontouching loops. Hence, the determinant Δ is given by

$$\Delta = 1 - (L_1 + L_2 + L_3)$$
$$= 1 - G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3$$

The cofactor Δ_1 of the determinant along the forward path connecting the input node and output node is obtained from Δ by removing the loops that touch this path. Since path P_1 touches all three loops, we obtain

$$\Delta_1 = 1$$

Therefore, the overall gain between the input $R(s)$ and the output $C(s)$, or the closed-loop transfer function, is given by

$$\frac{C(s)}{R(s)} = P = \frac{P_1\Delta_1}{\Delta}$$
$$= \frac{G_1G_2G_3}{1 - G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3}$$

which is the same as the closed-loop transfer function obtained by block diagram reduction. Mason's gain formula thus gives the overall gain $C(s)/R(s)$ without a reduction of the graph.

EXAMPLE 3–14 Consider the system shown in Figure 3–41. Obtain the closed-loop transfer function $C(s)/R(s)$ by use of Mason's gain formula.

In this system, there are three forward paths between the input $R(s)$ and the output $C(s)$. The forward path gains are

$$P_1 = G_1G_2G_3G_4G_5$$
$$P_2 = G_1G_6G_4G_5$$
$$P_3 = G_1G_2G_7$$

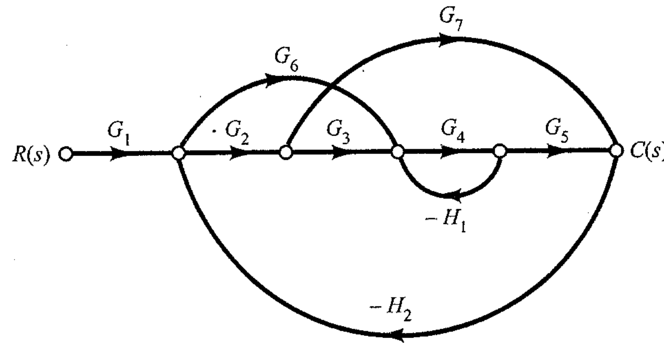


Figure 3–41
Signal flow graph for a system.

There are four individual loops, The gains of these loops are

$$L_1 = -G_4H_1$$
$$L_2 = -G_2G_7H_2$$
$$L_3 = -G_6G_4G_5H_2$$
$$L_4 = -G_2G_3G_4G_5H_2$$

Loop L_1 does not touch loop L_2. Hence, the determinant Δ is given by

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4) + L_1L_2 \tag{3–82}$$

The cofactor Δ_1, is obtained from Δ by removing the loops that touch path P_1. Therefore, by removing L_1, L_2, L_3, L_4, and L_1L_2 from Equation (3–82), we obtain

$$\Delta_1 = 1$$

Similarly, the cofactor Δ_2 is

$$\Delta_2 = 1$$

The cofactor Δ_3 is obtained by removing L_2, L_3, L_4, and L_1L_2 from Equation (3–82), giving

$$\Delta_3 = 1 - L_1$$

The closed-loop transfer function $C(s)/R(s)$ is then

$$\frac{C(s)}{R(s)} = P = \frac{1}{\Delta}(P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3)$$
$$= \frac{G_1G_2G_3G_4G_5 + G_1G_6G_4G_5 + G_1G_2G_7(1 + G_4H_1)}{1 + G_4H_1 + G_2G_7H_2 + G_6G_4G_5H_2 + G_2G_3G_4G_5H_2 + G_4H_1G_2G_7H_2}$$

Comments. The usual application of signal flow graphs is in system diagramming. The set of equations describing a linear system is represented by a signal flow graph by establishing nodes that represent the system variables and by interconnecting the nodes with weighted, directed, transmittances, which represent the relationships among the variables. Mason's gain formula may be used to establish the relationship between an input and an output. (Alternatively, the variables in the system may be eliminated one by one with reduction techniques.) Mason's gain formula is especially useful in reducing large and complex system diagrams in one step, without requiring step-by-step reductions.

Finally, it is noted that in applying the Mason's gain formula to a given system, one must be careful not to make mistakes in calculating the cofactors of the forward paths, Δ_k, since any errors. if they exist, may not easily be detected.

3–10 LINEARIZATION OF NONLINEAR MATHEMATICAL MODELS

Nonlinear Systems. A system is nonlinear if the principle of superposition does not apply. Thus, for a nonlinear system the response to two inputs cannot be calculated by treating one input at a time and adding the results.

Although many physical relationships are often represented by linear equations, in most cases actual relationships are not quite linear. In fact, a careful study of physical systems reveals that even so-called "linear systems" are really linear only in limited operating ranges. In practice, many electromechanical systems, hydraulic systems, pneumatic systems, and so on, involve nonlinear relationships among the variables. For example, the output of a component may saturate for large input signals. There may be a dead space that affects small signals. (The dead space of a component is a small range of input variations to which the component is insensitive.) Square-law nonlinearity may occur in some components. For instance, dampers used in physical systems may be linear for low-velocity operations but may become nonlinear at high velocities, and the damping force may become proportional to the square of the operating velocity.

Linearization of Nonlinear Systems. In control engineering a normal operation of the system may be around an equilibrium point, and the signals may be considered small signals around the equilibrium. (It should be pointed out that there are many exceptions to such a case.) However, if the system operates around an equilibrium point and if the signals involved are small signals, then it is possible to approximate the nonlinear system by a linear system. Such a linear system is equivalent to the nonlinear system considered within a limited operating range. Such a linearized model (linear, time-invariant model) is very important in control engineering.

The linearization procedure to be presented in the following is based on the expansion of nonlinear function into a Taylor series about the operating point and the retention of only the linear term. Because we neglect higher-order terms of Taylor series expansion, these neglected terms must be small enough; that is, the variables deviate only slightly from the operating condition.

Linear Approximation of Nonlinear Mathematical Models. To obtain a linear mathematical model for a nonlinear system, we assume that the variables deviate only slightly from some operating condition. Consider a system whose input is $x(t)$ and output is $y(t)$. The relationship between $y(t)$ and $x(t)$ is given by

$$y = f(x) \tag{3–83}$$

If the normal operating condition corresponds to $\bar{x}$, $\bar{y}$, then Equation (3–83) may be expanded into a Taylor series about this point as follows: