7
Stability of Control Systems

7.1 Background

As we have seen earlier that every system, for small amount of time has to pass through a transient period. Now whether system will reach to its intended steady state after passing through transients or not ? The answer to this question means to define whether system is stable or unstable. This is stability analysis.

For example, a meter is connected in a system to measure a particular parameter. Before showing the final reading, the pointer of meter will pass through the transients. The final reading is the steady state of the pointer. But during transients, it is possible that the pointer may become stationary due to certain problems in the moving system of that meter. So to achieve steady state, the system must pass through the transient period successfully. The analysis of, whether the given system can reach steady state; passing through the transients successfully is called **Stability Analysis** of the system.

7.2 Concept of Stability

Consider a system i.e. a deep container with an object placed inside it as shown in the Fig. 7.1.

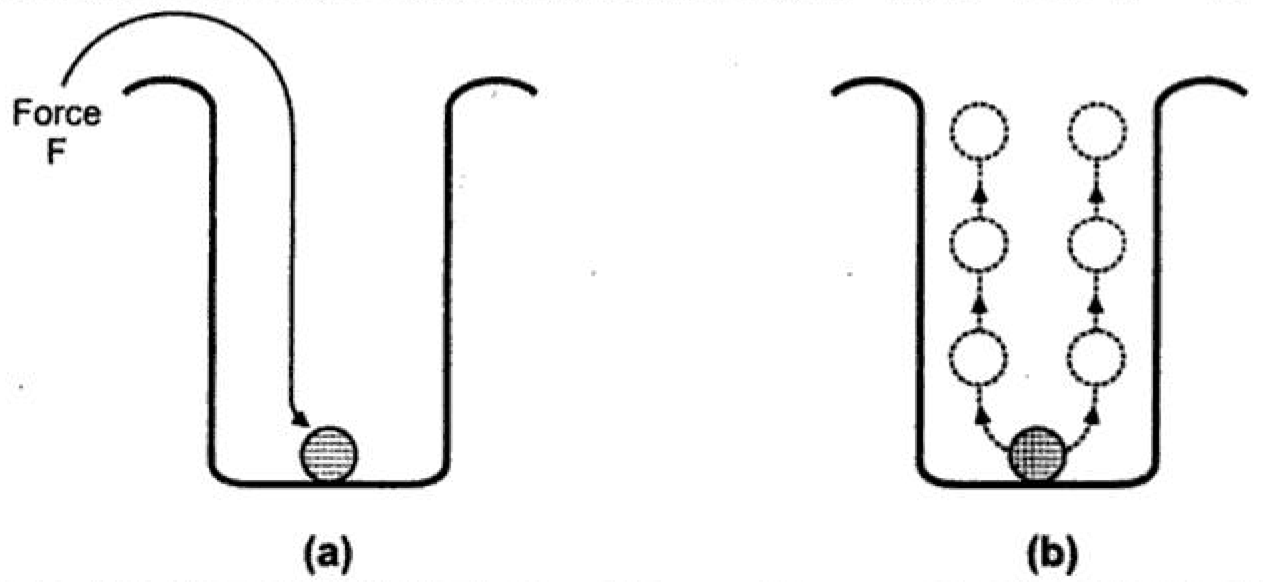


Fig. 7.1

Now if we apply a force to take out the object, as the depth of the container is more, it will oscillate and will settle down again at its original position.

If we assume that the force required to take out the object tends to infinity i.e. always object will oscillate when force is applied and will settle down but will not come out, such a system is called ***absolutely stable*** system. No change in parameters, disturbances, changes the output. As against this, consider a container which is pointed one, on which we try to keep a circular objects shown in the Fig. 7.2.

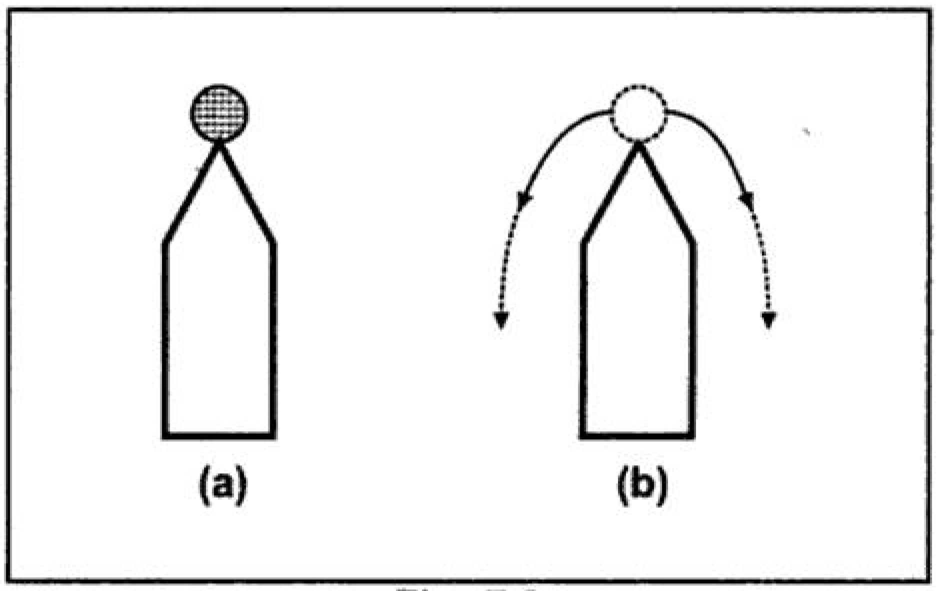


Fig. 7.2

In this case object will fall down without any external application of force. So if we try to keep the circular object, we will always fail to do so. Such system is called ***unstable system***.

While in certain cases the container is shallow then there exists a critical value of force for which object will come out of container.

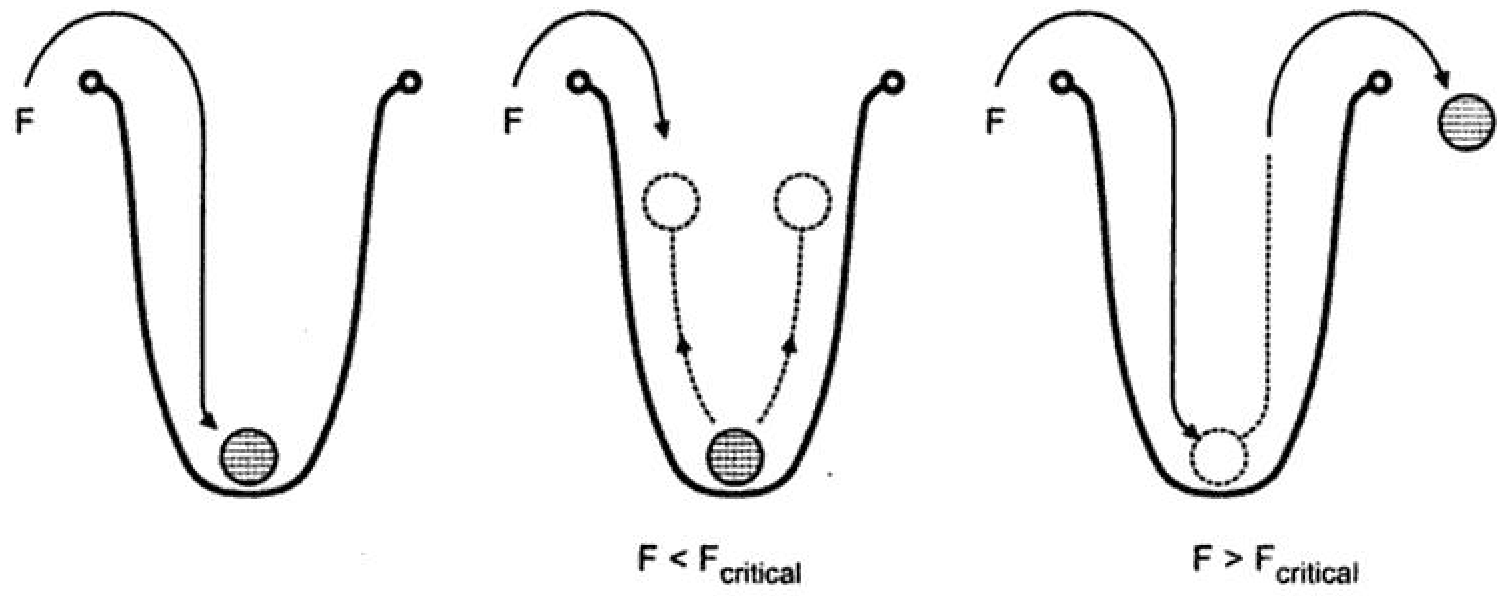


Fig. 7.3

As long as $F < F_{critical}$, object regains its original position but if $F > F_{critical}$, object will come out. Stability depends on certain conditions of the system hence system is called **conditionally stable system**.

There are few systems e.g. : pendulum where system keeps on oscillating when certain force is applied. Such systems are neither stable nor unstable and hence called ***critically stable or marginally stable systems***.

Now let us see on which factors exactly the stability depends in a control system.

7.3 Stability of Control Systems

The stability of a linear closed-loop system can be determined from the locations of closed loop poles in the s-plane.

e.g. : If system has closed loop T.F.

$$\frac{C(s)}{R(s)} = \frac{10}{(s+2)(s+4)}$$

Let us find out, output response for unit step input.

$$\therefore \quad R(s) = 1/s$$

$$\therefore \quad C(s) = \frac{10}{s(s+2)(s+4)} = \frac{A}{s}+\frac{B}{s+2}+\frac{C}{s+4}$$

$$\therefore \quad C(s) = 10\left\{\frac{1/8}{s}-\frac{1/4}{s+2}+\frac{1/8}{s+4}\right\} \quad \text{... Finding the partial fractions}$$

$$\therefore \quad C(s) = \frac{1.25}{s}-\frac{2.5}{s+2}+\frac{1.25}{s+4}$$

$$c(t) = \underbrace{1.25}_{\text{Steady state}} - \underbrace{2.5\,e^{-2t}+1.25\,e^{-4t}}_{\text{Transient}} = C_{ss} + c_t(t)$$

As closed loop poles are located in left half of s-plane, in output response there are exponential terms with negative indices i.e. e^{-2t} and e^{-4t}.

Now as $t \to \infty$ both exponential terms will approach to zero and output will be steady state output.

i.e. as $t \to \infty$, $c_t(t) = 0$

Transient output $= 0$

Such systems are called ***absolutely stable systems.***

Now transient terms are exponential terms with negative index because closed loop poles are located in left half of s-plane. For the above system under consideration, the closed loop poles are s = –2 and s = – 4 and the negative indices of exponential terms are also –2 and –4.

Key Point: *Thus if closed loop poles are located in left half, exponential indices in output are negative. And if indices are negative, exponential transient terms will vanish when* $t \to \infty$.

Now let us have a system with one closed loop pole located in right half of s-plane.

$$\frac{C(s)}{R(s)} = \frac{10}{(s-2)(s+4)}$$

Find out unit step response of above system.

$$C(s) = \frac{10}{s(s-2)(s+4)} = \left\{\frac{A}{s}+\frac{B}{s-2}+\frac{C}{s+4}\right\}$$

$$C(s) = -\frac{1.25}{s}+\frac{0.833}{s-2}+\frac{0.416}{s+4}$$

$$\therefore \quad c(t) = -1.25 + 0.833\, e^{+2t} + 0.416\, e^{-4t}$$

Now due to pole located in right half, there is one exponential term with positive index in transient output.

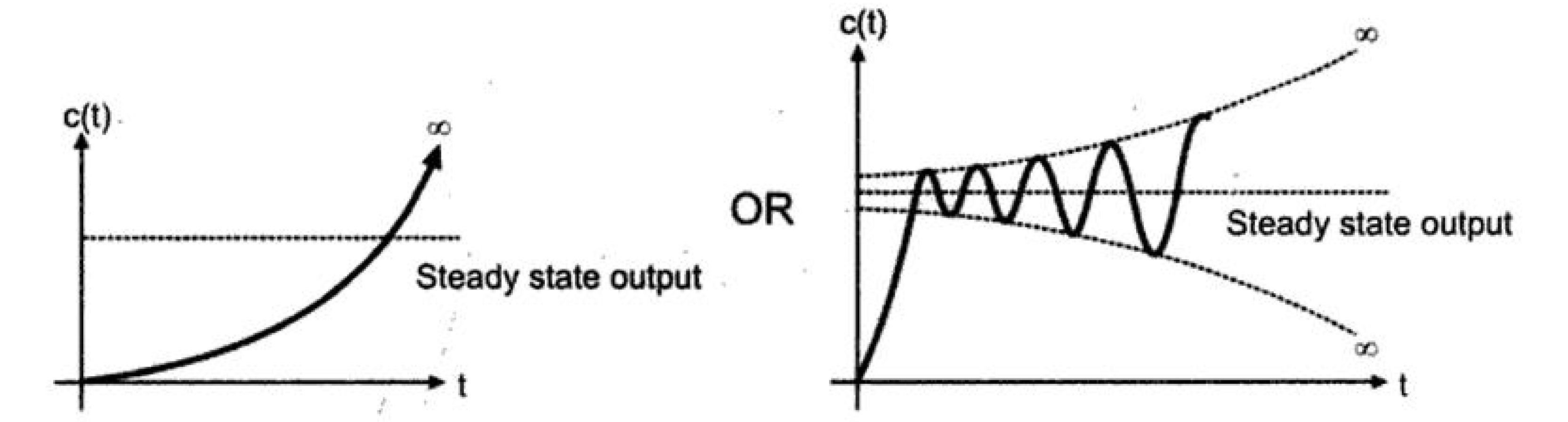
while $\quad c_{ss}(t) = -1.25$

t	c(t)
0	0
1	+ 4.91
2	+ 44.23
4	+ 2481.88
∞	∞

As it is clear from the table that instead of approaching to steady state value as $t \to \infty$, due to exponential term with positive index, transients go on increasing in amplitude. So such system is said to be unstable.

Key Point: *So it is clear that if any of the closed loop poles lie in right half of s-plane, then it gives the exponential term of positive index and due to that, transient response of increasing amplitude., making system unstable.*

In such systems output is uncontrollable and unbounded one. Output response of such systems is as shown in the Fig. 7.4.



(a) Increasing exponentially **(b) Increasing amplitude oscillations**

Fig. 7.4 Uncontrollable response

For such unstable systems, if input is removed output may not return to zero. And as soon as input power is turned on, output tends to ∞. If no saturation takes place in system and no mechanical stop is provided then system may get damaged and failed.

Remember that the stability depends on locations of closed loop poles. And the closed loop poles are the roots of the characteristic equation of the system.

So, | Closed loop poles = Roots of the characteristic equation |

If all the closed loop poles or roots of characteristic equation lies in left half of s-plane then in the output response there will be exponential terms with negative indices along with steady state terms. Such transient terms approach to zero as time advances. Eventually output reaches to equilibrium and attains steady state value. So transient terms in such systems may give oscillations but the amplitudes of such oscillations will be decreasing w.r.t. time and finally will vanish. So output response of such system can be shown as in the Fig. 7.5 (a) and (b).

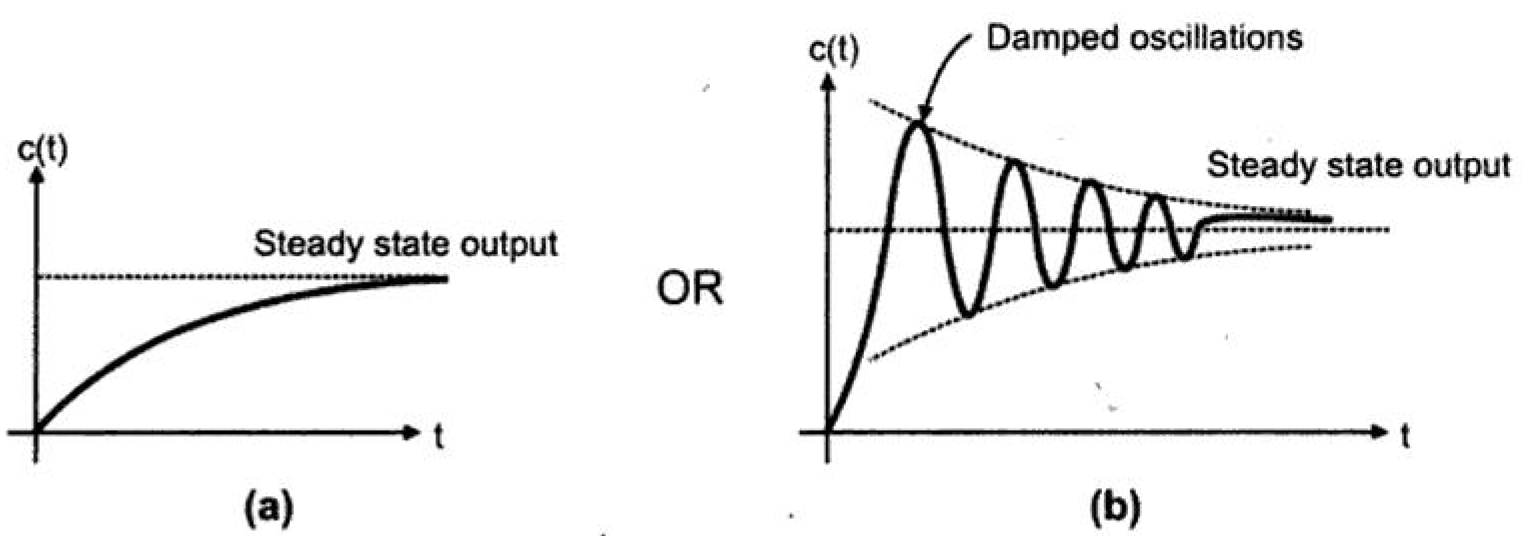


Fig. 7.5 Stable response

Definition of BIBO Stability : *This is Bounded Input Bounded Output stability (BIBO).*

A linear time invariant system is said to be stable if following conditions are satisfied :

i) When the system is excited by a bounded input, output is also bounded and controllable.

ii) In the absence of the input, output must tend to zero irrespective of the initial conditions.

Unstable System : *A linear time invariant system is said to be unstable if,*

i) For a bounded input it produces unbounded output.

ii) In absence of the input, output may not return to zero. It shows certain output without input.

Besides these two cases, if one or more pairs of simple **nonrepeated roots** of characteristic equation are located on the imaginary axis of the s-plane, but there are no roots in the right half of s-plane, the output response will be undamped sinusoidal oscillations of constant frequency and amplitude. Such systems are said to be critically or marginally stable systems.

Critically or Marginally Stable System : *A linear time invariant system is said to be critically or marginally stable if for a bounded input its output oscillates with constant frequency and amplitude. Such oscillations of output are called* ***undamped oscillations*** *or* ***sustained oscillations.***

For such systems, one or more pairs of **nonrepeated** roots are located on imaginary axis as shown in the Fig. 7.6 (b).

Output response of such systems is as shown in the Fig. 7.6 (a).

Key Point : *The stability or instability is a property of the system itself i.e. closed loop poles of the system and does not depend on input or driving function. The poles of input do not affect stability of system, they affect only steady state output.*

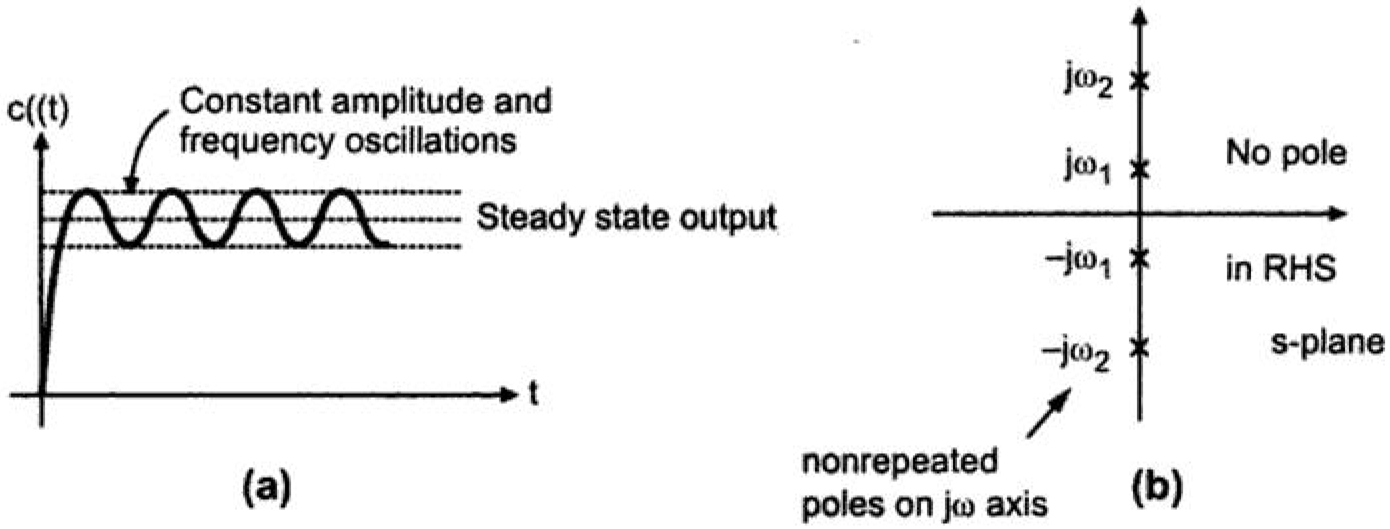
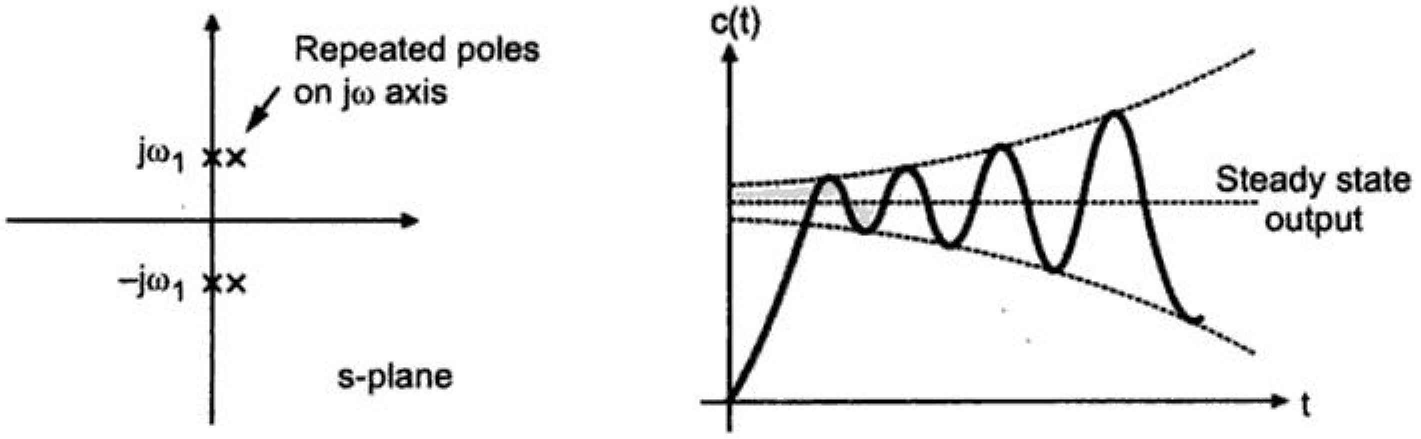


Fig. 7.6 Critically or marginally stable

Special Case : If there are **repeated roots** located purely on imaginary axis, system is said to be unstable.



(a) Location of poles **(b) Unstable system**

Fig. 7.7

Conditionally Stable System : A linear time invariant system is said to be conditionally stable if for a certain condition of a particular parameter of the system, its output is bounded one. Otherwise if that condition is violated output becomes unbounded and system becomes unstable i.e. stability of system depends on condition of parameter of the system. Such system is called **conditionally stable system.**

So s-plane can be divided into three distinct zones from stability point of view as shown in the Fig. 7.8.

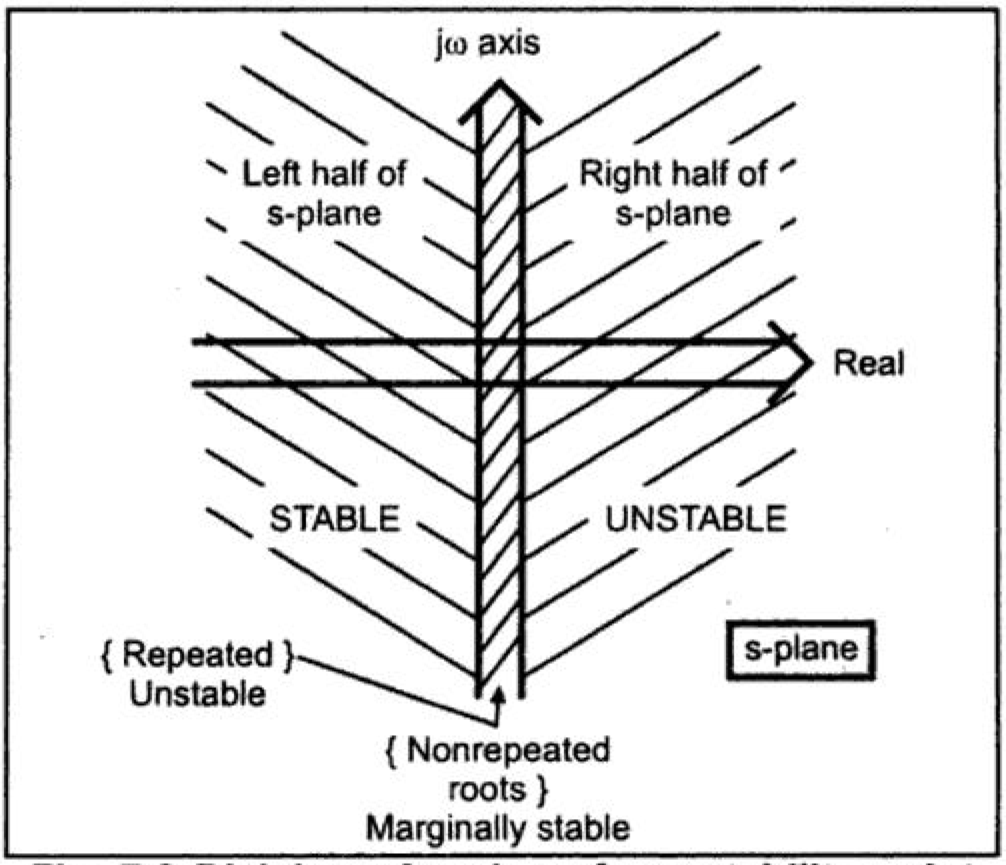


Fig. 7.8 Division of s-plane from stability point of view

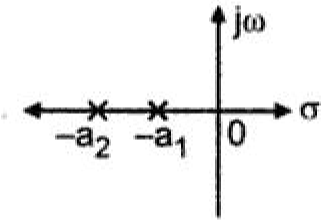
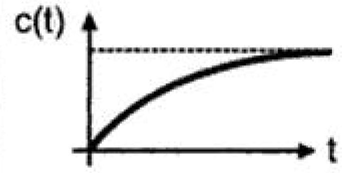
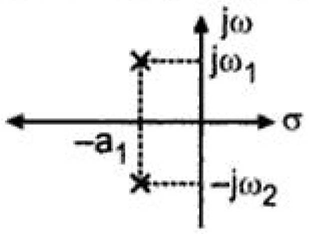
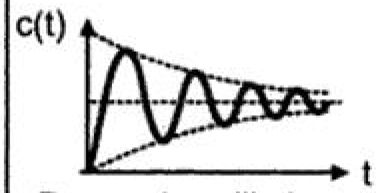
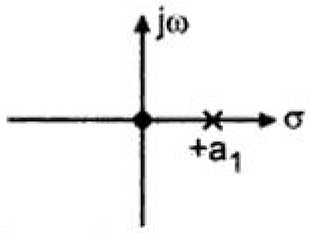
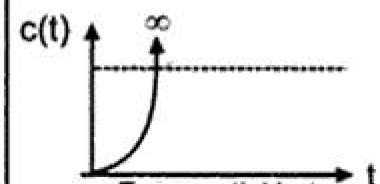
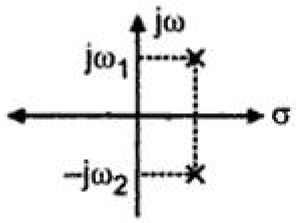
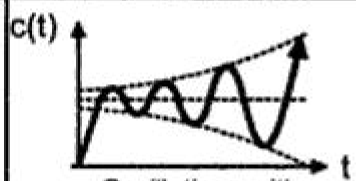
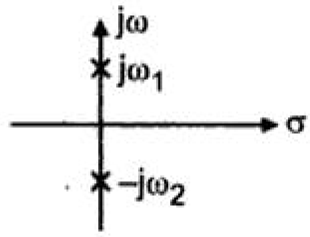
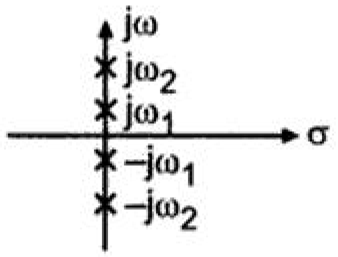
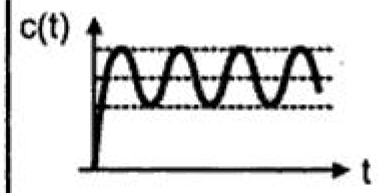
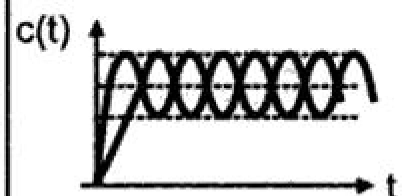
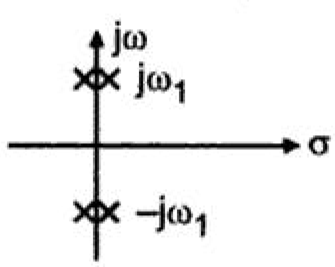
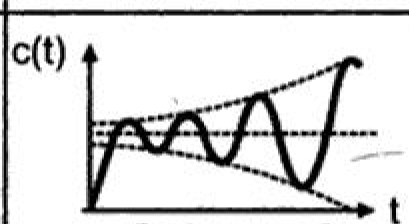
Sr. No.	Nature of closed loop poles	Locations of closed loop poles in s-plane	Step response	Stability condition
1.	Real, negative i.e. in L.H.S. of s-plane		 Pure exponential	Absolutely stable
2.	Complex conjugate with negative real part i.e. in L.H.S. of s-plane		 Damped oscillations	Absolutely stable
3.	Real, positive i.e. in R.H.S. of s-plane (Any one closed loop pole in right half irrespective of number of poles in left half of s-plane)		 Exponential but increasing towards ∞	Unstable
4.	Complex conjugate with positive real part i.e. in R.H.S. of s-plane		 Oscillations with increasing amplitude	Unstable
5.	Nonrepeated pair on imaginary axis without any pole in R.H.S. of s-plane	 or  Two nonrepeated pairs on imaginary axis.	 Frequency of oscillations = ω_1  Sustained oscillations with two frequency components ω_1 and ω_2	Marginally or critically stable Marginally or critically stable.
6.	Repeated pair on imaginary axis without any pole in R.H.S. of s-plane		 Oscillations of increasing amplitude	Unstable

Table 7.1 Closed loop poles and stability

7.4 Zero Input and Asymptotic Stability

Some systems in practice may get driven by the initial conditions, without any input applied to it. For example a series RC circuit with capacitor initially charged to some voltage. This initial voltage is enough to start the operation of the system. This initial voltage, without any external input, drives the current till capacitor gets fully discharged. The stability related to such a system which is under zero input condition but operated under initial condition is called **zero input stability** of the system. The current through RC circuit reduces to zero as capacitor gets fully discharged. The current in such a case is called **zero input response** of the system, which is only due to the initial conditions. From this, zero input stability can be defined as :

If the zero input response of the system subjected to the finite initial conditions, reaches to zero as time t approaches infinity, the system is said to be **zero input stable** otherwise it is called **zero input unstable.**

Mathematically if c(t) is the zero input response of the system then for zero input stability there exists a positive number M, which depends on set of finite initial conditions such that,

$$|c(t)| \leq M < \infty \text{ for all } t \geq t_0$$

and

$$\lim_{t \to \infty} |c(t)| = 0$$

As magnitude of zero input response reaches zero as $\lim t \to \infty$, the zero input stability is also called the **asymptotic stability.**

7.4.1 Remarks about Asymptotic Stability

Following are the important remarks about zero input or asymptotic stability,

1. The zero input or asymptotic stability depends on the roots of the characteristic equation i.e. closed loop poles of the systems.

2. All the requirements about the locations of roots of the characteristic equation related to BIBO stability are applicable to zero input or asymptotic stability. For zero input or asymptotic stability also, all the roots of the characteristic equation must be located in left half of s-plane.

3. If a system is BIBO stable, then it must be zero input or asymptotically stable.

Thus hereafter the system is said to be just stable, unstable or marginally stable, for all practical purposes.

Note that for nonrepeated pair of roots of the characteristic equation on jω axis, system is marginally stable. But an integrator having transfer function 1/s i.e. root located at origin is treated to be stable for all practical purposes as an exception.

7.5 Relative Stability

The system is said to be relatively more stable or unstable on the basis of settling time. System is said to be relatively more stable if settling time for that system is less than that of the other system.

The settling time of the root or pair of complex conjugate roots is inversely proportional to the real part of the roots.

So for the roots located near the jω axis, settling time will be large. As roots or pair of complex conjugate roots moves away from jω - axis i.e. towards left half of s-plane, settling time becomes lesser or smaller and system becomes more and more stable.

So relative stability of the system improves, as the closed loop poles move away from the imaginary axis in left half of s-plane.

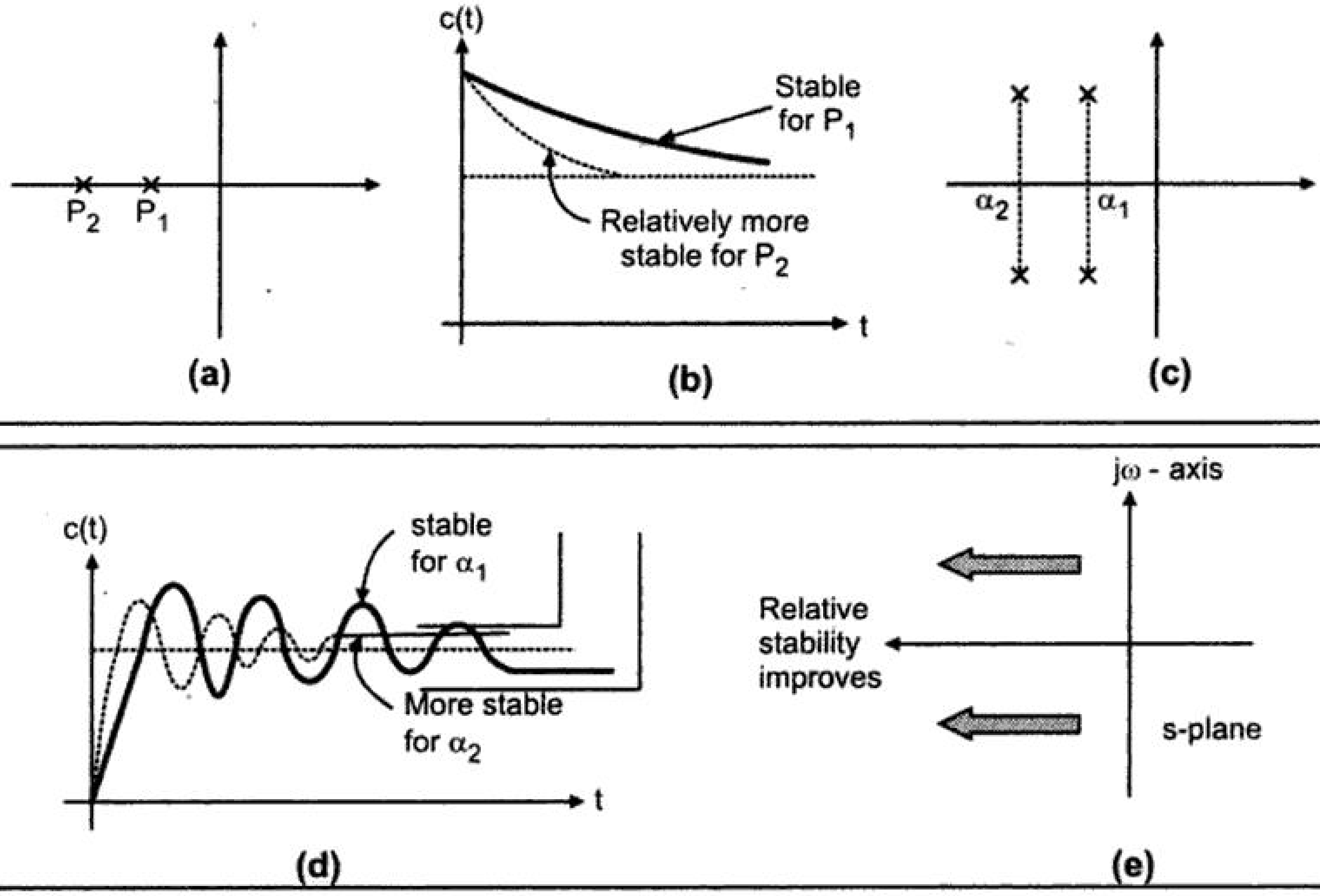


Fig. 7.9 Relative stability

7.6 Routh-Hurwitz Criterion

This represents a method of determining the location of poles of a characteristic equation with the respect to the left half and right half of the s-plane without actually solving the equation.

The T.F. of any linear closed loop system can be represented as,

$$\frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + ... + b_m}{a_0 s^n + a_1 s^{n-1} + ... + a_n} = \frac{B(s)}{F(s)}$$

where `a' and `b' are constants.

To find closed loop poles we equate F(s) = 0. This equation is called **characteristic equation** of the system.

i.e. $$F(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + ... + a_n = 0$$

Thus the roots of the characteristic equation are the closed loop poles of the system which decide the stability of the system.

7.6.1 Necessary Conditions

In order that the above characteristic equation has no root in right of s-plane, it is **necessary but not sufficient** that,

1) All the coefficients of the polynomial have the same sign.
2) None of the coefficient vanishes i.e. all powers of `s' must be present in descending order from `n' to zero.

These conditions are not sufficient.

7.6.2 Hurwitz's Criterion

The sufficient condition for having all the roots of characteristic equation in left half of s-plane is given by Hurwitz. It is referred as Hurwitz Criterion. It states that :

The necessary and sufficient condition to have all roots of characteristic equation in left half of s-plane is that the sub-determinants D_K , K = 1, 2, ...n obtained from Hurwitz's determinant `H' must all be positive.

Method of forming Hurwitz determinant :

$$H = \begin{vmatrix} a_1 & a_3 & a_5 & .... & a_{2n-1} \\ a_0 & a_2 & a_4 & .... & a_{2n-2} \\ 0 & a_1 & a_3 & .... & a_{2n-3} \\ 0 & a_0 & a_2 & .... & a_{2n-4} \\ 0 & 0 & a_1 & .... & a_{2n-5} \\ . & . & . & .... & . \\ . & . & . & .... & . \\ 0 & . & . & .... & a_n \end{vmatrix}$$

The order is n × n where n = order of characteristic equation. In Hurwitz determinant all coefficients with suffices greater than `n' or negative suffices must all be replaced by zeros. From Hurwitz determinant subdeterminants D_K, K= 1, 2, ... n must be formed as follows :

$$D_1 = |a_1| \qquad D_2 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} \qquad D_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} \qquad D_K = |H|$$

For the system to be stable, all the above determinants must be positive.

Ex. 7.1 *Determine the stability of the given characteristic equation by Hurwitz's method.* $F(s) = s^3 + s^2 + s^1 + 4 = 0$ *is characteristic equation.*

Sol. : $a_0 = 1,\ a_1 = 1,\ a_2 = 1,\ a_3 = 4,\ n = 3$

$$H = \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 4 \end{vmatrix}$$

$$D_1 = |1| = 1$$

$$D_2 = \begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} = -3$$

$$D_3 = \begin{vmatrix} 1 & 4 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 4 \end{vmatrix} = 4 - 16 = -12$$

As D_2 and D_3 are negative, given system is unstable.

7.6.3 Disadvantages of Hurwitz's Method

i) For higher order systems, to solve the determinants of higher order is very complicated and time consuming.

ii) Number of roots located in right half of s-plane for unstable system cannot be judged by this method.

iii) Difficult to predict marginal stability of the system.

Due to these limitations, a new method is suggested by the scientist Routh called Routh's method. It is also called Routh-Hurwitz method.

7.7 Routh's Stability Criterion

It is also called **Routh's array method** or **Routh-Hurwitz's Method**

Routh suggested a method of tabulating the coefficients of characteristic equation in a particular way. Tabulation of coefficients gives an array called **Routh's array.**

Consider the general characteristic equation as,

$$F(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots\ldots + a_n = 0$$

Method of forming an array :

s^n	a_0	a_2	a_4	a_6	....
s^{n-1}	a_1	a_3	a_5	a_7	
s^{n-2}	b_1	b_2	b_3		
s^{n-3}	c_1	c_2	c_3		
:	:	:	:		
s^0	a_n				

Coefficients for first two rows are written directly from characteristic equation.

From these two rows next rows can be obtained as follows.

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}, \quad b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}, \quad b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}$$

From 2^{nd} and 3^{rd} row, 4^{th} row can be obtained as

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}, \quad c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}$$

This process is to be continued till the coefficient for s^0 is obtained which will be a_n. From this array stability of system can be predicted.

7.7.1 Routh's Criterion

> ***The necessary and sufficient condition for system to be stable is "All the terms in the first column of Routh's array must have same sign. There should not be any sign change in the first column of Routh's array."***

If there are any sign changes existing then,

a) System is unstable.

b) The number of sign changes equals the number of roots lying in the right half of the s-plane.

Examine the stability of given equations using Routh's method :

Ex. 7.2 $s^3 + 6s^2 + 11s + 6 = 0$

Sol. : $a_0 = 1, \quad a_1 = 6, \quad a_2 = 11, \quad a_3 = 6, \quad n = 3$

s^3	1	11
s^2	6	6
s^1	$\frac{11 \times 6 - 6}{6} = 10$	0
s^0	6	

As there is no sign change in first column, **system is stable.**

Ex. 7.3 $s^3 + 4s^2 + s + 16 = 0$

Sol. : $a_0 = 1, \quad a_1 = 4, \quad a_2 = 1, \quad a_3 = 16$

s^3	1	1
s^2	+ 4	16
s^1	$\frac{4 - 16}{4} = -3$	0
s^0	+ 16	

As there are two sign changes, **system is unstable.**

Number of roots located in the right half of s-plane = number of sign changes = 2.