8

THE ROOT LOCUS TECHNIQUE

8.1 INTRODUCTION

We have observed in the previous chapter that there are two related points with respect to study of stability of a control system. First, all the poles of the closed-loop system must lie on the left-hand side of the imaginary axis in the s-plane. Second, the examination has to be made to see how close the poles are with the $j\omega$-axis of the s-plane. The second aspect provides us with information regarding the relative stability of the control system.

We have also seen that the poles are the roots of the characteristic equation. The characteristic equation is obtained by substituting the denominator of the closed-loop transfer function to zero. The characteristic equation is useful in the study of performance of a control system.

From the design viewpoint, by writing the differential equation of the system and solving the differential equation with respect to a controlled variable, the time response can be found out. Thus, we will know the accurate solution of the equation and hence the performance of the system. But this approach of solution may be difficult for even a slightly complex system. Efforts involved in determining the roots of the characteristic equation have been avoided by applying Routh–Hurwitz criterion as described in the previous chapter. But this criterion only tells the designer as to whether a system is stable or unstable. The designer of a control system cannot remain satisfied with this information because he is unable to indicate the degree of stability of the system. The degree of stability will tell about the amount of overshoot, settling time, etc. for an input, say a step or a ramp input.

Root locus technique is a powerful graphical method used for the analysis and design of a control system. This method of analysis not only indicates whether a system is stable or unstable but also shows the degree of stability of a stable system.

Root locus is a plot of the roots of the characteristic equation of the closed loop as a function of gain. The effect of adjusting the closed-loop gain of the system on its stability can be studied by root locus method of stability analysis. In root locus method, gain is assumed to be a parameter which is varied from 0 to $\propto$ and the movement of the poles in the s-plane is sketched.

8.2 ROOT LOCUS CONCEPT

Consider a second-order unity feedback control system shown in Figure 8.1.

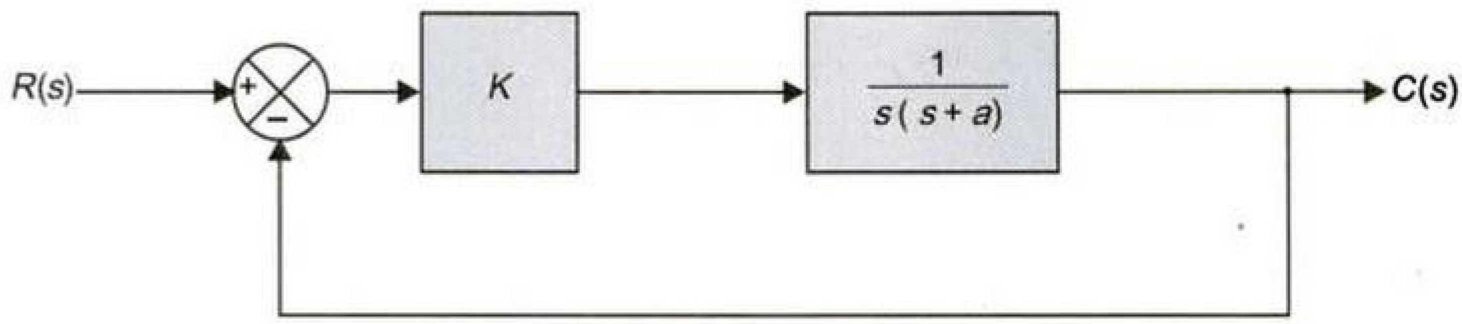


Fig. 8.1 *Block diagram of a second-order system.*

The open-loop transfer function of the system is given as

$$G(s) = \frac{K}{s(s+a)}$$

The characteristic equation is $s(s + a) = 0$.

It has two poles at $s = 0$ and $s = -a$.

The closed-loop transfer function of the system is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{K}{s^2 + as + K}$$

The characteristic equation of the system is

$$s^2 + as + K = 0 \tag{8.1}$$

The second-order system as above will be stable for positive values of a and K. Its dynamic behaviour will be controlled by the roots of the characteristic equation.

Roots of the characteristic equation are

$$s_1, s_2 = \frac{-a \pm \sqrt{a^2 - 4K}}{2}$$

or

$$s_1, s_2 = \frac{-a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 - K}$$

If we vary K from zero to infinity, the two roots (s_1, s_2) will describe a loci in the s-plane. The root locations for different values of K will change.

By examining the values of roots, that is s_1 and s_2, we observe that

(1) When $0 \le K < a^2/4$, the roots are real and distinct.

When $K = 0$ the roots are $s_1 = 0$ and $s_2 = -a$, which are the open-loop poles.

(2) When $K = a^2/4$, the roots are real and equal, that is, $s_1 = s_2 = -a/2$.

(3) When $a^2/4 < K < \infty$, the roots are complex conjugate with constant real part equal to $-a/2$.

The root loci plotted for changing values of K shown in Figure 8.2 indicates the following system behaviour.

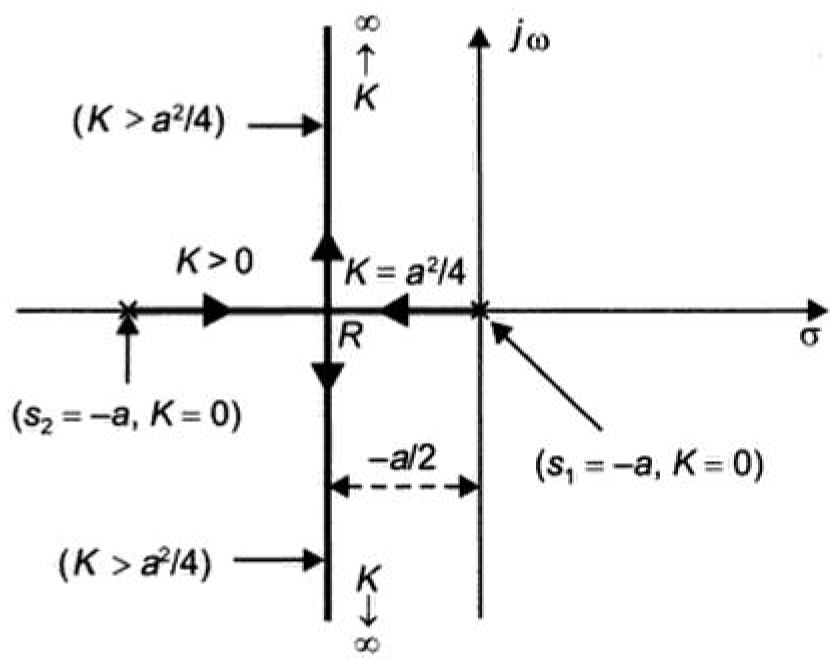


Fig. 8.2 *Root loci of a second-order closed-loop system with characteristic equation, $s^2 + as + K = 0$ and changing values of gain K.*

(1) The root locus plot has two branches starting at the two open-loop poles ($s_1 = 0$ and $s_2 = -a$) when the value of gain K is 0.

(2) As K is increased from 0 to $a^2/4$, the roots move along the real axis towards the point, R $(-a/2, 0)$ from opposite directions. The system will behave as an over-damped system. At $K = a^2/4$, the roots are $-a/2$, $-a/2$. At this value of K, the system will behave as critically damped system.

(3) At $K > a^2/4$, the roots become complex with real part equal to $-a/2$, that is, the roots break away from the real axis, become complex conjugate and move towards ∞ along the vertical line at $\sigma = -a/2$. Since the loci move away from the real axis, the system becomes under-damped. For this case, settling time is nearly constant as the real part is constant.

We have drawn the root locus by direct solution of the characteristic equation. For higher order systems, this procedure will become complicated and time consuming.

Evans in 1948 developed a simplified graphical technique for root locus plot which is described in the following. The characteristic equation of the closed-loop system shown in Figure 8.1 is $1 + G(s)H(s) = 0$

or

$$G(s)H(s) = -1 = -1 + j0 \tag{8.2}$$

(a) Magnitude criterion

From Equation (8.2), we see that the magnitude of the open-loop transfer function is equal to unity for all the roots of the characteristic equation, $G(s)H(s) = 1$. Magnitude criterion can determine the value of K for a point to be on the root locus.

(b) Angle criterion

The angle of the open-loop transfer function is an odd integral multiple of π.

$$\angle G(s)H(s) = \pm 180° \,(2q + 1);$$

where, $q = 0, 1, 2 \ldots$

The gain factor K does not affect the angle criterion.

For any point to be on the root locus in the s-plane, it has to satisfy both angle criterion and magnitude criterion. The magnitude criterion is checked after confirming the existence of the point on the root locus by applying the angle criterion. To understand this, let us consider an example where

$$G(s)H(s) = \frac{k}{s(s+1)(s+2)}$$

Let us examine whether $s = -0.5$ lies on the root locus or not.

First we apply angle criterion as

$\angle G(s)\,H(s)$ at $s = -0.5 = \pm 180°\,(2q + 1)$ where $q = 0, 1, 2, \ldots$

Here,

$$\angle G(s)\,H(s) = \frac{K}{(-0.5)(-0.5+1)(-0.5+2)} = \frac{K}{(-0.5+j0)(0.5+j0)(1.5+j0)}$$

$$= \frac{K\angle 0°}{180°\,0°\,0°} = -180°$$

Since angle criterion is satisfied, the point $s = -0.5$ lies on the root locus. Now, we also will check by applying the magnitude criterion to find the value of K for which the point $s = -0.5$ lies on the root locus.

Using magnitude condition,

$$|G(s)\,H(s)| = 1 \text{ at } s = -0.5$$

Here,

$$\frac{K}{|-0.5|\,|0.5|\,|1.5|} = 1$$

or

$$K = 0.375$$

For this value of K, point $s = -0.5$ lies on the root locus.

8.3 ROOT LOCUS CONSTRUCTION PROCEDURE

As we have seen, root locus is the graphical plot of the poles of a closed-loop system with respect to change in the gain parameter K of the system as s changes from 0 to ∞. The knowledge of open-loop poles and zeros are important here as the root locus always starts from open-loop poles and terminate on an open-loop zero or infinity. We will take up an example of plotting the root locus and along with that write the general rules or guidelines.

Example 8.1 Sketch the root locus of a control system whose transfer function is $G(s) = K/s\,(s + 1)$ with unity feedback.

Solution We have

$$G(s) = \frac{K}{s(s+1)} \quad \text{and } H(s) = 1$$

$$G(s)H(s) = \frac{K}{s(s+1)}$$

By examining the denominator of $G(s)$, we find that the number of open-loop poles $n = 2$. They are $s = 0$ and $s = -1$. As evident from the numerator, the number of open-loop zeros $m = 0$.

The closed-loop transfer function of the system is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} = \frac{K/s\,(s+1)}{1 + K/s(s+1)} = \frac{K}{s^2 + s + K}$$

The characteristic equation is

$$s^2 + s + K = 0$$

$$s_1, s_2 = \frac{-1 \pm \sqrt{1^2 - 4K}}{2}$$

$$= -0.5 \pm \sqrt{0.25 - K}$$

For $K > 0.25$, the roots are complex conjugates.

The root loci start at 0 and -1, that is, at the open-loop poles.

The number of branches or root loci = Number of open-loop poles

Number of asymptotes = Number of open-loop poles − Number of open-loop zeros

In this case, asymptotes = 2 − 0 = 2.

Determination of breakaway points:

Characteristic equation is

$$1 + G(s)H(s) = 0$$

Substituting, $$\frac{K}{s(s+1)} = -1$$

or, $$K = -s\,(s + 1)$$

or, $$K = -s^2 - s$$

We put $$\frac{dK}{ds} = 0$$

$\therefore$ $$\frac{dK}{ds} = -2s - 1 = 0$$

or, $$s = -\frac{1}{2} = -0.5$$

The two root loci, starting at 0 and –1 respectively approach each other and breakaway asymptotically at –0.5.

The value of K at the breakaway point on the real axis is calculated as

$$K = -s\,(s + 1)$$
$$= -\left[-\frac{1}{2}\left(-\frac{1}{2} + 1\right)\right]$$
$$= -\left[-\frac{1}{2} \times \frac{1}{2}\right]$$
$$= 0.25$$

For a value of $K = 0.25$, the two root loci meet at the real axis at $s = -0.5$ and breakaway at $\phi_A = 90°$ and 270° asymptotically as K increases beyond 0.25 towards infinity, where angles of asymptotes is

$$\phi_A = \frac{(2q + 1)180°}{n - m}$$

where n = number of open-loop poles and m = number of open-loop zeros.

$$= \frac{(2 \times 0 + 1) \times 180°}{2 - 0}$$
$$= 90°$$
$$q = 0, 1, 2, \ldots, (n - m) - 1$$

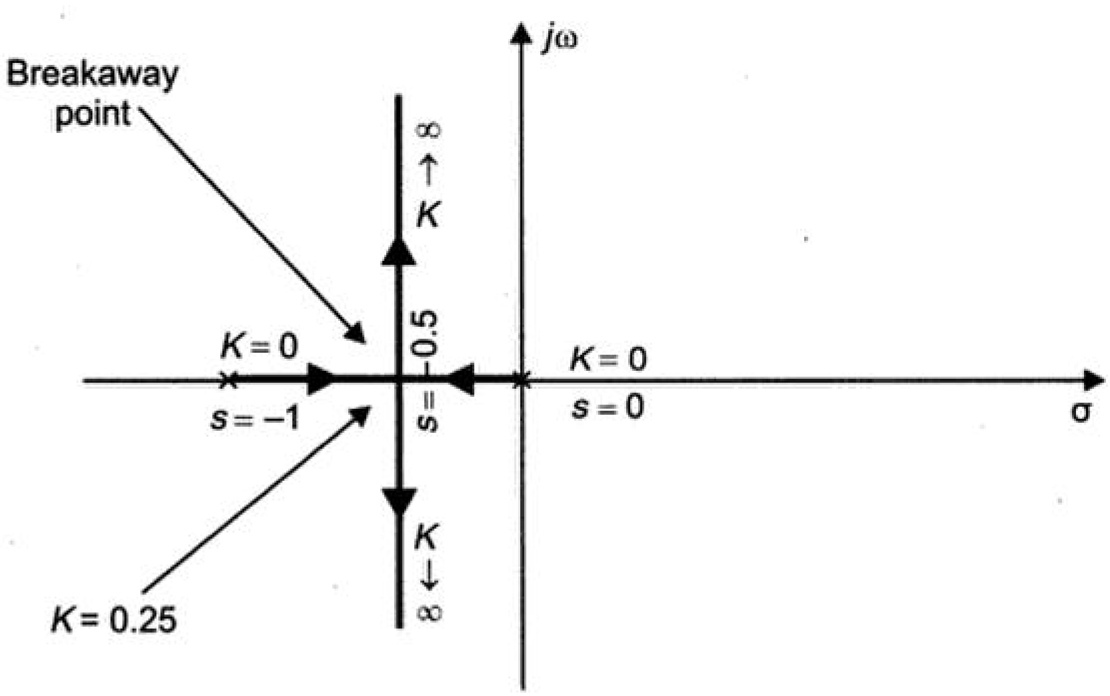


Fig. 8.3 *Root loci of $s^2 + s + K = 0$.*

and

$$\phi_A = \frac{(2 \times 1 + 1)180°}{2 - 1} = 270°$$

The root locus has been drawn as in Figure 8.3. It may be observed that the root locus is symmetrical about the real axis, that is σ axis. There is no existence of the root locus to the left pole at $s = -1$ on the real axis.

8.4 ROOT LOCUS CONSTRUCTION RULES

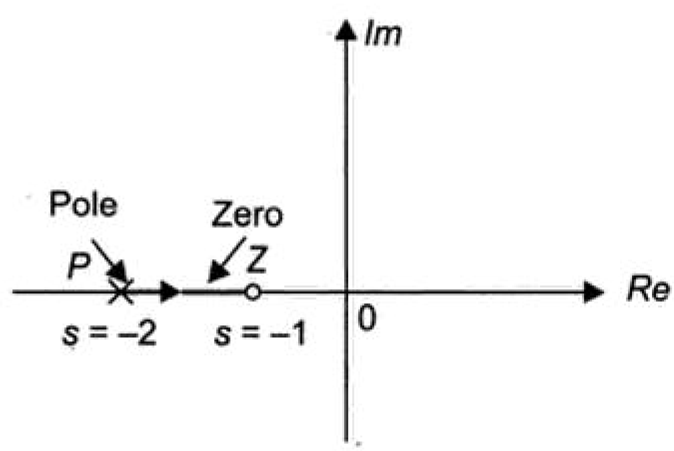
A mentioned earlier, root locus is the path of the roots of the characteristic equation, $1 + G(s) H(s) = 0$ traced out in s-plane as the system parameter (gain K) is changed.

The root locus diagram or plot can be completed using the following procedure. The procedure is presented in the form of certain rules.

(1) **Starting and termination of root locus**—From the open-loop transfer function, locate the poles and zeros. Each branch of the root locus originates from an open-loop pole with $K = 0$ and terminates either on an open-loop zero or at infinity as the value of K increases from 0 to ∞. In most cases, we will have more poles than zeros. If we have n poles and m zeros, and $n > m$, then $n - m$ branches of the root locus will reach infinity. Because the root loci originate at the poles, the number of root loci is equal to number of poles.

(2) **Root locus on the real axis**—The root locus on the real axis always lies in a section of the real axis to the left of an odd number of poles and zeros.

Let the open-loop transfer function of a control system be $G(s) = K(s + 1)/s + 2$. The pole is at $s = -2$ and the zero is at $s = -1$ as shown in Figure 8.4 (a). The root locus will start $s = -2$ and terminate at zero at $s = -1$. There is existence of root locus to the left of Z and no existence to the left of P on the real axis (root locus on real axis exists to the left of odd number of poles and zeros).



(a)

Fig. 8.4 (a) Location of poles and zeros and the root locus on the real axis.

(3) **Symmetry of the root locus**—The root loci must be symmetrical about the real axis because the complex roots appear in pairs.

(4) **The member of asymptotes and their angles with the real axis**—The $(n - m)$ branches of root loci move towards infinity. They do so along straight line asymptotes. The angle of asymptotes with respect to the real axis is given by

$$\phi_A = \frac{(2q + 1)}{n - m} 180°, \quad q = 0, 1, 2, \ldots$$

where n is the number of poles and m is the number of zeros.

(5) **Centroid of the asymptotes**—The linear asymptotes are centred at a point on the real axis. This is called the centroid which is given by the relation

$$\sigma_A = \frac{\Sigma \text{ real parts of poles} - \Sigma \text{ real parts of zeros}}{n - m}$$

(6) **Breakaway points**—The root locus breakaway from the real axis where a number of roots are available, normally, where two roots exist. The method of determining the breakaway point is to rearrange the characteristic equation in terms of K. We then evaluate $dK/ds = 0$ in order to find the breakaway point. Since the characteristic equation can have real as well as complex multiple roots, its root locus can have real as well as complex breakaway points. However, because of conjugate symmetry of root loci, the breakaway point must either be on the real axis or must occur in complex conjugate pairs.

(7) **Intersection of the root locus with the imaginary axis**—The point at which the locus crosses the imaginary axis, in case it does, is determined by applying Routh–Hurwitz criterion. The value of K for which the locus crosses the imaginary axis is calculated by equating the terms in the first column of the Routh array of s^1 and s^0 to zero.

(8) **Angle of departure of the root locus**—The angle of departure of the locus from a complex pole is calculated as

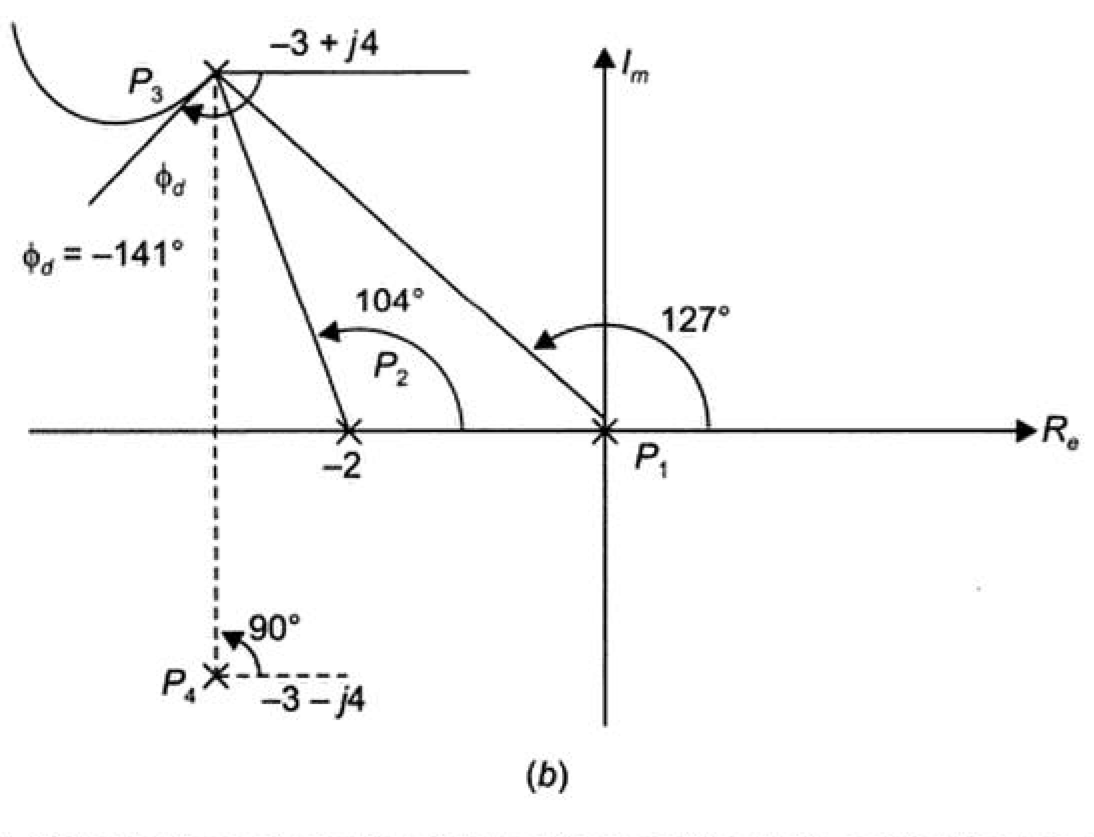
ϕ_d = 180° – sum of angles made by vectors drawn from the other poles to this pole + sum of angles made by vectors drawn from the zeros to this pole.

Let us consider an example. Let

$$G(s)H(s) = \frac{K}{s(s+2)(s^2+6s+2s)}$$

The poles are at $s_1 = 0$, $s_2 = -2$, $s_3 = -3 + j4$, $s_4 = -3 - j4$. There are no zeros. The positions of poles are shown in Figure 8.4 (b). The angle of departure of the root locus from the complex pole at P_3 is calculated as

$$\phi_d = 180° - (127° + 104° + 90°) + 0 = 180° - 321° = -141°$$



(b)

Fig. 8.4 (b) Calculation of angle of departure of the root locus from a complex pole.

8.5 ROOT LOCUS CONSTRUCTION RULES-ILLUSTRATED THROUGH EXAMPLES

The root locus construction rules have been explained in the previous section. However, for the sake of better understanding the rules are once again illustrated through examples.

The following rules are applicable in sketching the root locus plot.

Rule 1: Symmetry of root locus—Any root locus must be symmetrical about the real axis, that is, the upper half of the root locus diagram is exactly the same as the lower half about the real axis. This can be seen from any root locus diagram.

Rules 2 and 3: Starting and termination of root loci—Root locus will start from an open-loop pole with gain $K = 0$ and terminate either on an open-loop zero or to infinity with $K = \infty$.

Let us illustrate these rules with an example. Let open-loop transfer functions of control systems are

(1) $G(s)H(s) = \dfrac{K(s+1)}{(s+2)}$

(2) $G(s)H(s) = \dfrac{K}{s(s+1)}$

For (1), the root locus will start at the pole at $s = -2$ and terminate at zero at $s = -1$ as shown in Figure 8.5 (a).

For (2) the poles are at $s = 0$ and $s = -1$. As there is no zero, the locus will originate at $s = 0$ and $s = -1$ and approach towards each other and then break away to infinity as the value of gain K is increased continuously as shown in Figure 8.5 (b).

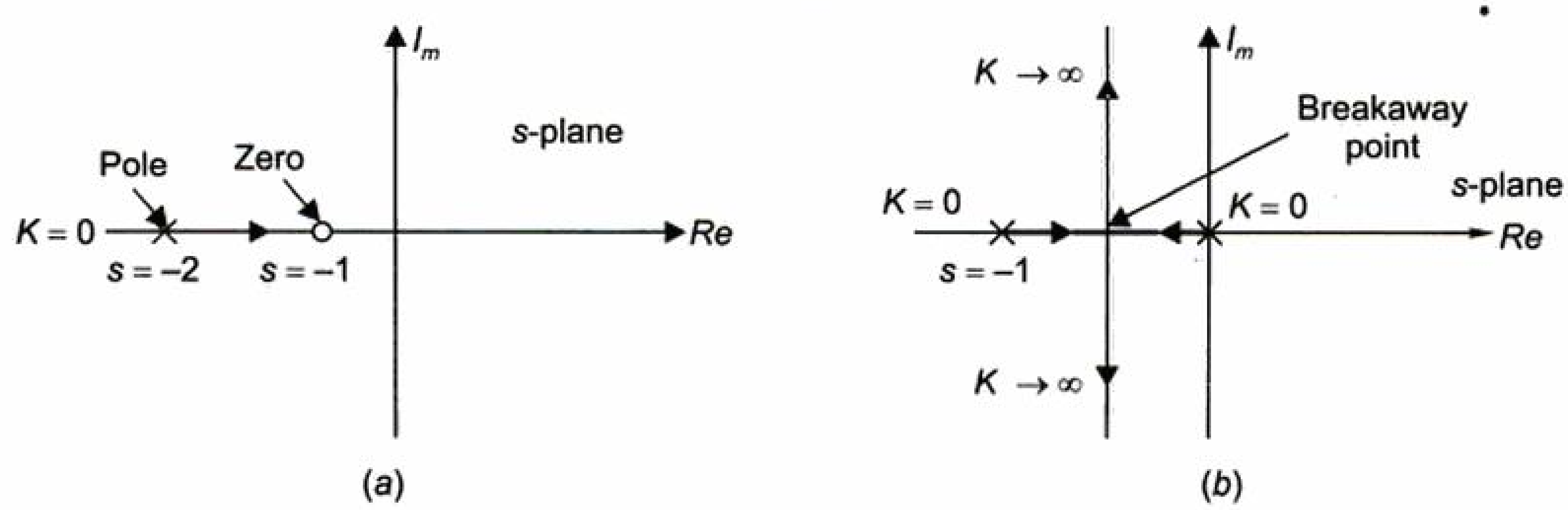


Fig. 8.5 *(a) Starting and termination of root locus where number of poles are equal to number of zeroes i.e. n = m; (b) Starting and termination of root locus when n > m.*

For

$$G(s)H(s) = \frac{K(s+1)}{s+2}$$

Rule 4: Number of root loci—If P is the number or poles and Z is the number of zeros in the transfer function $G(s)H(s)$, the number of root loci N will be as follows:

$$N = P \text{ if } P > Z$$
$$N = P = Z \text{ if } P = Z$$

For example, in the root locus shown under Rules 2 and 3, in Figure 8.5 (a) $P = 1$, $Z = 1$. Therefore, the number of root loci, $N = P = Z = 1$. And in Figure 8.5 (b), $P = 2$, $Z = 0$. Therefore, $N = 2$. The two root loci originating from origin and -1 respectively are approaching to ∞ in two directions.

Rule 5: Root loci on the real axis—The root locus on the real axis will lie in a section of the real axis to the left of an odd number of poles and zeros.

This rule is illustrated through the following examples:

(1) $$G(s)H(s) = \frac{K(s+2)(s+3)}{s(s+1)}$$

Its root locus is shown below.

There is no root locus on the real axis between P_2 and Z_1 because the number of poles and zeros to its right is even, that is 2. See Figure 8.6.

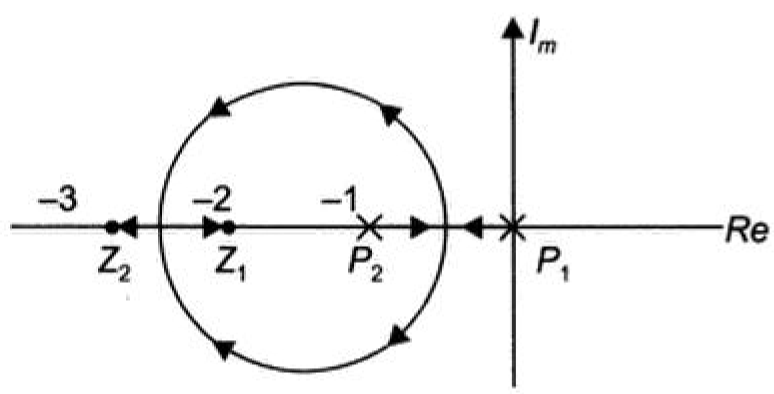


Fig. 8.6 *Location of root locus on the real axis.*

Similarly, beyond, z_2 to the left on the real axis there cannot be any root locus as the number of poles and zeros is even, that is 4 in this case.

Rule 6: The number of asymptotes and their angles with the real axis—As the value of K is increased to ∞, some branches of root locus from the real axis approach infinity along some asymptotic lines. These asymptotic lines are straight lines originating from the real axis make certain angles with the real axis. The total number of asymptotic lines and the angles they would make are calculated as follows:

Number of asymptotic lines asymptotes

$$= P - Z$$

where P is the number of poles and Z is the number of zeros of the open-loop transfer function, $G(s)H(s)$.

The angle of asymptotes with the real axis is

$$\phi_A = \frac{(2q+1)\,180^\circ}{P - Z} \quad \text{where } q = 0, 1, 2, \ldots$$

Let us consider,

$$G(s)H(s) = \frac{K}{s(s+2)}$$

Here, the number of poles $P = 2$; they are at $s = 0$ and $s = -2$ and number of zeros $Z = 0$.

Number of asymptotes $= P - Z = 2 - 0 = 2$

Let the angle of two asymptotes be ϕ_A and ϕ'_A respectively. Then,

$$\phi_A = \frac{(2 \times 0 + 1)180^\circ}{2 - 0} = 90^\circ \quad \text{for } q = 0$$

$$\phi'_A = \frac{(2 \times 1 + 1)180^\circ}{2 - 0} = 270^\circ \quad \text{for } q = 1$$

The root locus with the asymptotes are shown in Figure 8.7. One asymptote, LM is making 90° with the real axis and another asymptote LN is making an angle of 270° with the real axis. There are two root loci originating at the poles at $s = 0$ and $s = -2$. They approach each other and break away at L and approach towards infinity along the asymptotic lines as the value of K increases from 0 to ∞.

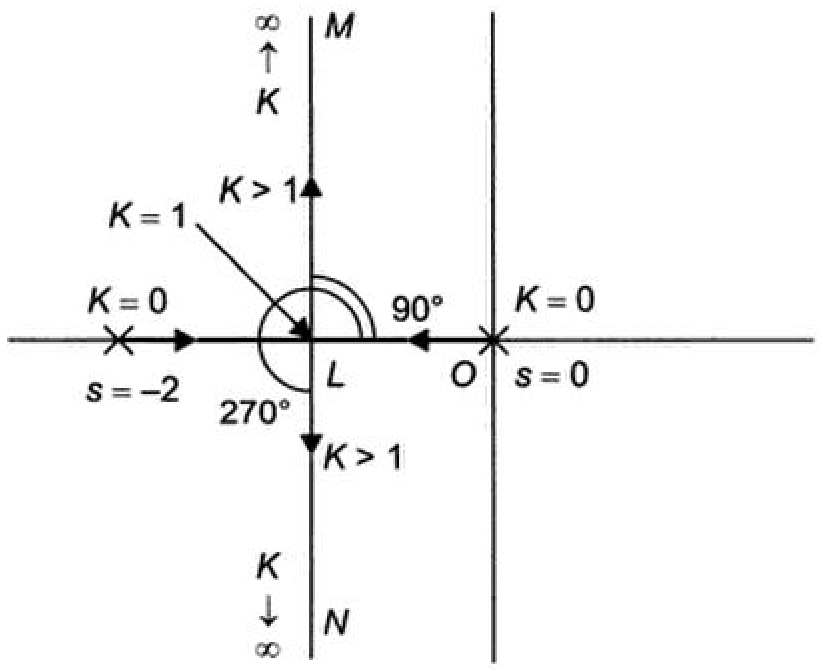


Fig. 8.7 The root locus with the asymptotes.

Rule 7: Centroid of the asymptotes—The point of intersection of the asymptotes with the real axis is called the centroid σ_A which is calculated as

$$-\sigma_A = \frac{\Sigma \text{ Real parts of poles} - \Sigma \text{ Real parts of zeros}}{P - z}$$

Let us consider the example of Rule 6 where $G(s) = K/s(s + 2)$

$$-\sigma_A = \frac{[0 - 2] - [0]}{2 - 0} = -\frac{2}{2} = -1$$

Therefore, the points L where the two asymptotes start is at a distance of -1 from the origin.

Let us consider another example,

$$G(s)H(s) = \frac{k}{s(s + 4)(s + 5)}$$

The poles are at $s = 0$, $s = -4$ and $s = -5$, which are shown below. There will be no part of the root locus to the left of B up to point A as there are even number of poles to the right of B. There will be part of root locus to the left of point A as there are odd number of poles to the right of A. There will be three root loci originating at the poles. They will terminate at zeros. If there is not zero present in $G(s)H(s)$, they will approach towards infinity as the value of K is increased.

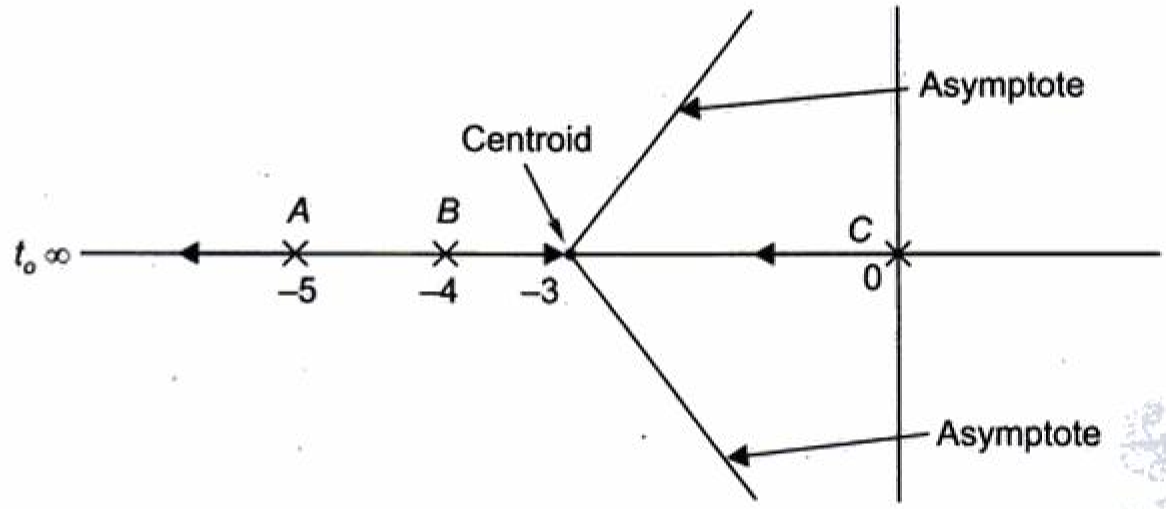


Fig. 8.8 *Centroid i.e. the point of intersection of the asymptotes.*

The root loci from B and C will approach each other along the real axis and break away towards infinity at a point on the real axis in between points B and C which is the centroid. The centroid of the asymptotes is calculated as

$$-\sigma_A = \frac{\text{Sum of real parts of poles} - \text{Sum of real parts of zeros}}{P - Z}$$

$$-\sigma_A = \frac{-5-4+0-0}{3-0} = \frac{-9}{3} = -3$$

Rule 8: Breakaway points—The root locus between two adjacent poles approaching each other break away on the real axis at a point on the real axis and move towards infinity as the value of K increases. For determining the breakaway point, we write the characteristic equation in terms of K and evaluate $dK/ds = 0$. The breakaway points will be either on the real axis or must occur as complex conjugate pairs. This is illustrated through examples as follows:

$$G(s)H(s) = \frac{K}{s(s+1)(s+2)}$$

The characteristic equations is $1 + G(s)H(s) = 0$,

or

$$s(s+1)(s+2) + K = 0$$

or

$$K = -s^3 - 3s^2 - 2s$$

$$\frac{dK}{ds} = -3s^2 - 6s - 2 = 0$$

that is,

$$3s^2 + 6s + 2 = 0$$

The roots are

$$s_1, s_2 = \frac{-6 \pm \sqrt{6^2 - 4 \times 3 \times 2}}{2 \times 3}$$

$$= -0.43, -1.57$$

The root locus sketch is shown in the following:

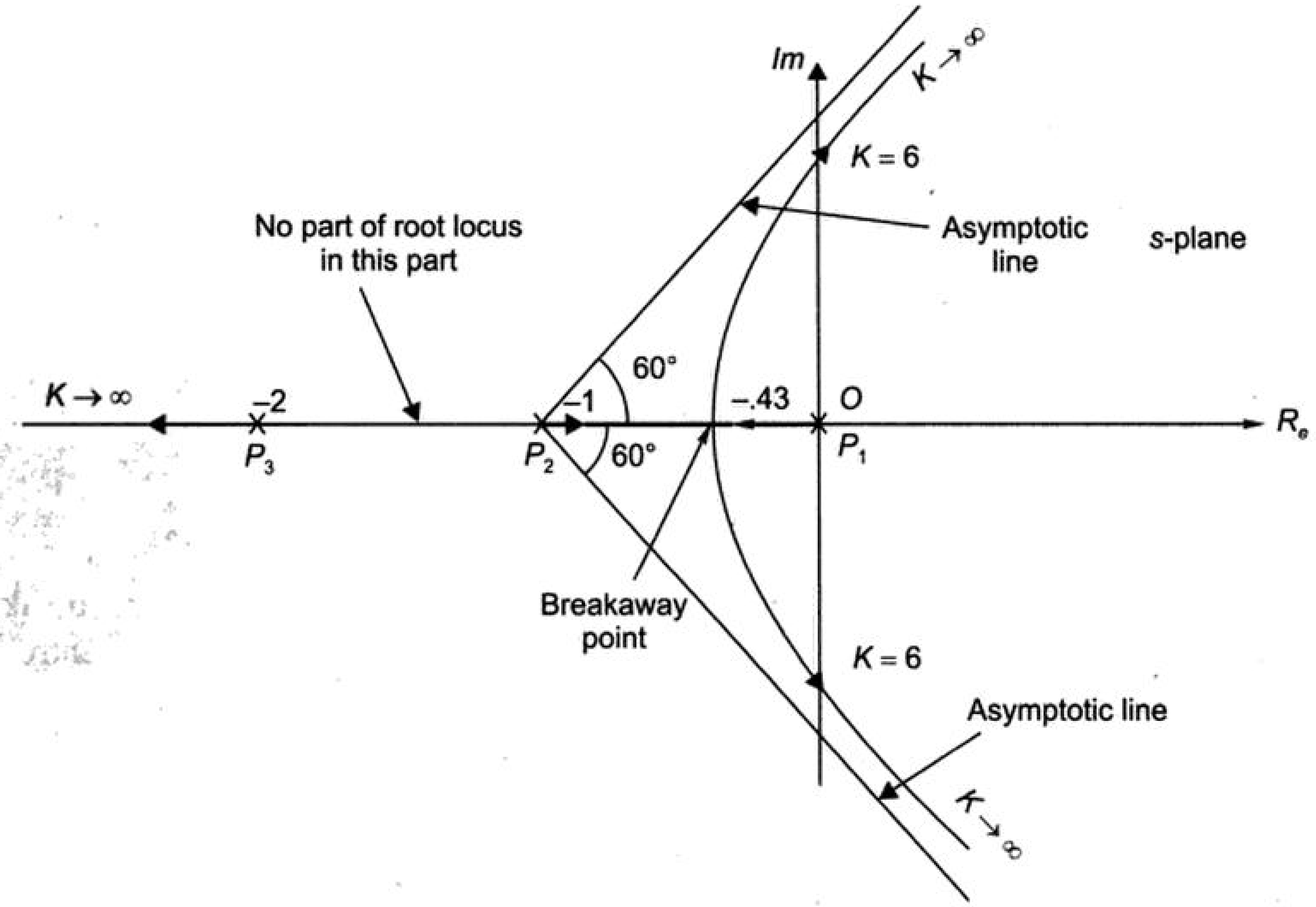


Fig. 8.9 *Determination of breakaway point of the root locus on the real axis.*

P_1, P_2 and P_3 are the positions of the poles at $s = 0$, $s = -1$ and $s = -2$, respectively. There is no zero in the numerator of the transfer function. The presence of root locus on the real axis will be to the left of odd number of poles and zeros. So there will be no part of the root locus between point –1 and –2 as has been shown in Figure 8.9. So, the breakaway point of root locus will be at –0.43 and not at –1.57.

The intersection of the root locus on the imaginary axis also has to be found out since the characteristic equation can have real as well as complex multiple roots.

The intersection of the root locus on the imaginary axis is calculated using the characteristic equation and forming the Routh's array as

$$s(s + 1)(s + 2) + K = 0$$

or

$$s^3 + 3s^2 + 2s + K = 0$$

The Routh array is represented as

s^3	1	2
s^2	3	K
s^1	$\dfrac{6 - K}{3}$	0
s^0	K	0

The value of K for which the root locus crosses the imaginary axis is calculated by equating the terms of the first column of Routh array of s^1 and s^0 to zero.

So we write,

$$\frac{6-K}{3}=0 \quad \text{and } K=0$$

Thus, $K=6$

The auxiliary equation is

$$3s^2+K=0$$

The roots of the auxiliary equation are dominant roots which are close to the imaginary axis or on the imaginary axis.

Thus,

$$3s^2=-K=-6$$

or

$$s^2=-2 \quad \text{or } s=j\sqrt{2}$$

Number of asymptotes = Number of open-loop poles
= 3.

The angles are

$$\phi_A=\frac{(2q+1)180^\circ}{P-z} \qquad q=0,1,2$$

$$=\frac{180}{3}=60^\circ \qquad \text{for } q=0$$

$$=\frac{(2+1)180^\circ}{3}=180^\circ \qquad \text{for } q=1$$

$$=\frac{(2\times 2+1)180^\circ}{3}=300^\circ \qquad \text{for } q=2\,2$$

Centroid

$$-\sigma_A=\frac{\Sigma \text{ Real parts of poles} - \Sigma \text{ Real parts of zeros}}{P-z}$$

$$=\frac{(-1-2)-0}{3-0}=-1$$

Thus, the complete root locus is as shown in Figure 8.10.

Rule 9: Angle of departure of the root locus from a complex pole and the angle of arrival at a zero—Angle of departure of the root locus from a complex pole is given as

ϕ_d = 180° – sum of angles of vectors drawn from other poles to this pole
+ sum of angles of vectors drawn to this pole from other zeros.

Let us illustrate this with an example.

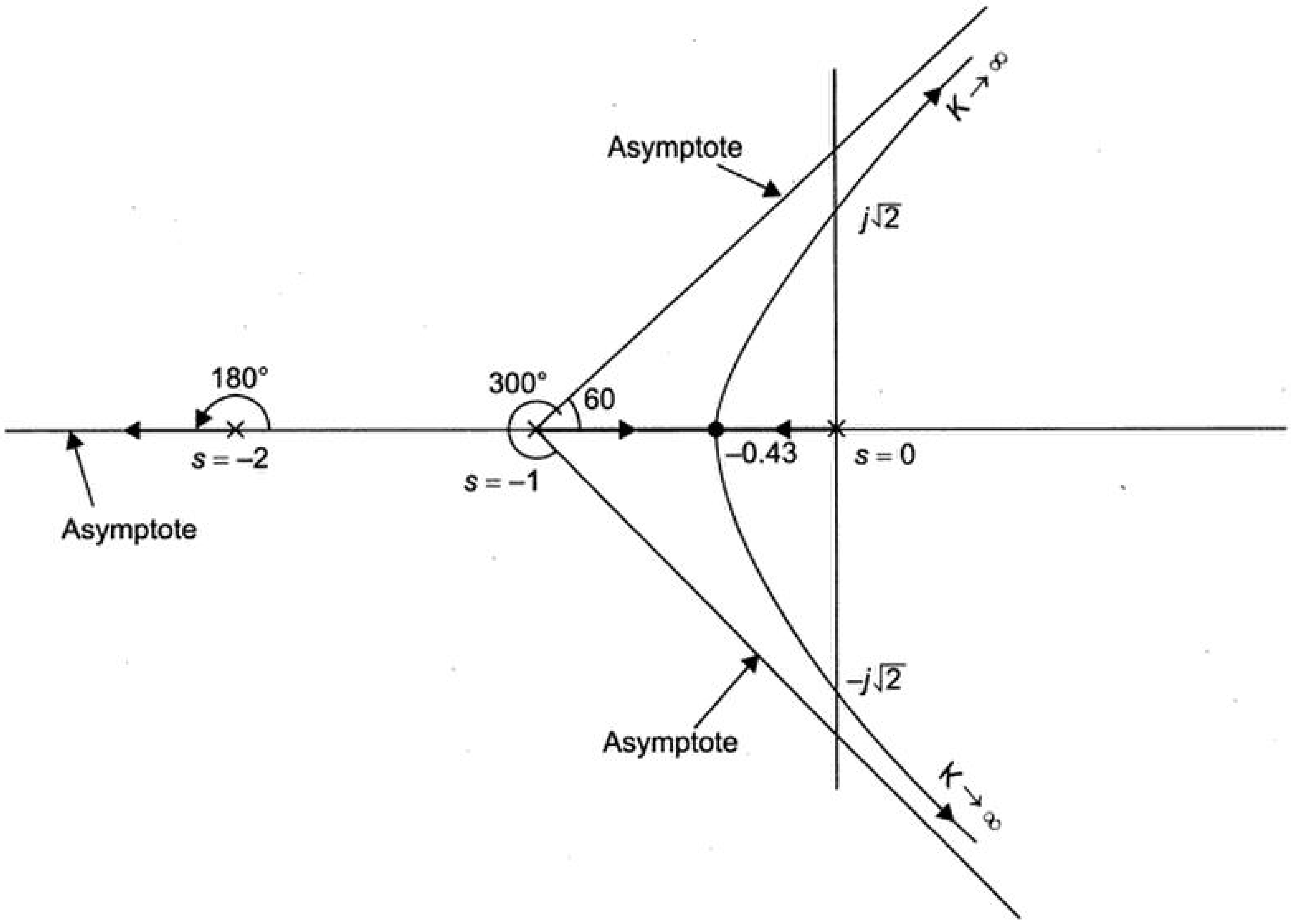


Fig. 8.10 *Determination of angle of departure of the root locus from a complex pole.*

Let

$$G(s)H(s) = \frac{K}{s(s+2)(s^2+6s+2s)}$$

The poles are at $s_1 = 0$, $s_2 = -2$, $s_3 = -3 + j4$ and $s_4 = -3 - j4$.

The positions of poles are shown in Figure 8.11. The angle of departure ϕ_A is calculated as,

$$\begin{aligned}\phi_A &= 180° - (127° + 104° + 90°)\\ &= 180° - 321° = -141°\end{aligned}$$

8.1.1 Additional Techniques

(1) **Determination of K on root loci**

For determining the value of K at any point on the root locus, we can use the following:

$$K = \frac{\text{Product of all vector lengths drawn from the poles of } G(s)H(s) \text{ to that point}}{\text{Product of all vector lengths drawn from the zeroes of } G(s)H(s) \text{ to that point}}$$

(2) **Ascertainment of any point to be on root locus**

For any point to be on the root locus in the s-plane, it has to satisfy the angle criterion and magnitude criterion. First, we apply the angle criterion to check that any point in s-plane which satisfied the angle condition has to be on the root locus. Magnitude condition is used only after confirming the existence of point on the root locus by angle condition.

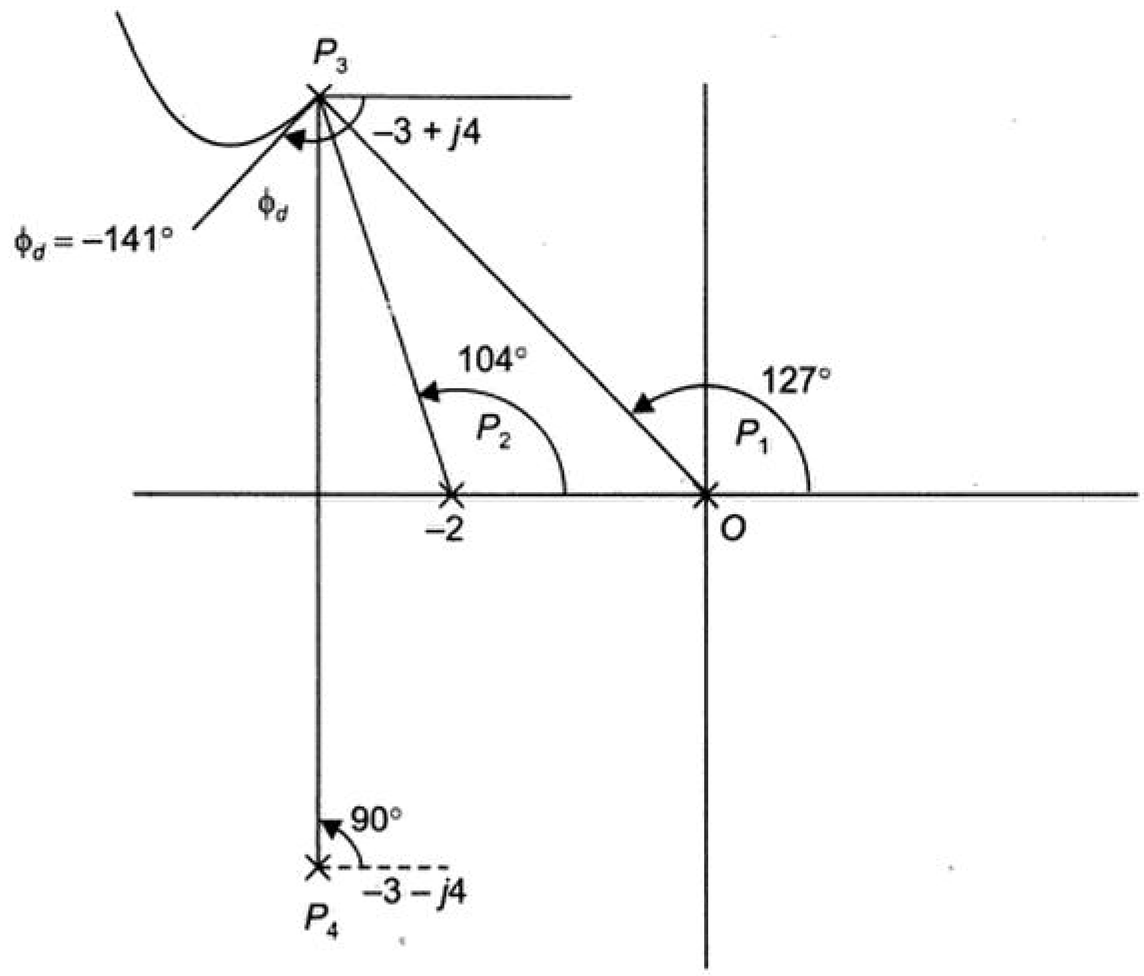


Fig. 8.11 *Method of determination of any point to be on the root locus.*

Let us consider, for example, where

$$G(s)H(s) = \frac{K}{s(s+1)(s+2)}$$

Let us examine whether $s = -0.5$ lies on the root locus or not.

First we apply angle criterion as

$$\angle G(s)H(s) \text{ at } s = -0.5 = \pm 180°(2q+1) \text{ where } q = 0, 1, 2, \ldots \qquad \text{(i)}$$

$$\angle G(s)H(s) = \frac{K}{(-0.5+j0)(-0.5+1)(-0.5+2)} = \frac{K}{(-0.5+j0)(0.5+j0)(1.5+j0)}$$

$$= \frac{K\angle 0°}{180°0°0°} = -180°$$

Thus, the angle condition as in (i) above is satisfied and the point $s = -0.5$ lies on the root locus. Now we will use magnitude condition as

$$|G(s)H(s)| = 1 \quad \text{at } s = -0.5$$

$$\frac{K}{|-0.5|\,|0.5|\,|1.5|} = 1$$

or

$$K = 0.375$$

for this value of K, point $s = -0.5$ lies on the root locus.

Example 8.2 A block diagram representation of a unity feedback control system is as follows.

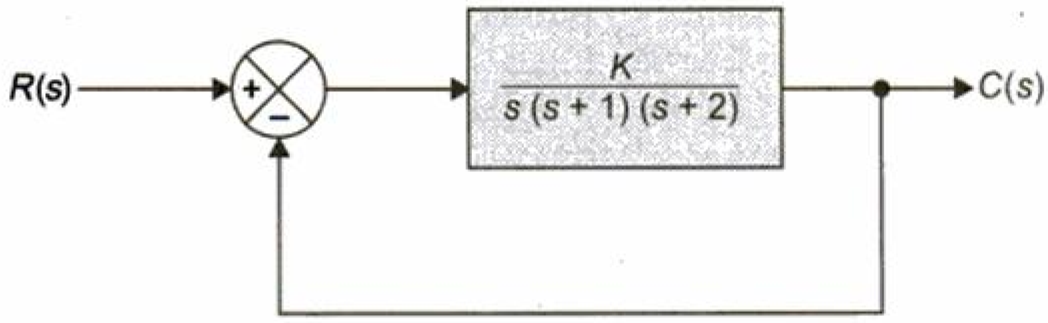


Fig. 8.12 *Block diagram of a feedback control system.*

For this system, sketch the root locus. Also determine the value of K so that the damping ratio, ξ, of a pair of complex conjugate closed-loop poles is 0.5.

Solution

$$G(s) = \frac{K}{s(s+1)(s+2)}$$

For determining the open-loop poles, we equate the denominator of $G(s)$ to 0.

$$\therefore \quad s(s+1)(s+2) = 0$$

(1) There are three open-loop poles at $s = 0$, $s = -1$ and $s = -2$.

(2) We know that the number of root locus asymptotes will be equal to number of open-loop poles minus the number of open-loop zeros. Here there is no open-loop zero.

There will be three branches of the root locus originating respectively at $s = 0$, $s = -1$ and $s = -2$.

(3) The three branches of the root locus will move towards infinity, as K increases, along the asymptotic lines whose angles with the real axis are

$$\phi_A = \frac{(2q+1)180°}{n-m}; \quad q = 0, 1, 2$$

$$= (2q+1)180° = 60°, 180°, 300°$$

(4) The root locus exist on the real axis between $s = 0$ and $s = -1$; and $s = -2$ moving toward ∞.

(5) The centroid $-\sigma_A$ is calculated as

$$-\sigma_A = \frac{\Sigma \text{ Real parts of poles} - \Sigma \text{ Real parts of zeros}}{\text{Number of poles} - \text{Number of zeros}}$$

$$= \frac{(-1-2)-0}{3-0}$$

$$= -1$$

(6) The breakaway points on the real axis is found by substituting $dK/ds = 0$.

The characteristic equation is

$$s(s+1)(s+2) + K = 0$$

or

$$K = -s^3 - 3s^2 - 2s$$

$$\frac{dK}{ds} = -3s^2 - 6s - 2 = 0$$

that is,

$$3s^2 + 6s + 2 = 0$$

$$s_1, s_2 = \frac{-6 \pm \sqrt{6^2 - 4 \times 3 \times 2}}{2 \times 3}$$

$$= -0.43, -1.57$$

(7) Intersection of the root locus on the imaginary axis is determined as follows.

The characteristic equation of the system is

$$s(s + 1)(s + 2) + K = 0$$

or

$$s^3 + 3s^2 + 2s + K = 0$$

The Routh array is

s^3	1	2
s^2	3	K
s^1	$\frac{6-K}{3}$	0
s^0	K	0

We know that the occurrence of a zero row in the Routh array indicates the presence of symmetrically located roots in the s-plane.

For this,

$$\frac{6-K}{3} = 0$$

or

$$K = 6$$

The auxiliary equation is

$$3s^2 + K = 0$$

or

$$3s^2 = -K = -6$$

or

$$s = \pm j\sqrt{2}$$

The position of poles, the asymptotes and the root locus plot have been shown in Figure 8.13.

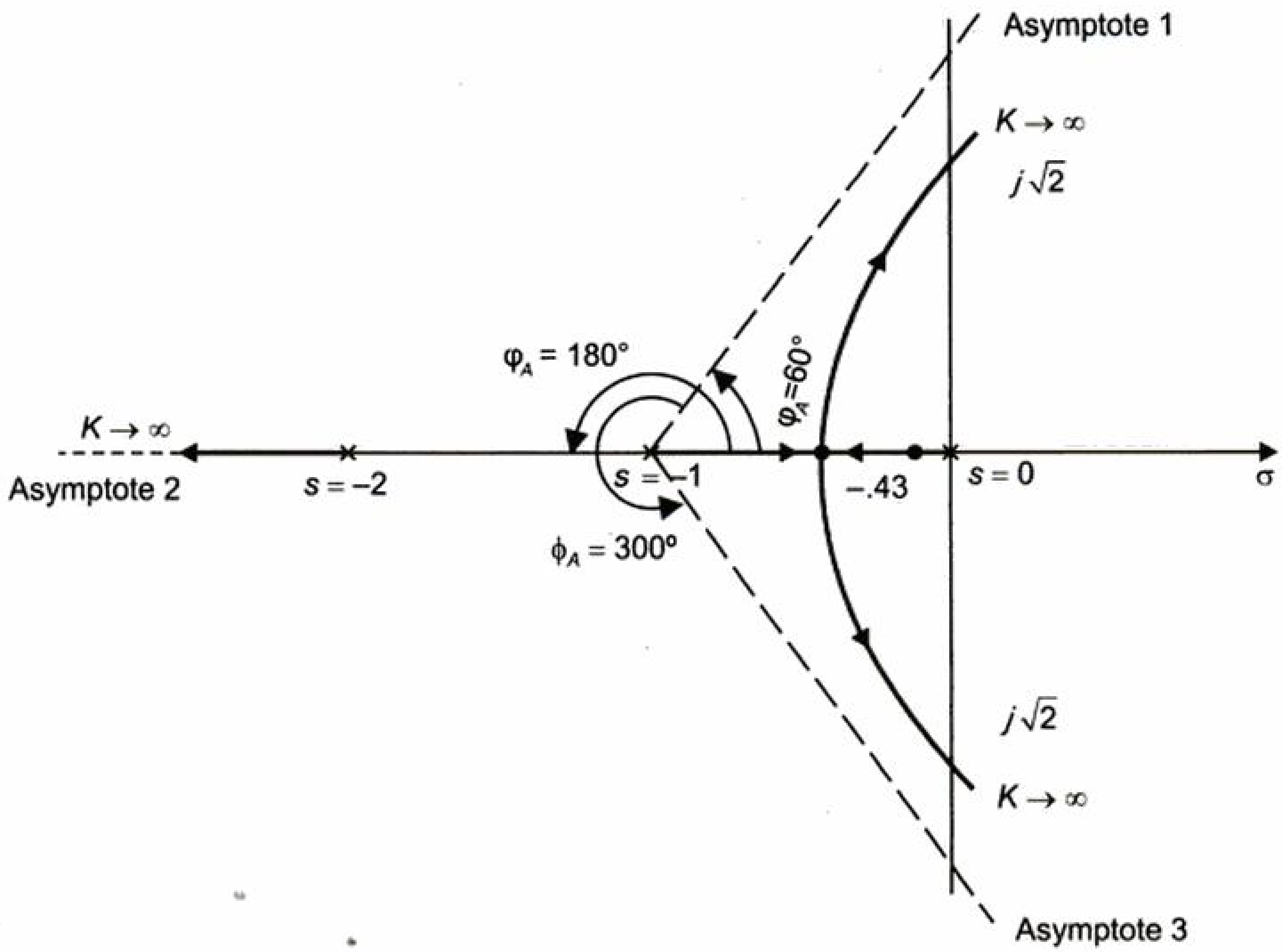


Fig. 8.13 Root locus plot for $s^3 + 3s^2 + 2s + K = 0$.

Note that for breakaway point at $s = -1.57$, the angle criterion is not satisfied and hence cannot be considered.

Example 8.3 The block diagram of a unity feedback control system is shown in Figure 8.14.

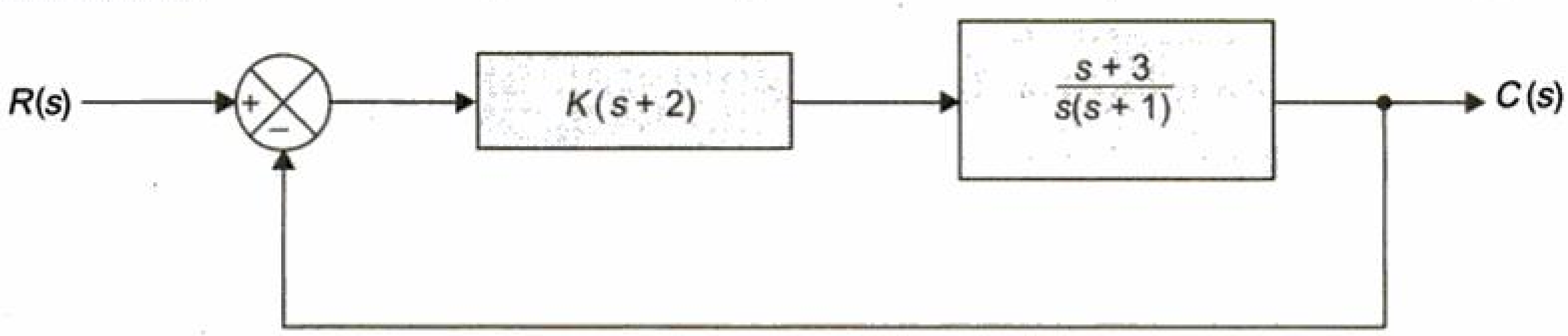


Fig. 8.14 Block diagram of a feedback control system.

Draw the root locus diagram for the above represented control system.

Solution The open-loop transfer function is

$$G(s)H(s) = \frac{K(s+2)(s+3)}{s(s+1)}$$

The number of open-loop poles is 2 at $s = 0$ and $s = -1$.

The number of open-loop zeros is 2 at $s = -2$ and $s = -3$.

Therefore, the number of root locus asymptotes = 2 – 2 = 0.

The number of root locus branches will be 2 originating at $s = 0$ and $s = -1$.

The root loci will terminate at the zeros at $s = -2$ and $s = -3$.

Now let us calculate the breakaway points. The characteristic equation is

$$1 + G(s)H(s) = 0$$

Substituting the actual values, we get

$$\frac{K(s+2)(s+3)}{s(s+1)} = -1$$

$$K = -\frac{s(s+1)}{(s+2)(s+3)} = \frac{-(s^2+s)}{s^2+5s+6}$$

We have to make $\dfrac{dK}{ds} = 0$

Thus,

$$\frac{dK}{ds} = \left[\frac{-(s^2+5s+6)(2s+1)+(s^2+5)(2s+5)}{(s^2+5s+6)^2}\right] = 0$$

or

$$2s^2 + 6s + 3 = 0$$

or

$$s_1, s_2 = -0.63, -2.36$$

The root loci branches originate at $s = 0$ and $s = -1$ and get terminated at zeros at $s = -2$ and $s = -3$. Thus, root locus exists between 0 and -1 and between -2 and -3. Thus, both the breakaway points $s = -0.63$ and $s = -2.36$ are valid (breakaway and break-in points).

The values of K at $s = -0.634$ and $s = -2.366$ are calculated as

$$K_1 = \left|\frac{(0.634)(0.366)}{(1.366)(2.366)}\right| = 0.07$$

and

$$K_2 = \left|\frac{(-2.366)(-1.366)}{(-0.366)(0.634)}\right| = 13.93$$

The root locus plot has been shown is Figure 8.15. The two root loci originate at $s = 0$ and at $s = -1$ on the real axis. As the value of K is increased, they approach each other and break away at $s = -0.634$ with value of $K = 0.07$. As the value of K increases, the two root loci makes semicircles and break in at $s = -2.366$ with value of $K = 13.93$. They get terminated at the two zeros at $s = -3$ and $s = -2$ as shown.

The root locus is a circle with its centre at -1.5 $\left(\text{i.e.} = \dfrac{2.366 - 0.634}{2} + 0.634\right)$

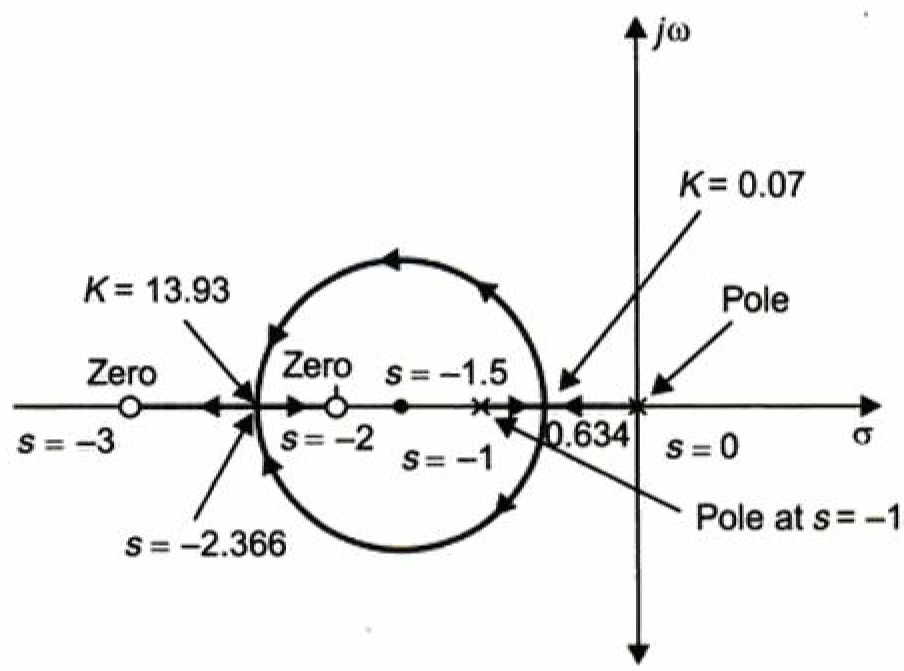


Fig. 8.15 *Root locus of* $G(s)\,H(s) = \dfrac{K(s+2)(s+3)}{s(s+1)}$.

Example 8.4 The transfer function of a system with unity feedback has been shown in Figure 8.16.

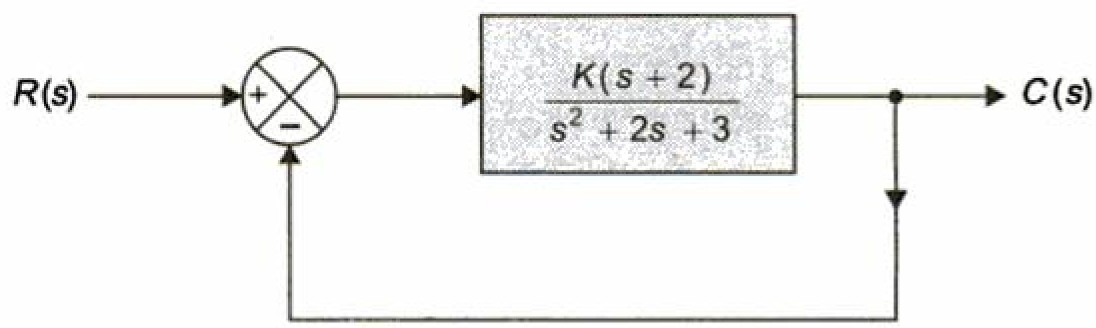


Fig. 8.16 *A unity feedback control system.*

Draw the root locus diagram for the above system.

Solution The open-loop poles are calculated from the equation $s^2 + 2s + 3 = 0$.

$$s_1, s_2 = \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 3}}{2}$$

$$= -1 \pm j\sqrt{2}$$

So, there are two open-loop poles. The open-loop zero is at $s = -2$.

The number of branches of the root locus is equal to the number of open-loop poles, that is 2.

$$\text{Number of asymptotes} = \text{Number of poles} - \text{Number of zeros}$$

$$= 2 - 1$$

$$= 1$$

Angle of the asymptote

$$\phi_A = \frac{(2q+1)180°}{n-m} \qquad q = 0$$

$$= \frac{180°}{1} = 180°$$

Now, let us determine the break-in point. For this, we take the characteristic equation of the closed-loop system as

$$1 + G(s)H(s) = 0$$

or

$$G(s)H(s) = -1$$

Substituting the values,

$$\frac{K(s+2)}{s^2+2s+3} = -1$$

or

$$K = -\frac{s^2+2s+3}{s+2}$$

Now let us calculate dK/ds and equate it to zero.

$$\frac{dK}{ds} = -\left[\frac{(s+2)(2s+2)-(s^2+2s+3)}{(s+2)^2}\right] = 0$$

$$= -\left[\frac{2s^2+6s+4-s^2-25-3}{(s+2)^2}\right] = 0$$

or

$$s^2 + 4s + 1 = 0$$

$$s_1, s_2 = -2 \pm \sqrt{3} = -3.732,\ -0.268$$

The value of K at $s = -3.732$ is

$$K = \left|\frac{s^2+2s+3}{s+2}\right| = \frac{(3.732)^2 + 2 \times 3.732 + 3}{3.732 + 2}$$

$$= 5.46$$

The break-in point at $s = -3.732$ lies between the position of zero at $s = -2$ and the infinity. At $s = -2$, the value of $K = \infty$.

Thus, the two root locus branches originate respectively from $-1 + j\sqrt{2}$ and $-1 - j\sqrt{2}$ and break in at the real axis at $s = -3.732$ as K increases to a value of 5.46 as shown in Figure 8.17.

The radius of the circle is $\sqrt{3}$, that is 1.732. The angle of take-off can be calculated as 145° as shown in Figure 8.17. Thus, the two branches starting from the open-loop zeros follow a circular path with the increase of gain factor K. As the value of K becomes 5.46, they meet at the real axis at or

$$K = -[s(s+2)(s+3)]$$
$$= -[s^3 + 5s^2 + 6s]$$

Substituting $dK/ds = 0$,

$$\frac{dK}{ds} = -[3s^2 + 10s + 6] = 0$$

$$3s^2 + 10s + 6 = 0$$

$$s = \frac{-10 \pm \sqrt{10^2 - 4 \times 3 \times 6}}{2 \times 3}$$

$$s = -\frac{10}{6} \pm \frac{5.3}{6} = -0.8, -2.54$$

The angle of asymptotes, ϕ_A, is calculated as

$$\phi_A = \frac{(2q+1)180°}{p-2} q = 0, 1, 2$$

$$= \frac{(2 \times 0 + 1)180°}{3} = 60° \quad \text{for } q = 0$$

for $q = 1$ and 2, $\phi_A = 180°$ and $300°$

The point of intersection of the asymptotes with the real axis, that is, the centroid σ_A, is calculated as

$$\sigma_A = \frac{\Sigma \text{ Poles} - \Sigma \text{ Zeros}}{P - Z} = \frac{(0-2-3)-0}{3-0}$$

$$= -\frac{5}{3} = -1.67$$

The value of K for which the system is stable is calculated by applying Routh–Hurwitz criterion using the characteristic equation.

Consider the characteristic equation,

$$s(s+2)(s+3) + K = 0$$

or

$$s^3 + 5s^2 + 6s + K = 0$$

The Routh array is

s^3	1	6
s^2	5	K
s^1	$\frac{30-K}{5}$	0
s^0	K	0

For stability, all the terms of the first column must be positive.

Hence, $K > 0$

and

$$\frac{30-K}{5} > 0, \text{ that is, } K < 30.$$

Therefore, the system is stable if $0 < K < 30$.

At $K = 30$, the system is oscillatory. For $K < 30$ all the roots will lie on the left-hand side of s-plane. If K exceeds 30, the roots will lie on the right-hand side of the s-plane. At $K = 30$, the locus will intersect the imaginary axis. The value of s at which the locus cuts the imaginary axis is calculated from the Routh array's second row as

$$5s^2 + K = 0$$

or

$$5s^2 + 30 = 0$$

or

$$s^2 = 6$$

or

$$s = \pm j\sqrt{6} = \pm j2.45 = \pm j\omega_n$$

The frequency of oscillation is equal to the distance from origin the point of intersection of the root locus with the $j\omega$ axis.

The frequency of oscillation, $\omega_n = 2.45$ rad/sec.

The root locus plot has been shown in Figure 8.22.

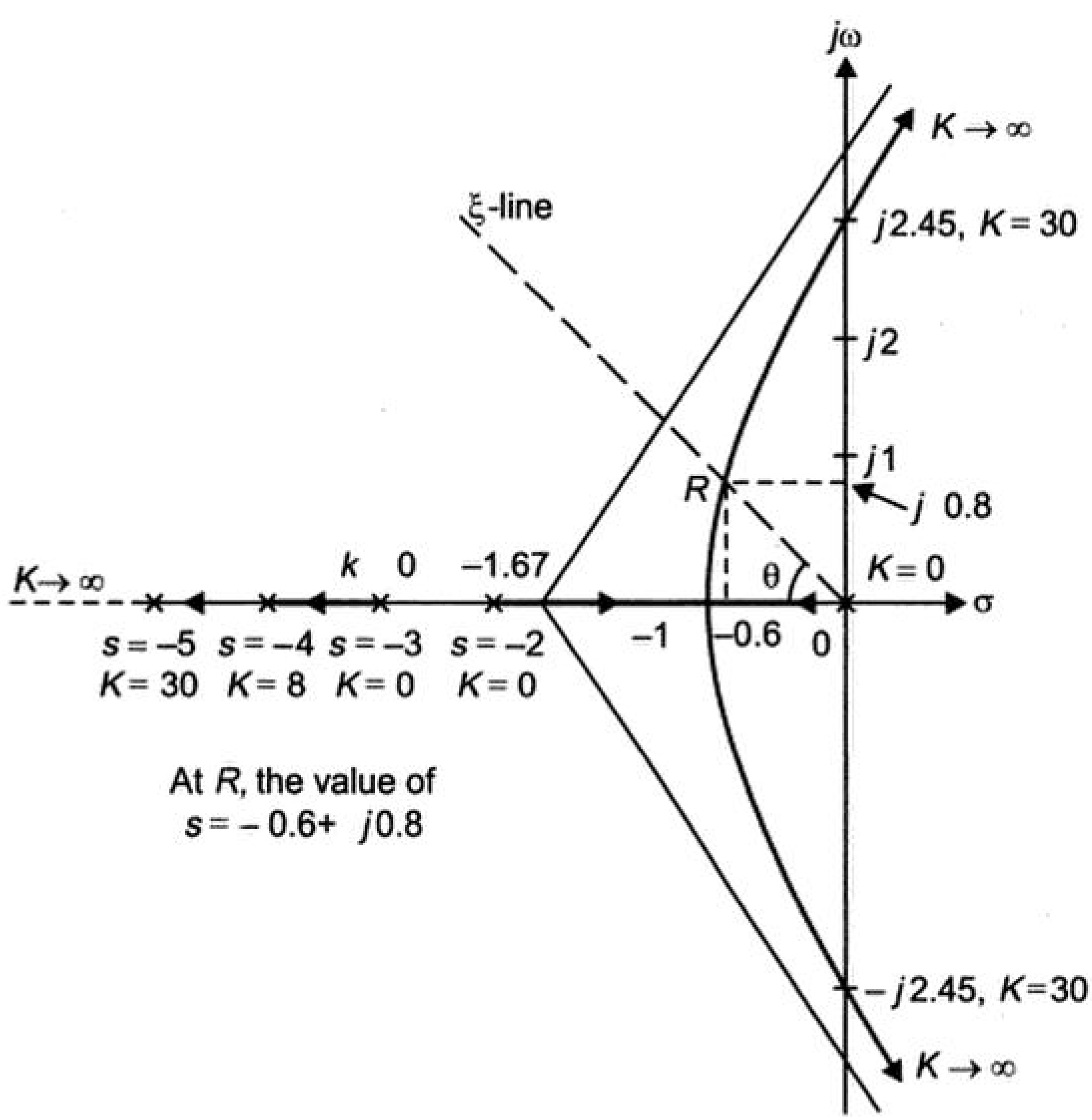


Fig. 8.22 *Root locus of unity feedback control system with* $G(s)\,H(s) = \dfrac{K}{s(s+2)(s+3)}$.