CHAPTER 7 Dynamic Programming

While considering the situations of allocation, transportation, assignment, scheduling and planning, it was assumed that the values of decision variables do not change over the planning horizon. Thus these problems were of static nature and were solved as specific situations occurring at a certain moment. However, we come across a number of situations where the decision variables vary with time, and these situations are considered to be dynamic in nature. The technique dealing with these types of problems is called *dynamic programming.* It will be shown in the body of the chapter that time element is not an essential variable, rather *any multistage situation in which a series of decisions are to be made is considered a dynamic programming problem.*

7.1 INTRODUCTION

In optimization problems involving a large number of decision variables or the inequality constraints, it may not be possible to use the methods of calculus for obtaining a solution. Classical mathematics handles the problems in a way to find the optimal values for all the decision variables simultaneously which for large problems rapidly increases the computations that become uneconomical or difficult to handle even by the available computers. The obvious solution is to split up the original large problem into small subproblems involving a few variables and that is precisely what the dynamic programming does. It uses recursive equations to solve a large, complex problem, broken into a series of interrelated decision stages (subproblems) wherein the outcome of the decision at one stage affects the decisions at the remaining stages.

Dynamic programming is a mathematical technique dealing with the optimization of multistage decision problems. The technique was originated in 1952 by Richard Bellman and G.B. Dantzig, and was initially referred to as the *stochastic linear programming*. Today dynamic programming has been developed as a mathematical technique to solve a wide range of decision problems and it forms an important part of every operation researcher's tool kit.

Though the originator of the technique, Richard Bellman, himself, has said, "we have coined the term 'dynamic programming' to emphasize that there are problems in which time plays an essential role", yet, in many dynamic programming problems time is not a relevant variable. For example, a decision regarding allocation of a fixed quantity of resources to a number of alternative uses constitutes one decision to be taken at one time, but the situation can be handled as a dynamic programming problem. As another instance, suppose a company has marked capital C to be spent on advertising its products through three different media *i.e.*, of newspaper, radio and television. In each media the advertisement can appear a number of times per week. Each appearance has associated with it certain costs and returns. How many times the product should be advertised in each media so that the returns are maximum and the total cost is within the prescribed limit ? In this situation time is not a variable, but the problem can be divided into stages and solved by dynamic programming.

7.2 DISTINGUISHING CHARACTERISTICS OF DYNAMIC PROGRAMMING

The important features of dynamic programming which distinguish it from other quantitative techniques of decision-making can be summarized as follows :

1. *Dynamic programming splits the original large problem into smaller subproblems (also called stages)* involving only a few variables, wherein the outcome of decision at one stage affects the decisions at the remaining stages.
2. *It involves a multistage process of decision-making*. The points at which decisions are called for are called stages. The stages may be certain time intervals or certain sub-divisions of the problems, for which independent feasible decisions are possible. Each stage can be thought of having a beginning and an end. The stages come in a sequence, the end of a stage forming the beginning of the next stage.
3. *In dynamic programming, the variable that links up two stages is called a state variable.* At any stage, the status of the problem can be described by the values the state variable can take. These values are referred to as states. Each stage may have, associated with it, a certain number of states. It is not essential to know about the previous decisions and how the states arise. This enables us to consider decisions one at a time.
4. *In dynamic programming the outcome of decisions depends upon a small number of variables*; that is, at any stage only a few variables should define the problem. For example, in the production smoothening problem, all that one needs to know at any stage is the production capacity, cost of production in regular and overtime, storage costs and the time remaining to the last decision.
5. *A stage decision does not alter the number of variables on which the outcome depends, but only changes the numerical value of these variables*. For the production smoothening problem, the number of variables which describe the problem *i.e.*, production capacity, production costs, storage costs and time to the last decision, remain the same at all stages. No variable is added or dropped. The effect to decision at any stage will be to alter the used production capacity, storage cost, production cost and time remaining to the last decision.
6. *Principle of Optimality. Dynamic programming is based on Bellman's Principle of Optimality, which states, "An optimal policy (a sequence of decisions) has the property that whatever the initial state and decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision*". This principle implies that a wrong decision taken at one stage does not prevent from taking of optimum decisions for the remaining stages. For example, in a production scheduling problem, wrong decisions made during first and second months do not prevent taking correct decisions during third, fourth month, etc. Using this principle of optimality, we find the best policy by solving one stage at a time, and then adding a series of one-stage-problems until the overall optimum of the original problem is attained.
7. *Bellman's principle of optimality forms the basis of dynamic programming technique.* With this principle in mind, *recursive equations* are developed to take optimal decision at each stage. A recursive equation expresses subsequent state conditions and it is based on the fact that a policy is 'optimal' if the decision made at each stage results in overall optimality over all the stages and not only for the current stage.
8. *Dynamic programming provides a systematic procedure wherein starting with the last stage of the problem and working backwards one makes an optimal decision for each stage of the problem*. The information for the last stage is the information derived from the previous stages. It may be noted that D.P. problems can also be solved by working forward *i.e.*, starting with the first stage and then working forward upto the last stage.

7.3 DYNAMIC PROGRAMMING APPROACH

Before discussing the solutions to numerical problems, it will be worthwhile to know a little more about some fundamental concepts of dynamic programming. The first concept is *stage*. As already discussed, the problem is broken down into subproblems and each subproblem is referred to as a stage. A stage signifies a portion of decision problem for which a separate decision can be made. At each stage there are a number of alternatives and the decision-making process involves the selection of one feasible alternative which may be called as *stage decision*. The stage decision may not be optimal for the considered stage, but contributes to make an overall optimal decision for the entire problem.

The other important concept is *state*. A state represents the status of the problem at a particular stage. The variables which specify the condition of decision process and summarize the current '*status*' of the system are called *state variables*. For example, in the capital budgeting problem, the capital is the state variable. The amount of capital allocated to the present stage and the preceding stages (or the capital remaining) defines the status of the problem. The number of state variables should be as small as possible. With the increase in number of state variables, increases the difficulty of problem solving.

The procedure adopted in the analysis of dynamic programming problems can be summarized as follows :

1. Define the problem variables, determine the objective function and specify the constraints.
2. Define the stages of the problem. Determine the state variables whose values constitute the state at each stage and the decision required at each stage. Specify the relationship by which the state at one stage can be expressed as a function of the state and decisions at the next stage.
3. Develop the recursive relationship for the optimal return function which permits computation of the optimal policy at any stage. Decide whether to follow the forward or the backward method to solve the problem. Specify the optimal return function at stage 1, since it is generally a bit different from the general optimal return function for the other stages.
4. Make a tabular representation to show the required values and calculations for each stage.
5. Find the optimal decision at each stage and then the overall optimal policy. There may be more than one such optimal policy.

7.4 FORMULATION OF DYNAMIC PROGRAMMING PROBLEMS

Consider a situation wherein a certain quantity 'R' of a resource (such as men, machines, money, materials, etc.) is to be distributed among 'n' number of different activities. The return 'P' depends upon the activities and the quantities of resource allotted to them and the objective is to maximize the total return.

If p_i (R_i) denotes the return form the ith activity with the resource R_i, then the total return may be expressed as

$$P(R_1, R_2, ..., R_n) = p_1(R_1) + p_2(R_2) + ... + p_n(R_n). \quad ... (7.1)$$

The quantity of resource R is limited, which gives rise to the constraint

$$R = R_1 + R_2 + ... + R_n, \quad R_i \geq 0, i = 1, 2, ..., n. \quad ... (7.2)$$

The problem is to maximize the total return given by equation (7.1) subject to constraint (7.2).

$$\text{If } f_n(R) = \max_{0 \leq R_i \leq R} [P(R_1, R_2, ..., R_n)] = \max_{0 \leq R_i \leq R} [p_1(R_1) + p_2(R_2) + ... + p_n(R_n)], \quad ... (7.3)$$

then f_n (R) is the maximum return from the distribution of the resource R to the n activities. Let us now allocate the resource to the activities, one by one, starting from the last *i.e.*, nth activity. An expression connecting f_n (R) and f_{n-1} (R) for arbitrary values of R and n may now be obtained with the help of principle of optimality. If R_n is the quantity of resource allocated to the nth activity such that $0 \leq R_n \leq R$, then regardless of the values of R_n, a quantity $(R - R_n)$ of the resource will be distributed amongst the remaining $(n - 1)$ activities. Let f_{n-1} $(R - R_n)$ denote the return from the $(n-1)$ activities, then the total return from all the n activities will be

$$p_n(R_n) + f_{n-1}(R - R_n).$$

An optimal choice of R_n will maximize the above function and thus the fundamental dynamic programming model may be expressed as

$$f_n(R) = \underset{0 \le R_n \le R}{\text{Max.}} [p_n(R_n) + f_{n-1}(R - R_n)],\ n = 2, 3, ..., \qquad ... (7.4)$$

where $f_1(R)$, when $n = 1$ is obtained from equation (7.3) as

$$f_1(R) = p_1(R). \qquad ... (7.5)$$

Equation (7.5) gives the return from the first activity when whole of the resource R is allotted to it. Once $f_1(R)$ is known, equation (7.4) provides a relation to evaluate $f_2(R)$, $f_3(R)$, ... This recursive process ultimately leads to the value of $f_{n-1}(R)$ and finally $f_n(R)$ at which the process stops.

EXAMPLE 7.4-1 (Employment Smoothening Problem)

A firm has divided its marketing area into three zones. The amount of sales depends upon the number of salesmen in each zone. The firm has been collecting the data regarding sales and salesmen in each area over a number of past years.

The information is summarized in table 7.1. For the next year firm has only 9 salesmen and the problem is to allocate these salesmen to three different zones so that the total sales are maximum.

TABLE 7.1

Profits in thousands of rupees

No. of salesmen	Zone 1	Zone 2	Zone 3
0	30	35	42
1	45	45	54
2	60	52	60
3	70	64	70
4	79	72	82
5	90	82	95
6	98	93	102
7	105	98	110
8	100	100	110
9	90	100	110

[*P.T.U. MBA May, 2002*]

Solution. In this problem the three zones represent the three stages and the number of salesmen represent the state variables.

Stage 1: We start with zone 1. The amount of sales corresponding to different number of salesmen allocated to zone 1 are given in table 7.1 and are reproduced in table 7.2.

TABLE 7.2

Zone 1

No. of salesmen :	0	1	2	3	4	5	6	7	8	9
Profit (000 '*of* ₹) :	30	45	60	70	79	90	98	105	100	90

Stage 2: Now consider the first two zones, zone 1 and 2. Nine salesmen can be divided among two zones in 10 different ways : as 9 in zone 1 and 0 in zone 2, 8 in zone 1 and 1 in zone 2, 7 in zone 1 and 2 in zone 2, etc. Each combination will have associated with it certain returns. The returns for all number of salesmen (total) 9, 8, 7, ..., 0 are shown in table 7.3.

For a particular number of salesmen, the profits for all possible combinations can be read along the diagonal. Max. profits are marked by*.

TABLE 7.3

Zone 1	x_1:	0	1	2	3	4	5	6	7	8	9
	$f_1(x_1)$:	30	45	60	70	79	90	98	105	100	90
Zone 2 x_2	$f_2(x_2)$										
0	35	65*	80*	95*	105*	114	125*	133	140	135	125
1	45	75	90	105*	115*	124	135*	143*	150	145	
2	52	82	97	112	122	131	142	150	157		
3	64	94	109	124	134	143*	154*	162			
4	72	102	117	132	142	151	162				
5	82	112	127	142	152	161					
6	93	123	138	153	163*						
7	98	128	143	158							
8	100	130	145								
9	100	130									

Stage 3: Now consider the distribution of *9* salesmen in three zones 1, 2 and 3. The decision at this stage will result in allocating certain number of salesmen to zone 3 and the remaining to zone 2 and 1 combined; and then by following the backward process, they will be distributed to zones 2 and 1.

For total of 9 salesmen to be allocated to the three zones, the returns are shown in table 7.4 below.

TABLE 7.4

No. of salesmen :	0	1	2	3	4	5	6	7	8	9
Total profit $f_2(x_2) + f_1(x_1)$:	65	80	95	105	115	125	135	143	154	163
Salesmen in zone 2 + *zone* 1:	0+0	0+1	0+2	0+3	1+3	0+5	1+5	3+4	3+5	6+3
$(x_2 + x_1)$				1+2				1+6		
No. of salesmen in Zone 3:	9	8	7	6	5	4	3	2	1	0
Profit $f_3(x_3)$:	110	110	110	102	95	82	70	60	54	42
Total profit $f_3(x_3) + f_2(x_2) + f_1(x_1)$:	175	190	205	207	210*	207	205	203	208	205

From table 7.4, the maximum profit for 9 salesmen is ₹ 2,10,000 if 5 salesmen are allotted to zone 3 and from the remaining four, 1 is allotted to zone 2 and 3 to zone 1.

EXAMPLE 7.4-2 (Capital Budgeting Problem)

A manufacturing company has three sections producing automobile parts, bicycle parts and sewing machine parts respectively. The management has allocated ₹ 20,000 for expanding the production facilities. In the auto parts and bicycle parts sections, the production can be increased either by adding new machines or by replacing some old inefficient machines by automatic machines. The sewing machine parts section was started only a few years back and thus the additional amount can be invested only by adding new machines to the section. The cost of adding and replacing the machines, along with the associated expected returns in the different sections is given in table 7.5. Select a set of expansion plans which may yield the maximum return.

TABLE 7.5

Alternatives		Auto parts section		Bicycle parts section		Sewing machine parts section	
		Cost (₹)	Return (₹)	Cost (₹)	Return (₹)	Cost (₹)	Return (₹)
1.	No expansion	0	0	0	0	0	0
2.	Add new machines	4,000	8,000	8,000	12,000	2,000	8,000
3.	Replace old m/cs	6,000	10,000	12,000	18,000	—	—

Solution

Here each section of the company is a stage. At each stage there are a number of alternatives for expansion. Capital represents the state variable. Let us consider the first stage – the auto parts section. There are three alternatives : no expansion, add new machines and replace old machines. The amount that may be allocated to stage 1 may vary from 0 to ₹ 20,000; of course, it will be overspending if it is more than ₹ 6000. The returns of the various alternatives is given in table 7.6.

TABLE 7.6

Stage 1 : Auto parts section

State x_1 (000' of ₹)	Evaluation of alternatives (Values in thousands of rupees) 1	2	3	Optimal solution	
	Cost $C_{11} = 0$ Return	Cost $C_{12} = 4$ Return	Cost $C_{13} = 6$ Return	Optimal Return	Decision
0	0	—	—	0	1
2	0	—	—	0	1
4	0	8	—	8	2
6	0	8	10	10	3
8	0	8	10	10	3
10	0	8	10	10	3
12	0	8	10	10	3
14	0	8	10	10	3
16	0	8	10	10	3
18	0	8	10	10	3
20	0	8	10	10	3

When the capital allocated is zero or ₹ 2,000, only first alternative (no expansion) is possible. Return is, of course, zero. When the amount allocated is ₹ 4,000, alternatives 1 and 2 are possible with returns of ₹ 0 and ₹ 8,000. So we select alternative 2 and when the amount allocated is ₹ 6,000, all the three alternatives are possible, giving returns of zero, ₹ 8,000 and ₹ 10,000 respectively. So we select alternative 3 with return of ₹ 10,000 and so on.

Stage 2: Let us now move to stage 2. Here, again, three alternatives are available. The computations are carried out in table 7.7.

TABLE 7.7

Stage 2 : Bicycle parts section (+ Auto parts section)

State x_2 (000'*of* ₹)	*Evaluation of alternatives (Values in thousands of rupees)*			*Optimal solution*	
	1	2	3		
	Cost $C_{21} = 0$ *Return*	*Cost* $C_{22} = 8$ *Return*	*Cost* $C_{23} = 12$ *Return*	*Optimal Return*	*Decision*
0	0 + 0 = 0	—	—	0	1
2	0 + 0 = 0	—	—	0	1
4	0 + 8 = 8	—	—	8	1
6	0 + 10 = 10	—	—	10	1
8	0 + 10 = 10	12 + 0 = 12	—	12	2
10	0 + 10 = 10	12 + 0 = 12	—	12	2
12	0 + 10 = 10	12 + 8 = 20	18 + 0 = 18	20	2
14	0 + 10 = 10	12 + 10 = 22	18 + 0 = 18	22	2
16	0 + 10 = 10	12 + 10 = 22	18 + 8 = 26	26	3
18	0 + 10 = 10	12 + 10 = 22	18 + 10 = 28	28	3
20	0 + 10 = 10	12 + 10 = 22	18 + 10 = 28	28	3

Here state x_2 represents the total amount allocated to the current stage (stage 2) and the preceding stage (stage 1). Similarly, the return also is the sum of the current stage and the preceding stage (Principle of optimality). Thus when $x_2 <$ ₹ 8,000, only the first alternative (no expansion) is possible. But with $x_2 =$ ₹ 8,000, a return of ₹ 12,000 is possible by selecting the second alternative (add new machines). With $x_2 =$ ₹ 12,000, three alternatives are possible with the maximum return of ₹ 20,000 from alternative 2. The optimal policy consists of a set of two decisions, namely adopt alternative 2 at second stage (table 7.7) and again alternative 2 at the first stage (table 7.6).

Stage 3: The computations for stage 3 are given in table 7.8.

TABLE 7.8

Stage 3 : Sewing m/c parts section (+ Bicycle parts section + Auto parts section)

State x_3 (000'*of* ₹)	*Evaluation of alternatives (Values in thousands of rupees)*		*Optimal solution*	
	Cost $C_{31} = 0$ *Return*	*Cost* $C_{32} = 2$ *Return*	*Optimal Return*	*Decision*
0	0 + 0 = 0	—	0	1
2	0 + 0 = 0	8 + 0 = 8	8	2
4	0 + 8 = 8	8 + 0 = 8	8	1, 2
6	0 + 10 = 10	8 + 8 = 16	16	2
8	0 + 12 = 12	8 + 10 = 18	18	2
10	0 + 12 = 12	8 + 12 = 20	20	2
12	0 + 20 = 20	8 + 12 = 20	20	1, 2
14	0 + 22 = 22	8 + 20 = 28	28	2
16	0 + 26 = 26	8 + 22 = 30	30	2
18	0 + 28 = 28	8 + 26 = 34	34	2
20	0 + 28 = 28	8 + 28 = 36	36	2

For x_3 = 20,000, the optimal decision for stage 3 is alternative 2, which gives a total return of ₹ 36,000. This involves a cost of ₹ 2,000 and leaves ₹ 18,000 to be allotted for stages 2 and 1 combined. From table 7.7, for allocation of ₹ 18,000, alternative 3 is to be chosen which costs ₹ 12,000. For remaining sum of ₹ 6,000, from table 7.6, decision alternative 3 is to be selected. Thus the optimal policy of expanding production facilties is 3–3–2, which can be elaborated as : replace old machines with automatics in auto parts section, replace old machines with automatics in bicycle parts section, and add new machines to the sewing machines parts section. This policy gives the optimal return of ₹ 36,000.

EXAMPLE 7.4-3

The owner of a chain of four grocery stores has purchased six crates of fresh strawberries. The following table gives the estimated profits at each store when it is allocated various number of boxes :

TABLE 7.9

Stores

		1	*2*	*3*	*4*
	0	*0*	*0*	*0*	*0*
	1	*4*	*2*	*6*	*2*
	2	*6*	*4*	*8*	*3*
Number of	*3*	*7*	*6*	*8*	*4*
boxes	*4*	*7*	*8*	*8*	*4*
	5	*7*	*9*	*8*	*4*
	6	*7*	*10*	*8*	*4*

The owner does not wish to split crates between stores, but is willing to make zero allocations. Find the allocation of six crates so as to maximize the profits.

[*P.T.U. M.Tech. April, 2012; Roorkee M.Sc. (Math.) 1977; Delhi M.Sc. (Math.) 1979; B.I.T. Ranchi B.Sc. (Prod.) 1984*]

Solution

This problem is similar to the allocation of salesmen to different zones. Here stores represent the stages and number of boxes represent the state variables. Thus the problem involves 4 stages and 6 state variables. Let x_1, x_2, x_3 and x_4 be the number of crates allocated to the 4 stores and $f_1(x_1)$, $f_2(x_2)$, $f_3(x_3)$ and $f_4(x_4)$ be the respective profits. Then the problem is

$$\text{maximize } Z = f_1(x_1) + f_2(x_2) + f_3(x_3) + f_4(x_4),$$
$$\text{subject to } \quad x_1 + x_2 + x_3 + x_4 \leq 6,$$
$$\text{where } \quad x_1, x_2, x_3, x_4 \text{ are non-negative integers.}$$

Stage 1: The estimated profits corresponding to different number of boxes allocated to store 1 are given in table 7.9 and are reproduced in table 7.10.

TABLE 7.10

Store 1

No. of boxes, x_1	:	0	1	2	3	4	5	6
Profit $f_1(x_1)$	:	0	4	6	7	7	7	7

Stage 2: Now consider the first two stores, store 1 and 2. Six boxes can be divided among the two sotres in 7 different ways : as 6 in store 1 and 0 in store 2, 5 in store 1 and 1 in store 2, etc. Each combination will have associated with it certain profits. The profits for all the total number of boxes, such as 6, 5, 4, ..., 0 are shown in table 7.11.

TABLE 7.11

Store 1	x_1:	0	1	2	3	4	5	6
	$f_1(x_1)$:	0	4	6	7	7	7	7
Store 2 x_2	$f_2(x_2)$							
0	0	0*	4*	6*	7	7	7	7
1	2	2	6*	8*	9	9	9	
2	4	4	8*	10*	11	11		
3	6	6	10*	12*	13			
4	8	8	12*	14*				
5	9	9	13					
6	10	10						

For a particular number of boxes, the profits for all possible combinations can be read along the diagonal. Maximum profits are marked by *. Thus the optimal profits and corresponding allocations of boxes to the two stores are given by :

TABLE 7.12

Boxes	:	0	1	2	3	4	5	6
$f_2(x_2) + f_1(x_1)$	:	0	4	6	8	10	12	14
$x_2 + x_1$	:	0 + 0	0 + 1	1 + 1	2 + 1	3 + 1	4 + 1	4 + 2
				0 + 2	1 + 2	2 + 2	3 + 2	

Stage 3: Now consider the distribution of 6 boxes to three stores 1, 2 and 3. The decision at this stage will result in allocating certain number of boxes to store 3 and the remaining to stores 2 and 1 combined and then by following the backward process, they will be distributed to stores 2 and 1. The profits for all the total number of boxes, such as 6, 5, 4, ..., 0 are shown in table 7.13.

TABLE 7.13

Store 1+2	*Boxes* (x_2+x_1)	0	1	2	3	4	5	6
	$f_2(x_2)+f_1(x_1)$	0	4	6	8	10	12	14
Store 3 x_3	$f_3(x_3)$							
0	0	0*	4	6	8	10	12	14
1	6	6*	10*	12*	14*	16*	18*	
2	8	8	12*	14*	16*	18*		
3	8	8	12	14	16			
4	8	8	12	14				
5	8	8	12					
6	8	8						

For any particular number of boxes, the profits for all possible combinations can be read along the diagonal. Maximum profits are marked by *. Thus the optimal profits and corresponding allocations of boxes to the three stores are given by table 7.14.

TABLE 7.14

Boxes	:	0	1	2	3	4	5	6
$f_3(x_3) + f_2(x_2) + f_1(x_1)$	:	0	6	10	12	14	16	18
$x_3 + (x_2 + x_1)$	:	0 + 0	1 + 0	1 + 1	2 + 1 1 + 2	2 + 2 1 + 3	2 + 3 1 + 4	2 + 4 1 + 5

Stage 4: Now consider the distribution of 6 boxes to four stores. The corresponding profits for all possible combinations are given in table 7.15.

TABLE 7.15

Boxes $\sum_{j=1}^{3} x_j$	:	0	1	2	3	4	5	6
$\sum_{j=1}^{3} f_j(x_j)$	:	0	6	10	12	14	16	18
x_4	:	6	5	4	3	2	1	0
$f_4(x_4)$	:	4	4	4	4	3	2	0
$\sum_{j=1}^{4} f_j(x_j)$	:	4	10	14	16	17	18^*	18^*

Thus the maximum possible profit is 18 for $x_4 = 1$ or 0. Going back, eight optimal allocations can be traced, each yielding profit of 18.

TABLE 7.16

		x_1^*	x_2^*	x_3^*	x_4^*
Possible optimal allocations	1	1	2	2	1
	2	1	3	1	1
	3	1	3	2	0
	4	1	4	1	0
	5	2	1	2	1
	6	2	2	1	1
	7	2	2	2	0
	8	2	3	1	0

EXAMPLE 7.4.4

An oil company has 8 units of money available for exploration of three sites. If oil is present at a site, the probability of finding it depends upon the amount allocated for exploiting the site, as given below.

TABLE 7.17

	Units of money allocated								
	0	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
Site 1	*0.0*	*0.0*	*0.1*	*0.2*	*0.3*	*0.5*	*0.7*	*0.9*	*1.0*
Site 2	*0.0*	*0.1*	*0.2*	*0.3*	*0.4*	*0.6*	*0.7*	*0.8*	*1.0*
Site 3	*0.0*	*0.1*	*0.1*	*0.2*	*0.3*	*0.5*	*0.8*	*0.9*	*1.0*

The probability that the oil exists at sites 1, 2 and 3 is 0.4, 0.3 and 0.2 respectively. Find the optimal allocation of money.

[*Kuru. U.M.Tech. (Mech.) May, 1994; P.U.M.E. (Mech.) Dec., 1988*]

Solution. In this oil exploration problem, the objective is to maximize the probability of finding oil by allocating the available amount of money to the three potential oil sites.

Let x_1, x_2 and x_3 be the units of money allocated to the sites 1, 2 and 3 respectively, and p_1 (x_1), p_2 (x_2) and p_3 (x_3) be the corresponding probabilities of finding oil, if it exists. Then actual probabilities of finding oil at the three sites are p_1 $(x_1) \times 0.4$, p_2 $(x_2) \times 0.3$ and p_3 $(x_3) \times 0.2$.

Thus the objective function can be written as

$$\text{maximize } Z = 0.4\, p_1 (x_1) + 0.3\, p_2 (x_2) + 0.2\, p_3 (x_3),$$

subject to constraint $x_1 + x_2 + x_3 \leq 8$,

where x_1, x_2, x_3 are non-negative integers.

The probabilities of finding the oil, taking into consideration the availabilities of oil at different sites, in the percentage form can be expressed as below.

TABLE 7.18

Units of money allocated

	0	1	2	3	4	5	6	7	8
Site 1, $f_1(x_1)$:	0	0	4	8	12	20	28	36	40
Site 2, f_2 (x_2):	0	3	6	9	12	18	21	24	30
Site 3, f_3 (x_3):	0	2	2	4	6	10	16	18	20

Here the three sites are regarded as the three stages and the money allocated is the state variable.

Stage 1: We start with site 1. The actual probabilities of finding the oil when expressed as percentages are shown in table 7.19.

TABLE 7.19

Site 1

Units of money allocated:	0	1	2	3	4	5	6	7	8
f_1 (x_1):	0	0	4	8	12	20	28	36	40

Stage 2: Now consider the first two sites 1 and 2. Eight units of money can be divided among the two sites in 9 different ways as shown in table 7.20.

TABLE 7.20

	x_1	0	1	2	3	4	5	6	7	8
	$f_1(x_1)$	0	0	4	8	12	20	28	36	40
x_2	$f_2(x_2)$									
0	0	0*	0	4	8	12*	20*	28*	36*	40*
1	3	3*	3	7	11	15	23	31	39	
2	6	6*	6	10	14	18	26	34		
3	9	9*	9	13	17	21	29			
4	12	12*	12	16	20	24				
5	18	18	18	22	26					
6	21	21	21	25						
7	24	24	24							
8	30	30								

The opitmal values of f_2 $(x_2) + f_1$ (x_1) are given in table 7.21.

TABLE 7.21

Units of money :	0	1	2	3	4	5	6	7	8
$f_2(x_2) + f_1(x_1)$:	0	3	6	9	12	20	28	36	40
$x_2 + x_1$:	0+0	1+0	2+0	3+0	4+0 0+4	0+5	0+6	0+7	0+8

Stage 3: Now consider the allocation of 8 units of money to the three sites. The corresponding probabilities expressed as percentages are shown in table 7.22.

TABLE 7.22

Units of money :	0	1	2	3	4	5	6	7	8
$f_2(x_2) + f_1(x_1)$:	0	3	6	9	12	20	28	36	40
$x_2 + x_1$:	0+0	1+0	2+0	3+0	4 + 0 0 + 4	0+5	0+6	0+7	0+8
x_3 :	8	7	6	5	4	3	2	1	0
$f_3(x_3)$:	20	18	16	10	6	4	2	2	0
$f_3(x_3) + f_2(x_2) + f_1(x_1)$:	20	21	22	19	18	24	30	38	40

Thus the maximum probability is 40%, which is obtained if $x_3 = 0$, $x_2 = 0$ and $x_1 = 8$ *i.e.*, if entire 8 units of money are allocated to site 1 only.

EXAMPLE 7.4-5

A company manufacturing a certain product has a contract of supplying 40 units at the end of month 1 and 60 units at the end of month 2. The cost of manufacturing x units in any month is c (x) = 100x + 0.4x². The company has enough production facilities to manufacture 100 units a month. If the company produces more than 40 units in month 1, any excess units can be carried over to month 2. However, there is an inventory carrying cost of ₹ 1.60 for each unit carried over from month 1 to 2. How many units should the company produce each month to minimize the total cost assuming that there is no initial inventory ?

Solution. Here month 1 and month 2 are the two stages and the number of units to be produced each month are the state variables. Let x_1 and x_2 be the number of units of the product to be produced in month 1 and 2 respectively and W be the amount of inventory at the end of month 1, which is to be carried to month 2.

Then $x_1 = 40 + W$ and $x_2 = 60 - W$.

Cost incurred in month 1, $c_1(x_1) = 100x_1 + 0.4x_1^2 = 100(40 + W) + 0.4(40 + W)^2$

$= 4{,}640 + 132W + 0.4W^2$,

and cost incurred in month 2, $c_2(x_2) = 100x_2 + 0.4x_2^2 = 100(60 - W) + 0.4(60 - W)^2$

$= 7{,}440 - 148W + 0.4W^2$

∴ Total cost incurred in the two months, including the inventory carrying cost,

$$c_1(x_1) + c_2(x_2) = 12{,}080 - 16W + 0.8W^2 + 1.60W$$

This cost is minimum if $\frac{d}{dW}(12{,}080 - 16W + 0.8W^2 + 1.60W) = 0$

or if $-16 + 1.6W + 1.60 = 0$

or if $W = 9$ units. ∴ $x_1 = 49$ units, $x_2 = 51$ units.

Minimum total cost = $12{,}080 - 16 \times 9 + 0.8 \times 9^2 + 1.60 \times 9$ = ₹ 12,015.20.

EXAMPLE 7.4-6

A manufacturer has entered into a contract for the supply of the following number of units of a product at the end of each month:

Month :	*Jan.*	*March*	*August*	*October*	*November*	*December*
No. of units :	*10*	*5*	*20*	*3*	*6*	*30*

The units manufactured during a month are available for supply at the end of the month or they may be kept in storage at a cost of ₹ 2 per unit per month. Each time the manufacture of a batch of units is undertaken, there is a set-up cost of ₹ 400. Determine the production schedule which will minimize the total cost. [*P.T.U. M.Tech. Dec., 2011; P.U. B.E. T.I.T. Dec., 2008*]

Solution. Here the six months represent the 6 stages and number of units to be manufactured are the state variables. We shall start from the last month of December and move backwards.

Month of December

The best decision is to produce 30 units with a cost of ₹ 400 towards the set-up cost and there is no storage cost.

Month of November

There are two alternatives :

1. Produce (6 + 30) = 36 units to satisfy the demand of November and December.
   Total cost = ₹ (400 + 30 × 2 × 1) = ₹ 460.
2. Produce 6 units in Nov. + 30 units in Dec. involving 2 set-ups and no storage cost.
   Total cost = ₹ (400 + 400) = ₹ 800.

∴ The optimum decision is to produce 36 units in Nov. and no units in Dec.

Month of October

Various alternatives are :

1. Produce (3 + 6 + 30) = 39 units in Oct.
   Total cost = ₹ (400 + 6 × 2 × 1 + 30 × 2 × 2) = ₹ 532.
2. Produce (3 + 6) = 9 units in Oct. and 30 units in Dec.
   Total cost = ₹ (400 × 2 + 6 × 2 × 1) = ₹ 812.
3. Produce 3 units in Oct. and 36 units in Nov.
   Total cost = ₹ (400 × 2 + 30 × 2 × 1) = ₹ 860.

Note that as per the decision made in Nov., producing 3 units in Oct., 6 in Nov. and 30 in Dec. is already ruled out as it involves higher cost.

Thus optimum decision is to produce 39 units in Oct. and nothing in Nov. and Dec.

Month of August

The various possible alternatives are :

1. Produce (20 + 3 + 6 + 30) = 59 units in August.
   Total cost = ₹ (400 + 3 × 2 × 2 + 6 × 2 × 3 + 30 × 2 × 4) = ₹ 688.
2. Produce (20 + 3 + 6) = 29 units in August and 30 in Dec.
   Total cost = ₹ [(400 × 2) + 3 × 2 × 2 + 6 × 2 × 3] = ₹ 848.
3. Produce (20 + 3) = 23 units in August and 36 in Nov.
   Total cost = ₹ [400 × 2 + 3 × 2 × 2 + 30 × 2 × 1] = ₹ 872.
4. Produce 20 units in August and 39 units in Oct.
   Total cost = ₹ [400 × 2 + 6 × 2 × 1 + 30 × 2 × 2] = ₹ 932.

Thus optimum decision is to produce 59 units in August and none in the following months.

Month of March

The various possible alternatives are :

1. Produce all (5 + 20 + 3 + 6 + 30) = 64 units in March.
   Total cost = ₹ [400 + 20 × 2 × 5 + 3 × 2 × 7 + 6 × 2 × 8 + 30 × 2 × 9]
   = ₹ 1,278.
2. Produce (5 + 20 + 3 + 6) = 34 units in March and 30 in Dec.
   Total cost = ₹ [400 × 2 + 20 × 2 × 5 + 3 × 2 × 7 + 6 × 2 × 8] = ₹ 1,138.
3. Produce (5 + 20 + 3) = 28 units in March and 36 units in Nov.
   Total cost = ₹ [400 × 2 + 20 × 2 × 5 + 3 × 2 × 7 + 30 × 2 × 1] = ₹ 1,102.

4. Produce (5 + 20) = 25 units in March and 39 units in Oct.

   Total cost = ₹ [400 × 2 + 20 × 2 × 5 + 6 × 2 × 1 + 30 × 2 × 2] = ₹ 1,132.

5. Produce 5 units in March and 59 units in August.

   Total cost = ₹ [400 × 2 + 3 × 2 × 2 + 6 × 2 × 3 + 30 × 2 × 4] = ₹ 1,088.

   ∴ The optimum decision is to produce 5 units in March and 59 units in August. The total cost involved is ₹ 1,088.

Month of January

The various possible alternatives are :

1. Produce all 74 units in January.

   Total cost = ₹ [400 + 5 × 2 × 2 + 20 × 2 × 7 + 3 × 2 × 9 + 6 × 2 × 10 + 30 × 2 × 11]

   = ₹ 1,534.

2. Produce (10 + 5 + 20 + 3 + 6) = 44 units in January and 30 units in December.

   Total cost = ₹ [400 × 2 + 5 × 2 × 2 + 20 × 2 × 7 + 3 × 2 × 9 + 6 × 2 × 10]

   = ₹ 1,274.

3. Produce (10 + 5 + 20 + 3) = 38 units in Janaury and 36 units in Nov.

   Total cost = ₹ [400 × 2 + 5 × 2 × 2 + 20 × 2 × 7 + 3 × 2 × 9 + 30 × 2 × 1]

   = ₹ 1,214.

4. Produce (10 + 5 + 20) = 35 units in January and 39 units in Oct.

   Total cost = ₹ [400 × 2 + 5 × 2 × 2 + 20 × 2 × 7 + 6 × 2 × 1 + 30 × 2 × 2]

   = ₹ 1,232.

5. Produce (10 + 5) = 15 units in January and 59 units in August.

   Total cost = ₹ [400 × 2 + 5 × 2 × 2 + 3 × 2 × 2 + 6 × 2 × 3 + 30 × 2 × 4]

   = ₹ 1,108.

6. Produce 10 units in January, 5 units in March and 59 units in August.

   Total cost = ₹ [400 × 3 + 3 × 2 × 2 + 6 × 2 × 3 + 30 × 2 × 4] = ₹ 1,488.

Thus optimum decision is to produce 15 units in Jan. and 59 units in August.

Therefore, the best production schedule that will minimize total cost and satisfy the demand from January till December is to produce 15 units in January and 59 units in August.

EXAMPLE 7.4-7

A salesman is planning a business tour from Mumbai to Kolkata in the course of which he proposes to cover one city from each of the company's different marketing zones en route. As he

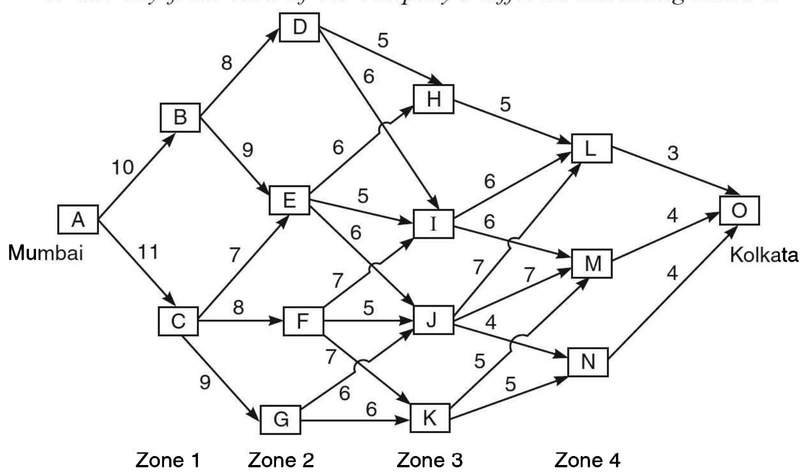

Fig. 7.1

has limited time at his disposal, he has to complete his tour in the shortest possible time. The network in Fig. 7.1 shows the number of days' time involved for covering any of the various intermediate cities (time includes travel as well as working time). Determine the optimum tour plan.

Solution. Starting from A, the cities of various marketing zones may be considered as distinct stages.

Stage 1 : B or C ?
Stage 2 : D, E, F or G ?
Stage 3 : H, I, J or K ?
Stage 4 : L, M or N ?
Stage 5 : Best route to O.

Stage 1: At this stage it is not known whether B lies on the overall shortest route; but if it does, the shortest route from A to B is AB.

$$\left.\begin{array}{l}\text{A to B} = 6\\ \text{A to C} = 11\end{array}\right\}\text{ the only routes.}$$

Stage 2: It is not known whether D lies on the overall shortest route; but if it does, the only route from A is ABD = 10 + 8 = 18.

Similarly, ABE = 10 + 9 = 19
ACE = 11 + 7 = 18
ACF = 11 + 8 = 19
ACG = 11 + 9 = 20.

From the above, shortest routes are :

$$\left.\begin{array}{l}\text{A to D} = 18\\ \text{A to E} = 18\\ \text{A to F} = 19\\ \text{A to G} = 20.\end{array}\right\}$$

Stage 3: It is not known whether H lies on the overall shortest route; but if it does, is it through D or E ?

Both D and E are reached in 18 days by the quickest route from A (from the optimal result from stage 2).

Therefore ADH = 18 + 5 = 23
AEH = 18 + 6 = 24
Similarly, ADI = 18 + 6 = 24
AEI = 18 + 5 = 23
AFI = 19 + 7 = 26
AEJ = 18 + 6 = 24
AFJ = 19 + 5 = 24
AGJ = 20 + 6 = 26
AFK = 19 + 7 = 26
AGK = 20 + 6 = 26.

From the above, shortest routes from A are

$$\left.\begin{array}{l}\text{A to H} = 23\\ \text{A to I} = 23\\ \text{A to J} = 24\\ \text{A to K} = 26.\end{array}\right\}$$

Stage 4: Proceeding in the same way as for stage 3, we have

AHL = 23 + 5 = 28
AIL = 23 + 6 = 29

$$\begin{aligned} AJL &= 24 + 7 = 31 \\ AIM &= 23 + 6 = 29 \\ AJM &= 24 + 7 = 31 \\ AKM &= 26 + 5 = 31 \\ AJN &= 24 + 4 = 28 \\ AKN &= 26 + 5 = 31. \end{aligned}$$

∴ The shortest routes from A are

$$\left.\begin{aligned} \text{A to L } &= 28 \\ \text{A to M} &= 29 \\ \text{and A to N} &= 28. \end{aligned}\right\}$$

Final stage: There are three alternatives to reach O from the 4th stage *viz*. LO, MO and NO. Using the optimal times at 4th stage,

$$\left.\begin{aligned} ALO &= 28 + 3 = 31 \\ AMO &= 29 + 4 = 33 \\ ANO &= 28 + 4 = 32. \end{aligned}\right\}$$

Thus the shortest time from A to O = 31. Now we retrace the steps backwards along the network to identify the intermediate cities along the shortest route.

A – O — Final
A – L – O — Stage 4
A – H – L – O — Stage 3
A – D – H – L – O — Stage 2
A – B – D – H – L – O — Optimal route.

The problem of finding the shortest route is known as the *stage coach problem.*

EXAMPLE 7.4-8

A dealer has to dispose of certain goods within 5 weeks' time. The market prices are fluctuating from week to week. It is estimated that the chances of getting ₹ 2,000 for the whole stock are 45%, chances of getting ₹ 2,500 are 35% and chances for getting ₹ 3,000 are 20%. If the goods are not sold in the first 4 weeks, then they will have to be disposed of in the fifth week at the prevailing market price in that week. When should the stocks be sold ?

[*P.T.U. M.Tech. April, 2012*]

Solution. The five weekly periods can be treated as five stages and the sale prices are the state variables. At each stage the dealer has to make a choice among the alternatives 'sell' and 'wait'. If the prevailing market price is more than that he expects in the following weeks, he should 'sell' and if it is less, he should 'wait'.

These types of problems can be conveniently handled by starting from the last stage and solving by the *backward induction* or *backward process.*

Fifth Week (*stage*): As it is essential to sell the stocks by the fifth week, he will get ₹ 2,000 or ₹ 2,500 or ₹ 3,000 depending upon the price prevailing in the market in that week.

Fourth Week (*stage*): The prevailing price may be ₹ 2,000, ₹ 2,500 or ₹ 3,000. If it is ₹ 3,000, he will naturally sell the stock. If it is less than ₹ 3,000, he may decide to sell or wait. He will sell if the market price in 4th week is more than the expected return in the 5th week, otherwise he will wait.

$$\begin{aligned} \text{Now expected return in the 5th week } &= ₹\ [2{,}000 \times .45 + 2{,}500 \times .35 + 3{,}000 \times .20] \\ &= ₹\ 2{,}375. \end{aligned}$$

Thus if the market price in the 4th week is ₹ 3,000 or ₹ 2,500 he should sell; otherwise he should wait.

Third week (*stage*): If the prevailing price is ₹ 3,000 he should sell. If the price is less, he will still sell if he can get more than the expected return in the 4th week; otherwise he will wait.

Expected return in the 4th week = ₹ [3,000 × .20 + 2,500 × .35 + 2,375 × .45]
= ₹ 2,544.

Thus if the market price in the 3rd week is ₹ 3,000, he should sell; otherwise he should wait.

Second week (*stage*): If the prevailing price is ₹ 3,000 he should sell. If not, he should see the expected return in the third week.

Expected return in the 3rd week = ₹ [3,000 × 0.20 + 2,544 × 0.35 + 2,544 × 0.45]
= ₹ 2,635.

So, he should wait.

First week (*stage*): If the prevailing price is ₹ 3,000 he should sell; if not, expected return in the 2nd week will determine his decisioin.

Expected return in the 2nd week = ₹ [3000 × .20 + 2,635 × .35 + 2,635 × .45]
= ₹ 2,708.

So, he should wait.

Thus we reach at the following optimal decision policy: The dealer should sell goods if the market price in the first, second and third weeks is ₹ 3,000; otherwise wait. In the fourth week, if he can get ₹ 2,500 or ₹ 3,000 he should sell; if it is less than ₹ 2,500 he should wait. If the goods remain unsold upto the 5th week he should dispose of the stock at whatever value he gets in that week.

EXAMPLE 7.4-9 (Cargo Loading Problem)

In a cargo loading problem, there are 4 items of different weights/unit and different value/unit as given below.

Item (i)	*Weight / unit (w_i, kg/unit)*	*Value / unit (p_i, ₹/unit)*
1	1	1
2	3	5
3	4	7
4	6	11

The maximum cargo load is restricted to 17. How many units of each item be loaded to maximize the value ? *[P.U.B.E(T.I.T.) Nov., 2005]*

Solution. It is a four-stage problem, each item represents a stage. The state of the system is represented by the weight capacity available for allocation to stages 1, 2, 3, 4 and is denoted by x_i which varies from 0 to 17. If a_i is the number of units of item i, then the problem is

$$\text{maximize } Z = \sum_{i=1}^{4} a_i\, p_i\,,$$

$$\text{subject to } \sum_{i=1}^{4} a_i\, w_i \le W\,.$$

Stage 1: Here, $w_1 = 1$ kg/unit, p_1 = Re. 1/unit; $\dfrac{W}{w_1} = \dfrac{17}{1} = 17.$

$\therefore\ a_1 = 0, 1, 2, ..., 17.$

Stage 2: Here, $w_2 = 3$kg/unit, p_2 = ₹ 5/unit ; $\dfrac{W}{w_2} = \dfrac{17}{3}$

$= 5.67$ (= 5, integral value).

$\therefore\ a_2 = 0, 1, 2, ..., 5.$

TABLE 7.23

x_i	Stage 1		Stage 2		Stage 3		Stage 4		$f_i^*(x_i)$
	$w_1 = 1, p_1 = 1$ $a_1 = 0,1,2, ...,17$		$w_2 = 3, p_2 = 5$ $a_2 = 0,1,2, ..., 5$		$w_3 = 4, p_3 = 7$ $a_3 = 0,1,2,3,4$		$w_4 = 6, p_4 = 11$ $a_4 = 0,1,2$		
	a_1	$f_1(x_1)$	a_2	$f_2(x_2)$	a_3	$f_3(x_3)$	a_4	$f_4(x_4)$	
0	0	0*	0	—	0	—	0	—	0
1	1	1*	0	—	0	—	0	—	1
2	2	2*	0	—	0	—	0	—	2
3	3	3	1	5+0=5*	0	—	0	—	5
4	4	4	1	5+1=6	1	7+0=7*	0	—	7
5	5	5	1	5+2=7	1	7+1=8*	0	—	8
6	6	6	2	10+0=10	1	7+2=9	1	11+0=11*	11
7	7	7	2	10+1=11	1	7+5=12*	1	11+1=12*	12
8	8	8	2	10+2=12	2	14+0=14*	1	11+2=13	14
9	9	9	3	15+0=15	2	14+1=15	1	11+5=16*	16
10	10	10	3	15+1=16	2	14+2=16	1	11+7=18*	18
11	11	11	3	15+2=17	2	14+5=19*	1	11+8=19*	19
12	12	12	4	20+0=20	3	21+0=21	2	22+0=22*	22
13	13	13	4	20+1=21	3	21+1=22	2	22+1=23*	23
14	14	14	4	20+2=22	3	21+2=23	2	22+2=24*	24
15	15	15	5	25+0=25	3	21+5=26	2	22+5=27*	27
16	16	16	5	25+1=26	4	28+0=28	2	22+7=29*	29
17	17	17	5	25+2=27	4	28+1=29	2	22+8=30*	30

Stage 3: Here, $w_3 = 4$ kg/unit, $p_3 = ₹\ 7$/unit; $\frac{W}{w_3} = \frac{17}{4} = 4.25.$

∴ $a_3 = 0, 1, 2, 3, 4.$

Stage 4: Here, $w_4 = 6$ kg/unit, $p_4 = ₹\ 11$/unit; $\frac{W}{w_4} = \frac{17}{6} = 2.83.$

∴ $a_4 = 0, 1, 2.$

Let $f_1(x_1), f_2(x_2), f_3(x_3)$ and $f_4(x_4)$ be the values of the loaded items at stage 1, 2, 3 and 4 respectively. The computation for different stages are given in table 7.23.

As seen from the table, for total load of 17 kg, the maximum value of cargo items is ₹ 30 (= 22 + 8 = 22 + 7 + 1), which is achieved if we load 1 unit of item 1, 1 unit of item 3 and 2 units of item 4.

EXAMPLE 7.4-10 (Selection of Advertising Media)

A cosmetics manufacturing company is interested in selecting the advertising media for its product and the frequency of advertising in each media. The data collected over the past two years regarding the frequency of advertising in three media of newspaper, radio and television and the related sales of the product give the following results:

TABLE 7.24

	Expected sales in thousands of rupees		
Frequency/week	*Television*	*Radio*	*Newspaper*
1	220	150	100
2	275	250	175
3	325	300	225
4	350	320	250

The cost of advertising in newspaper is ₹ 500 per appearance, while in radio and in television, it is ₹ 1,000 and ₹ 2,000 respectively. The budget provides ₹ 4,500 per week for advertisement. The problem is of determining the optimal combination of advertising media and advertising frequency.

Solution

The problem can be decomposed into three stages corresponding to the three media of advertising. In each media four alternatives (frequencies) are possible. Each alternative, has associated with it certain cost and return (expected sales). Here again, the capital marked for allocation to different media is the state variable. A combination of media and frequency is to be selected in such a way as to maximize the total sales with expenditure not exceeding the specified limit of ₹ 4,500.

Let us consider the advertising media of television as the first stage. If x_1 is the capital allocated to stage 1, and R_{1j} (C_{1j}) is the return (expected sales) corresponding to cost C_{1j}, then, the optimal return is

$$f_1(x_1) = \max [R_{1j}(C_{1j})],$$
$$j = 0, 1, 2, 3, 4, \text{ with } 0 \le x_1 \le C.$$

By applying this equation at various levels of expenditure, the various alternatives are evaluated and the one giving the largest expected sales is selected. The selected frequencies and the optimal return for different values of x_1 are given in table 7.25.

TABLE 7.25

Stage 1

State x_1	*Cost per appearance = ₹ 2,000* *Return (in thousands of rupees)*	*Frequency*
500	—	0
1,000	—	0
1,500	—	0
2,000	220	1
2,500	220	1
3,000	220	1
3,500	220	1
4,000	275	2
4,500	275	2

For x_1 = 0, ₹ 500, ₹ 1,000 and ₹ 1,500; it is not possible to advertise in this media, since the cost of one appearance per week is ₹ 2,000. For x_1 = ₹ 2,000, ₹ 2,500, ₹ 3,000 and ₹ 3,500, the product can be advertised only once, giving a return of ₹ 2,20,000.

With x_1 = ₹ 4,000, ₹ 4,500, two appearances can occur giving a return of ₹ 2,75,000.

TABLE 7.26

Stage 2

Cost per appearance = ₹ 1,000
Return in thousands of rupees

State x_2	*0*	*1*	*2*	*3*	*4*	*Return*	*Freq.*
500	0	0	—	—	—	0	0
1,000	0	**150**	—	—	—	150	1
1,500	0	**150+0**	—	—	—	150	1
2,000	220	150+0	**250**	—	—	250	2
2,500	220	150+0	**250+0**	—	—	250	2
3,000	220	**150+220**	250+0	300	—	370	1
3,500	220	**150+220**	250+0	300+0	—	370	1
4,000	275	150+220	**250+220**	300+0	320	470	2
4,500	275	**150+220**	250+220	300+0	320+0	470	2

Now let us move to the second stage. Again for advertising in radio, four alternatives (frequencies) are possible. Here, the state x_2 will signify the expenditure incurred at the first stage and at the current stage.

At any value of state x_2 $(0 \leq x_2 \leq C)$,

$$\text{optimal return } f_2(x_2) = \max [R_{2j}(C_{2j}) + f_1(x_1)], \text{ for } j = 0 \text{ to } 4$$
$$= \max [R_{2j}(C_{2j}) + f_1(x_2 - C_{2j})], \text{ for } j = 0 \text{ to } 4.$$

The evaluation of altrenatives is carried in the tabular form shown in table 7.26. To illustrate, at x_2 = 3,000, four alternatives are possible, *i.e.*, do not advertise, advertise once, twice or thrice. It is not possible to advertise four times because that needs a sum of ₹ 4,000. If we do not purchase any advertisement (frequency = 0), the amount of ₹ 3,000 can purchase one advertisement in media T, giving expected sales of ₹ 2,20,000. If one advertisement is purchased in media R, this will cost ₹ 1,000, and with amount of ₹ 2,000 left one advertisement can be purchased in media T, giving total return of ₹ (150 + 220) × 1,000 = ₹ 3,70,000. If two advertisements are purchased in R, costing ₹ 2,000, the balance amount of ₹ 1,000 will be of no use in media T, and thus will give total sales as ₹ (250 + 0) × 1,000 = ₹ 2,50,000. The maximum return comes when we purchase one advertisement in media R. This is the optimal decision for x_2 = ₹ 3,000.

Now we move to the third stage.

$$f_3(x_3) = \max [R_{3j}(C_{3j}) + f_2(x_2)], \text{ for } j = 0 \text{ to } 4$$
$$= \max [R_{3j}(C_{3j}) + f_2(x_3 - C_{3j})], \text{ for } j = 0 \text{ to } 4.$$

TABLE 7.27

Cost per appearance = ₹ 500

Stage 3 *Return in thousands of rupees*

State x_3	*0*	*1*	*2*	*3*	*4*	*Optimal Decision Total sales*	*Frequency*
500	0	**100**	—	—	—	100	1
1,000	150	100+0	**175**	—	—	175	2
1,500	150	**100+150**	175+0	225	—	250	1
2,000	250	100+150	**175+150**	225+0	250	325	2
2,500	250	100+250	175+150	**225+150**	250+0	375	3
3,000	370	100+250	**175+250**	225+150	250+150	425	2
3,500	370	100+370	175+250	**225+250**	250+150	475	3
4,000	470	100+370	**175+370**	225+250	250+250	545	2
4,500	470	100+470	175+370	**225+370**	250+250	595	3

The computations are given in table 7.27. For the allocated capital of ₹ 4,500, the maximum sales that can be expected are of ₹ 5,95,000. From table 7.27, the optimal decision is : purchase three advertisements in newspaper. This will cost ₹ 1,500. The amount left is ₹ 3,000, and corresponding to that at stage 2, the optimal decision is : purchase one advertisement in radio. This costs ₹ 1,000 which leaves behind an amount of ₹ 2,000 which can purchase one advertisement in television (stage 1).

Similarly, if the firm wants to spend only ₹ 4,000 per week, the optimal policy will be : purchase two advertisements in newspaper costing ₹ 1,000, one in radio costing ₹ 1,000, and one in television costing ₹ 2,000. This will give an optimal expected sales worth ₹ 5,45,000.

EXAMPLE 7.4-11

A man is engaged in buying and selling identical items. He operates from a warehouse that can hold 500 items. Each month he can sell any quantity that he chooses to stock at the beginning of the month. Each month, he can buy as much as he wishes for delivery at the end of the month, *so long as his stock does not exceed 500 items. For the next four months he has the following error-free forecasts of cost and sale prices :*

Month :	*1*	*2*	*3*	*4*
Cost :	*27*	*26*	*24*	*28*
Sale price :	*28*	*25*	*25*	*27*

If he currently has a stock of 200 units, what quantities should he sell and buy in the next four months ? Find the solution using dynamic programming.

[*Agra (M.Stat.) 1974; I.S.I. Dip. (Oprn. Mgmt.) 1978; Kuru.U., 1979*]

Solution

The problem can be analysed by treating the four months as the four stages.

Let x_i be the number of items to be sold during the month i,
y_i be the number of items to be ordered during the month i,
b_i be the stock level in the beginning of month i,
p_i be the sale price in month i,
c_i be the purchase price in month i,
w be the warehouse capacity, which is 500.

The problem will be solved starting with the 4th month and then proceeding backward.

If $f_n\ (b_n)$ is the return when there are n more months to follow and the initial stock at the beginning of month n is b_n, then n varies from 1 to 4 as the months vary from 4th to 1st.

n :	4	3	2	1
Month :	I	II	III	IV

The recursive equations can be written as

At stage 1:

$$f_1\ (b_n) = \max_{x_n, y_n}\ [p_1 x_n - c_1 y_n],$$

where $x_n \leq b_n$ and $b_n - x_n + y_n \leq W$.

For any other stage,

$$f_n\ (b_n) = \max_{x_n, y_n}\ [p_n x_n - c_n y_n + f_{n-1}\ (b_n - x_n + y_n)].$$

When n = 1,

$$f_1\ (b_1) = \max_{x_1, y_1}\ [p_1 x_1 - c_1 y_1].$$

Since c_1 is positive, to maximize $f_1\ (b_1)$, $y_1 = 0$; and since no stock should be left at the end of the 4th month, the amount to be sold during the month should be equal to be the amount at the beginning of the month *i.e.*, $x_1 = b_1$.

$$\therefore\quad f_1\ (b_1) = p_1 b_1 = 27 b_1.$$

When n = 2,

$$\begin{aligned} f_2\ (b_2) &= \max_{x_2, y_2}\ [p_2 x_2 - c_2 y_2 + f_1\ (b_1)] \\ &= \max_{x_2, y_2}\ [p_2 x_2 - c_2 y_2 + f_1\ (b_2 - x_2 + y_2)] \\ &= \max_{x_2, y_2}\ [25 x_2 - 26 y_2 + 27(b_2 - x_2 + y_2)] \\ &= \max_{x_2, y_2}\ [y_2 - 2x_2 + 27 b_2]. \end{aligned}$$

Since

$$\begin{aligned} y_2 &\leq W - b_2 + x_2 \\ &\leq 500 - b_2 + x_2, \end{aligned}$$

$$\begin{aligned} f_2\ (b_2) &= \max_{x_2}\ [500 - b_2 + x_2 - 2x_2 + 27 b_2] \\ &= \max_{x_2}\ [26 b_2 - x_2 + 500]. \end{aligned}$$

To maximize $f_2(b_2)$, x_2 can be taken as zero.

$$\therefore \quad f_2(b_2) = 26b_2 + 500.$$

When n = 3,

$$f_3(b_3) = \max_{x_3, y_3} [p_3x_3 - c_3y_3 + f_2(b_2)]$$

$$= \max_{x_3, y_3} [25x_3 - 24y_3 + 26b_2 + 500].$$

Now $\quad b_2 = b_3 - x_3 + y_3.$

$$\therefore \quad f_3(b_3) = \max_{x_3, y_3} [25x_3 - 24y_3 + 26(b_3 - x_3 + y_3) + 500]$$

$$= \max_{x_3, y_3} [26b_3 - x_3 + 2y_3 + 500],$$

and $\quad y_3 \le 500 - b_3 + x_3.$

$$\therefore \quad f_3(b_3) = \max_{x_3} [26b_3 - x_3 + 2(500 - b_3 + x_3) + 500]$$

$$= \max_{x_3} [24b_3 + x_3 + 1{,}500].$$

Now to maximize $f_3(b_3)$, x_3 should be maximum permissible, which is b_3 since $x_3 \le b_3$.

$$\therefore \quad f_3(b_3) = 25b_3 + 1{,}500.$$

When n = 4,

$$f_4(b_4) = \max_{x_4, y_4} [p_4x_4 - c_4y_4 + f_3(b_3)]$$

$$= \max_{x_4, y_4} [28x_4 - 27y_4 + 25b_3 + 1{,}500].$$

But $\quad b_3 = b_4 - x_4 + y_4.$

$$\therefore \quad f_4(b_4) = \max_{x_4, y_4} [28x_4 - 27y_4 + 25(b_4 - x_4 + y_4) + 1{,}500]$$

$$= \max_{x_4, y_4} [25b_4 + 3x_4 - 2y_4 + 1{,}500].$$

To maximize $f_4(b_4)$, y_4 should be zero as $y_4 \ge 0$, and x_4 which is $\le b_4$, should at maximum be equal to b_4.

i.e., $\quad y_4 = 0$ and $x_4 = b_4$, which gives

$$f_4(b_4) = 28b_4 + 1{,}500.$$

Now, it is given that stock level at the beginning of the first month is 200.

Thus $\quad b_4 = 200.$

$$\therefore \quad f_4(b_4) = 28 \times 200 + 1{,}500 = 7{,}100,$$

$$x_4 = 200, \; y_4 = 0,$$

$$b_3 = b_4 - x_4 + y_4 = 0,$$

$$x_3 = b_3 = 0,$$

$$y_3 = 500 - b_3 + x_3 = 500,$$

$$b_2 = b_3 - x_3 + y_3 = 500,$$

$$x_2 = 0, \; y_2 = 500 - b_2 + x_2 = 0,$$

$$b_1 = b_2 - x_2 + y_2 = 500,$$

$$x_1 = b_1 = 500, \text{ and}$$

$$y_1 = 0.$$