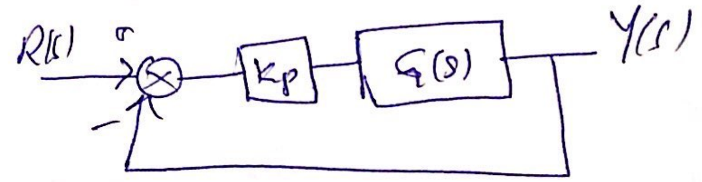
Time Response (Example)

Ex: The plant t/f G(s) is given as

$$G(s) = \frac{1}{s(s+4)}$$

if this plant is kept in a unity feedback configuration with a proportional controller $K_p = 16$, then find t_r, t_s and M_p?

Solution :->



$$H(s) = \frac{Y(s)}{R(s)} = \frac{K_p G(s)}{1 + K_p G(s)} = \frac{16 \frac{1}{s(s+4)}}{1 + \frac{16}{s(s+4)}}$$

$$= \frac{16}{s^2 + 4s + 16} \rightarrow (1)$$

The std. form is

$$H(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \rightarrow (2)$$

Compare (1) & (2)

$$\omega_n^2 = 16 \Rightarrow \omega_n = 4$$

$$2\xi\omega_n = 4 \Rightarrow \xi\omega_n = 2 \Rightarrow \xi = \frac{2}{4} = 0.5$$

(2)

$$\boxed{\xi = 0.5 \quad \& \quad \omega_n = 4}$$

$$t_r \cong \frac{1.8}{\omega_n} = \frac{1.8}{4} = 0.45 \text{ Sec}$$

$$t_s = \frac{4.6}{\delta} = \frac{4.6}{\xi\omega_n} = \frac{4.6}{0.5 \times 4} = 2.3 \text{ Sec}$$

$$M_p = 100 \times e^{-\pi \xi/\sqrt{1-\xi^2}} \%$$

$$= 100 \times e^{-\pi \frac{0.5}{\sqrt{1-0.5^2}}} = ?\ \%$$

→ The specifications we've studied so far (i.e., t_r, t_s and M_p) are transient response specifications.

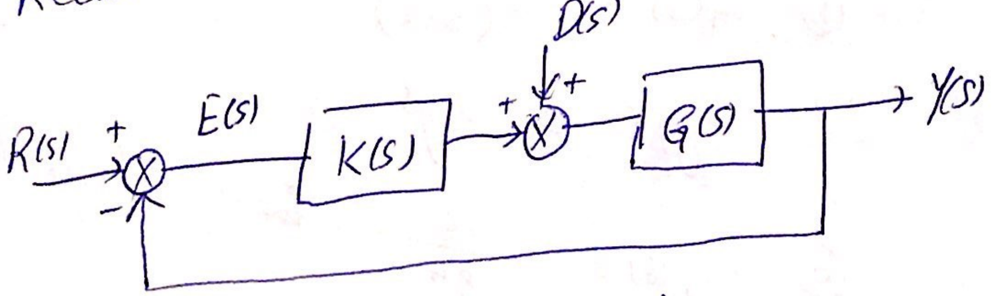
→ In tracking references we also keep in mind the steady state.

Steady State :→ "When all the transients die out and the sys. comes to rest or uniform motion".

⇒ "We need (ideally) zero error in steady state",

Steady State Error

Recall the block diagram



→ A control sys. is required s.t

$E(s) \longrightarrow 0$ i.e., $Y(s) = R(s)$

$R(s) - Y(s) = 0$ $\quad (y(t) = r(t))$

$$\frac{E(s)}{R(s)} = \frac{1}{1 + G(s)K(s)}$$

$$\frac{E(s)}{D(s)} = -\frac{G(s)}{1+G(s)K(s)}$$

⇒ We want $E(s)$ to be 0 ~~even~~ in the presense of disturbance. (Robustness)

System Type :→ Number of integrators in a system define the type of sys. e.g.,

(a) $G(s) = \frac{1}{s+1}$, no integrator

type 0 sys.

(b) $G(s) = \frac{1}{s(s+2)}$, One integrator

type 1 sys.

(c) $G(s) = \frac{10}{s^2}$, two integrators

type 2 sys.

Steady State error (E_{ss})

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)K(s)}$$

$$E(s) = \frac{1}{1+G(s)K(s)} R(s)$$

By FVT

$$\boxed{E_{ss} = \lim_{s \to 0} s E(s)}$$

---

1 $E_{ss} = ?$ when $R(s)$ is unit step i.e., $R(s) = \frac{1}{s}$

$$E_{ss} = \lim_{s \to 0} \cancel{s} \frac{1}{\cancel{s}} \frac{1}{1+G(s)K(s)}$$

$$E_{ss} = \frac{1}{1+G(0)K(0)} \rightarrow 1$$

if $G(s)K(s)$ is type 0 then

$E_{ss} \neq 0$ ∴ For $E_{ss} = 0$ we need

$G(0)K(0) = \infty$ which is possible only

if $G(s)K(s)$ has an integrator

i.e., an integrator is required either

in the plant ~~of in~~ $G(s)$ or in the

controller $K(s)$.

Now E_{ss} due to $D(s)$ $[D(s) = \frac{1}{s}]$

$$E_{ss} = -\lim_{s \to 0} \cancel{s} \frac{1}{\cancel{s}} \cdot \frac{G(s)}{1 + G(s)K(s)}$$

$$\frac{E(s)}{D(s)} = -\frac{G}{1+GK}$$

$$= -\lim_{s \to 0} \frac{1}{\frac{1}{G(s)} + K(s)}$$

$$= -\frac{1}{\frac{1}{G(0)} + K(0)}$$

→ Now an integrator in the plant is not enough.

→ In order ~~that~~ for $E_{ss} = 0$ we must have an integrator in the controller $K(s)$.

② E_{ss} in the presence of unit Ramp $\left(\frac{1}{s^2}\right)$ i/ps

$$E(s) = \frac{1}{s^2} \frac{1}{1 + G(s)K(s)}$$

$$E_{ss} = \lim_{s \to 0} s \frac{1}{s^2} \frac{1}{1 + G(s)K(s)}$$

$$= \lim_{s \to 0} \frac{1}{s + sG(s)K(s)}$$

ⓐ if $G(s)K(s)$ is type 0 then $E_{ss} = \infty$

ⓑ if $G(s)K(s)$ is typ 1 then $sG(0)K(0)$ may be some finite value (There will be a finite error in this case)

(c) if $G(s)K(s)$ is type 2

then $\lim_{s\to 0} s\,G(s)K(s) = \infty$

$\Rightarrow$ $E_{ss} = 0$

Ramp disturbace $D(s) = \frac{1}{s^2}$

$$\frac{E(s)}{D(s)} = \frac{-G(s)}{1+G(s)K(s)}$$

$$E(s) = -D(s)\,\frac{G(s)}{1+G(s)K(s)}$$

$$E_{ss} = -\lim_{s\to 0} s\,\frac{1}{s^2}\,\frac{G(s)}{1+G(s)K(s)} \quad \because D(s) = \frac{1}{s^2}$$

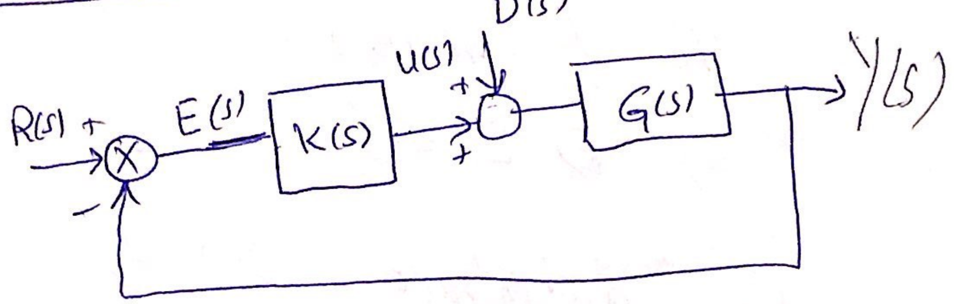
$$= -\lim_{s\to 0} \frac{G(s)}{s + sG(s)K(s)}$$

$$= -\lim_{s\to 0} \frac{1}{\frac{s}{G(s)} + sK(s)}$$

Now for complete disturbance rejection we need at least two integrators in the controller $K(s)$.

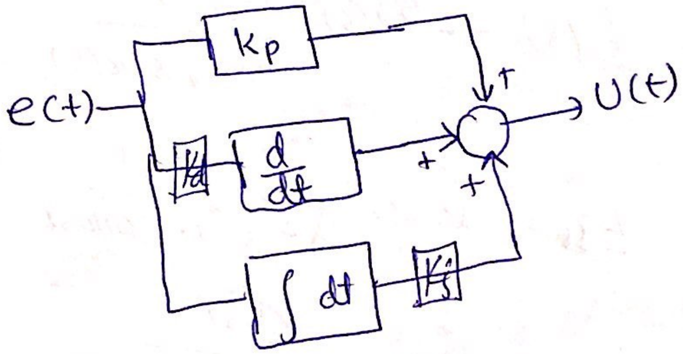
The same procedure can be carried out for parabolic i/ps.

PID Controller



$$\boxed{U(t) = K_p e(t) + K_d \dot{e}(t) + K_i \int e(t)\,dt} \rightarrow (1)$$

↳ PID control in Time-domain.



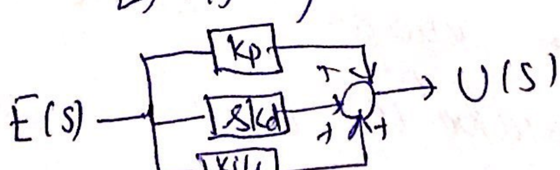
Taking laplace of (1)

$$U(s) = K_p E(s) + K_d s E(s) + \frac{K_i}{s} E(s)$$

$$= \left[K_p + K_d s + \frac{K_i}{s} \right] E(s)$$

$$\boxed{\frac{U(s)}{E(s)} = K_p + K_d s + \frac{K_i}{s}}$$

↳ T/f of PID.



→ Now each component of PID has a role to play.

e.g (a) $\int e(t)dt$ The integral term $\int e(t)dt \neq 0$ and keep increasing until $e(t) = 0$ (that's why integral component is imp. for steady state error elimination (good tracking)

(b) The derivative Term $\boxed{\frac{d}{dt} e(t)}$

⇒ As we know that $\frac{d}{dt} e(t)$ will be bigger if $e(t)$ changes more.

→ So derivative term is required to add damping to the sys.

(c) Proportional component $e(t)$

$K_p e(t) \uparrow$ if $|e(t)| \uparrow$

i.e., in order to make the error vanish quikkly (to make sys fast) we need bigger K_p.

$K_p \uparrow \longrightarrow$ Sys. is fast

$K_d \uparrow \longrightarrow$ Sys dampout quickly

$K_i \uparrow \longrightarrow$ Low steady state error.

Example :→ The plant t/f is given as

$$G(s) = \frac{1}{s(s+2)}$$

If $k(s) = k_p$ then

① Find E_{ss} to step Reference? $R(s) = \frac{1}{s}$

② " " " " Disturbance? $D(s) = \frac{1}{s}$

Solution :→ ①

$$\frac{E(s)}{R(s)} = \frac{1}{1 + G(s)K(s)}$$

$$E(s) = R(s) \frac{1}{1 + G(s)K(s)}$$

$$= \frac{1}{s} \frac{1}{1 + G(s)K(s)}$$

$$E_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s \frac{1}{s} \frac{1}{1 + G(s)K_p}$$

$$E_{ss} = \frac{1}{1+G(0)K_p}$$

$$G(0) = \frac{1}{0(0+2)} = \infty$$

So

$$E_{ss} = \frac{1}{1+\infty} = 0$$

$E_{ss} = 0$ in tracking a reference step

2

$$\frac{E(s)}{D(s)} = \frac{-G(s)}{1+G(s)K_p}$$

$$E_{ss} = -\lim_{s \to 0} s \frac{1}{s} \frac{G(s)}{1+G(s)K_p}$$

$$= -\lim_{s \to 0} \frac{G(s)}{1+G(s)K_p}$$

$$= -\lim_{s \to 0} \frac{1}{\frac{1}{G(s)} + K_p}$$

$$= -\frac{1}{\frac{1}{G(0)} + K_p} = -\frac{1}{0+K_p} \quad \because G(0) = \infty$$

$$\boxed{E_{ss} = -\frac{1}{K_p}} \neq 0$$

⇒ Conclusion:→ Integral control is required to reject disturbance.

Example :→ For the given plant

$$G(s) = \frac{2}{s(s+2)}$$

Design a controller s.t

(a) $t_r = 4 \text{ sec}$

(b) $M_p = 10\%$

Solution

$$\frac{Y(s)}{R(s)} = \frac{G(s)K(s)}{1 + G(s)K(s)}$$

$$= \frac{2K(s)}{s^2 + 2s + K(s)} \rightarrow (1)$$

in (1) if $K(s) = K_p$ then we can only alter the rise time so

let $K(s) = K_p + K_d s$.

then

$$\frac{E(s)}{R(s)} = \frac{2(K_p + K_d s)}{s^2 + 2s + K_p + K_d s}$$

$$= \frac{2(K_p + K_d s)}{s^2 + (2 + K_d)s + K_p} \rightarrow (2)$$

Now we'll compare the denominator of 2 with the standard 2nd order

t/f
$$H(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \rightarrow 3$$

$$\boxed{\omega_n^2 = k_p} \rightarrow 4 \qquad \boxed{K_d = 2\xi\omega_n - 2} \rightarrow 5$$

$$2 + K_d = 2\xi\omega_n \Rightarrow \boxed{\xi = \frac{2 + K_d}{2\omega_n}}$$

Notice that though the numerators of 2 and 3 are not a like and the formulas

$$t_r \cong \frac{1.8}{\omega_n} \quad \text{and} \quad M_p = e^{-\pi\xi/\sqrt{1-\xi^2}}$$

are valid for 3 only. Yet using these formulas we can get a very good starting point for further iterations.

→ From give specs.

$$\frac{1.8}{\omega_n} = 4 \Rightarrow \omega_n = \frac{1.8}{4}$$

put this in 4

$\left(\frac{1.8}{4}\right)^2 = K_p$ and similarly

we can workout K_d for ξ.

$$K_d = 2\xi\omega_n - 2$$

$$= 2\xi\left(\frac{1.8}{4}\right) - 2$$

↳ ξ will come from

$$0.1 = e^{-\pi \xi/\sqrt{1-\xi^2}}$$

$$\ln(0.1) = \frac{-\pi\xi}{\sqrt{1-\xi^2}}$$

$$\left(\sqrt{1-\xi^2}\right)\ln(0.1) = -\pi\xi$$

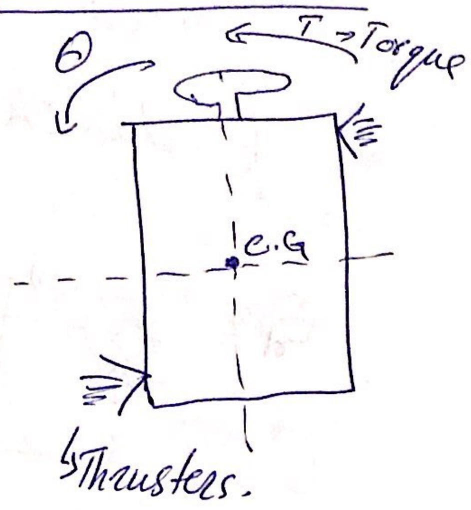
$$(1-\xi^2)\left(\ln(0.1)\right)^2 = \pi^2\xi^2$$

$$\left(\ln(0.1)\right)^2 - \xi^2\left(\ln(0.1)\right)^2 - \pi^2\xi^2 = 0$$

$$\left(\ln(0.1)\right)^2 = \xi^2\left[\left(\ln(0.1)\right)^2 - \pi^2\right]$$

$$\xi = \sqrt{\frac{\left(\ln(0.1)\right)^2}{\ln(0.1)^2 - \pi^2}} \quad \text{approx.}$$

Example (Satellite attitude Control)



① Design a controller
s.t
$t_r \leq 2\,sec$
$M_p \leq 5\%$?

② $E_{ss} = 0$ in the presence of step disturbance?

Solution

EOM:

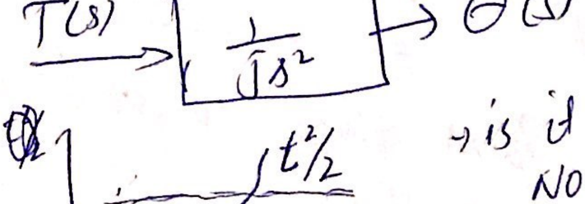
$$T(t) = J\ddot{\theta}(t)$$

$\hookrightarrow$ Moment of inertia.

$$\boxed{\frac{\Theta(s)}{T(s)} = \frac{1}{Js^2}}$$

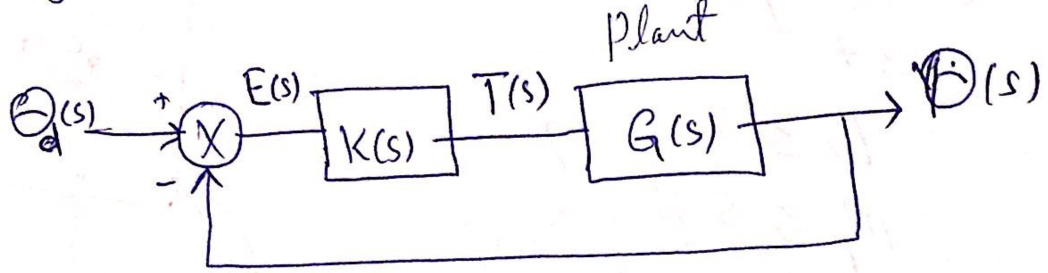
$\hookrightarrow$ Double integrator plant

In open loop



$\to$ is it stable?
NO
$\to$ Do you like the response?

⇒ If we want to track a desired angle we'll need feedback.



$$\frac{\Theta(s)}{\Theta_d(s)} = \frac{K(s)G(s)}{1+K(s)G(s)}$$

and let $K(s) = K_p + K_d s$
i.e., PD control.

then (For simplicity assume $J = 1$)

$$\frac{\Theta(s)}{\Theta_d(s)} = \frac{K_p + K_d s}{s^2 + K_d s + K_p} \rightarrow ①$$

$$\frac{\omega_n^2}{s^2+2\xi\omega_n}$$

Comparing the denominator of ① with the std. one $(s^2+2\xi\omega_n s+\omega_n^2)$

$$\boxed{K_p = \omega_n^2 \text{ and } K_d = 2\xi\omega_n} \rightarrow ②$$

Now from given transient response specifications

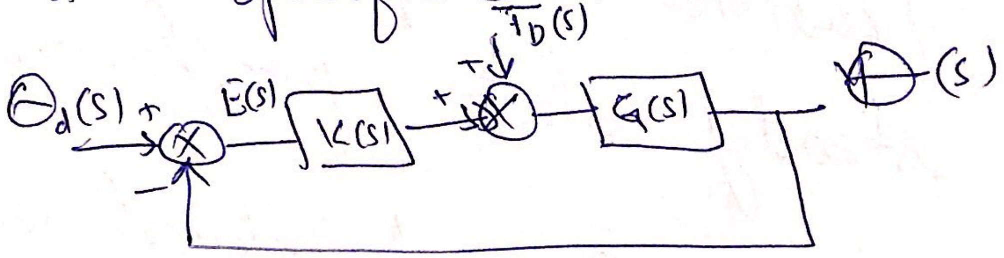
$$t_r \approx \frac{1.8}{\omega_n} \leq 2 \Rightarrow \omega_n \geq \frac{1.8}{2}$$

$$\omega_n \geq 0.9 \text{ rad}$$

and $M_p \leq 5\% \Rightarrow \xi \geq 0.707$.

Put this ω_n & ξ in (2) will give values of K_p and K_d.

→ However, we also have steady state specification.



$$\frac{E(s)}{\quad} = \frac{\Theta_d(s)}{1+G(s)K(s)} - \frac{T_D(s)\,G(s)}{1+G(s)K(s)}$$

$\frac{E}{T_D}(s)$

$$\Theta_d(s) = \frac{1}{s}, \quad T_D(s) \to \frac{1}{s}$$

$$e_{ss} = \lim_{s\to 0}\left[s\frac{1}{s}\frac{1}{1+G(s)K(s)} - s\frac{1}{s}\frac{G(s)}{1+G(s)K(s)} \right]$$

$$= \frac{1}{1+G(0)K(0)} - \lim_{s\to 0}\frac{1}{\frac{1}{G(s)}+K(s)}$$

$$= 0 - \frac{1}{0+K_p} \qquad \because K(0) = K_d\,0 + K_p$$

$\because G(s) = \frac{1}{s^2}$

$G(0) = \infty$

Now we can increase Kp to reduce Ess however, increasing Kp will destroy the transient response

$\left(-\frac{1}{K_p}\right)$

⇒ PD control can't alone cater for both the transient response and steady state specifications.

⇒ So we'll need integral action

$$K(s) = K_p + K_d s + \frac{K_i}{s}$$

$$1 + G(s)K(s) = 1 + \frac{1}{s^2}\left(\frac{K_p s + K_d s^2 + K_i}{s}\right)$$

$$= \frac{s^3 + K_d s^2 + K_p s + K_i}{s^3}$$

Now

$$\frac{E(s)}{T_D(s)} = \frac{-G(s)}{1 + G(s)K(s)} = -\frac{1/s^2}{\frac{s^3 + K_d s^2 + K_p s + K_i}{s^3}}$$

$$= -\frac{s}{s^3 + K_d s^2 + K_p s + K_i}$$

E_{ss} for step disturbance $T_d(s) = \frac{1}{s}$

$$E_{ss} = \lim_{s \to 0} s \frac{1}{s} \frac{s}{s^3 + K_d s^2 + K_p s + K_i} = 0$$

$\Rightarrow$ Notice that Now

$$\frac{\Theta(s)}{\Theta_d(s)} = \frac{G(s)K(s)}{1 + G(s)K(s)}$$

$$= \frac{K_d s^2 + K_p s + K_i}{\boxed{s^3 + K_d s^2 + K_p s + K_i}}$$

3^{rd} order.

So Now the co-efficient matching business is not valid any more. However, we know ~~that~~ the guidlines i.e.,

$K_p \uparrow \longrightarrow t_r \downarrow$

$K_d \uparrow \longrightarrow M_p \downarrow$

$K_i \uparrow \longrightarrow e_{ss} \downarrow$