**50. High School Grades of First-Year College Students** Forty-seven percent of first-year college students enrolled in 2005 had an average grade of A in high school compared to 20% of first-year college students in 1970. Choose 6 first-year college students at random enrolled in 2005. Find the probability that

*a.* All had an A average in high school 0.011

*b.* None had an A average in high school 0.022

*c.* At least 1 had an A average in high school 0.978

Source: www.census.gov

**51. Rolling a Die** If a die is rolled 3 times, find the probability of getting at least 1 even number. $\frac{7}{8}$

**52. Selecting a Flower** In a large vase, there are 8 roses, 5 daisies, 12 lilies, and 9 orchids. If 4 flowers are selected at random, find the probability that at least 1 of the flowers is a rose. Would you consider this event likely to occur? Explain your answer. 0.678; yes the event is a little more likely to occur than not since the probability is about 68%.

## Extending the Concepts

**53.** Let $A$ and $B$ be two mutually exclusive events. Are $A$ and $B$ independent events? Explain your answer. No, since $P(A \cap B) = 0$ and does not equal $P(A) \cdot P(B)$.

**54. Types of Vehicles** The Bargain Auto Mall has the following cars in stock.

| | SUV | Compact | Mid-sized |
|---|---|---|---|
| Foreign | 20 | 50 | 20 |
| Domestic | 65 | 100 | 45 |

Are the events "compact" and "domestic" independent? Explain. No, since $P(C|D) \neq P(C)$.

**55. College Enrollment** An admissions director knows that the probability a student will enroll after a campus visit is 0.55, or $P(E) = 0.55$. While students are on campus visits, interviews with professors are arranged. The admissions director computes these conditional probabilities for students enrolling after visiting three professors, DW, LP, and MH.

$$P(E|\text{DW}) = 0.95 \qquad P(E|\text{LP}) = 0.55 \qquad P(E|\text{MH}) = 0.15$$

Is there something wrong with the numbers? Explain.

**56. Commercials** Event $A$ is the event that a person remembers a certain product commercial. Event $B$ is the event that a person buys the product. If $P(B) = 0.35$, comment on each of these conditional probabilities if you were vice president for sales.

*a.* $P(B|A) = 0.20$

*b.* $P(B|A) = 0.35$

*c.* $P(B|A) = 0.55$

## 4–4 Counting Rules

Many times a person must know the number of all possible outcomes for a sequence of events. To determine this number, three rules can be used: the *fundamental counting rule,* the *permutation rule,* and the *combination rule.* These rules are explained here, and they will be used in Section 4–5 to find probabilities of events.

The first rule is called the **fundamental counting rule.**

### The Fundamental Counting Rule

**Objective 5**

Find the total number of outcomes in a sequence of events, using the fundamental counting rule.

**Fundamental Counting Rule**

In a sequence of $n$ events in which the first one has $k_1$ possibilities and the second event has $k_2$ and the third has $k_3$, and so forth, the total number of possibilities of the sequence will be

$$k_1 \cdot k_2 \cdot k_3 \cdots k_n$$

*Note:* In this case *and* means to multiply.

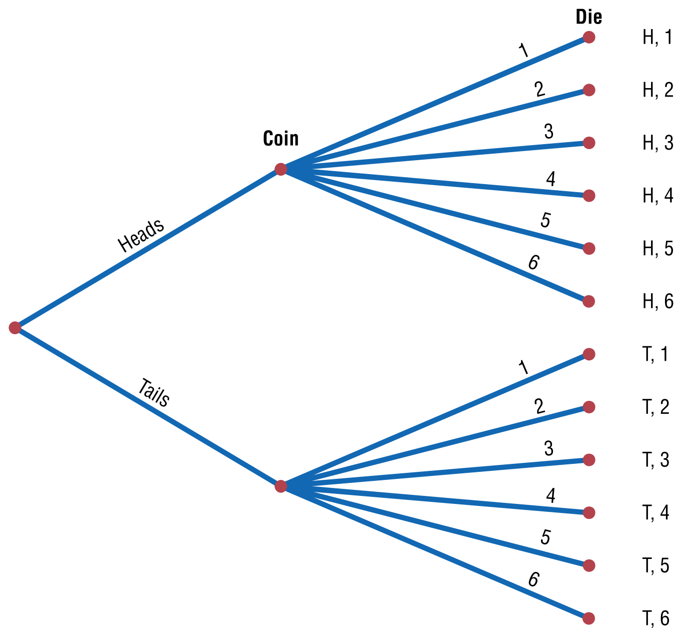
Examples 4–38 through 4–41 illustrate the fundamental counting rule.

## Example 4–38

### Tossing a Coin and Rolling a Die

A coin is tossed and a die is rolled. Find the number of outcomes for the sequence of events.

**Figure 4–8**
**Complete Tree Diagram for Example 4–38**




*Interesting Fact*

Possible games of chess: $25 \times 10^{115}$.

**Solution**

Since the coin can land either heads up or tails up and since the die can land with any one of six numbers showing face up, there are $2 \cdot 6 = 12$ possibilities. A tree diagram can also be drawn for the sequence of events. See Figure 4–8.

## Example 4–39

### Types of Paint

A paint manufacturer wishes to manufacture several different paints. The categories include

| | |
|---|---|
| Color | Red, blue, white, black, green, brown, yellow |
| Type | Latex, oil |
| Texture | Flat, semigloss, high gloss |
| Use | Outdoor, indoor |

How many different kinds of paint can be made if you can select one color, one type, one texture, and one use?

**Solution**

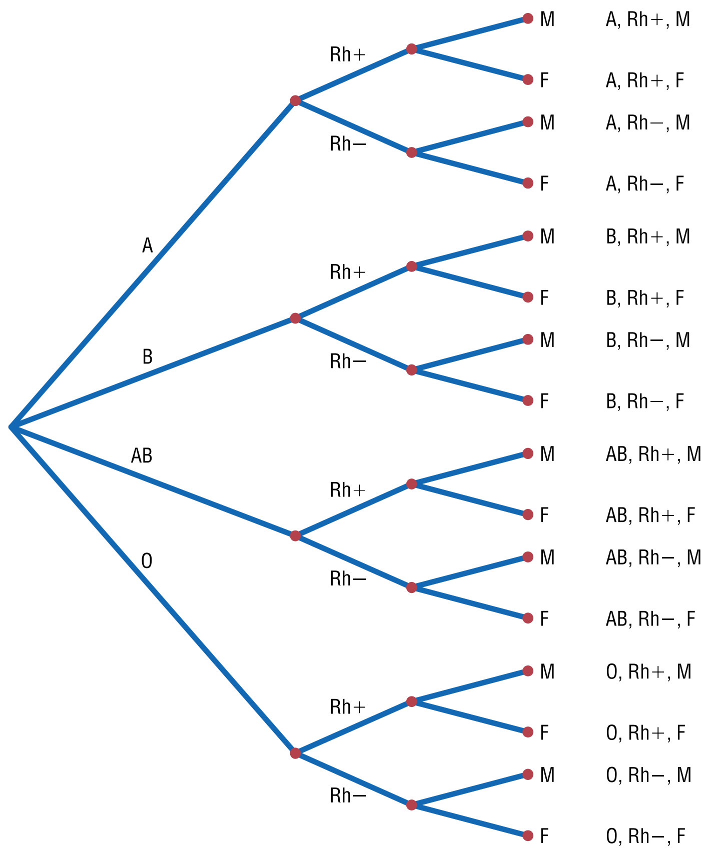
You can choose one color and one type and one texture and one use. Since there are 7 color choices, 2 type choices, 3 texture choices, and 2 use choices, the total number of possible different paints is

| **Color** | | **Type** | | **Texture** | | **Use** | |
|---|---|---|---|---|---|---|---|
| $\boxed{7}$ | $\cdot$ | $\boxed{2}$ | $\cdot$ | $\boxed{3}$ | $\cdot$ | $\boxed{2}$ | $= 84$ |

## Example 4–40 Distribution of Blood Types

There are four blood types, A, B, AB, and O. Blood can also be Rh+ and Rh−. Finally, a blood donor can be classified as either male or female. How many different ways can a donor have his or her blood labeled?

**Figure 4–9**
Complete Tree Diagram for Example 4–40




### Solution

Since there are 4 possibilities for blood type, 2 possibilities for Rh factor, and 2 possibilities for the gender of the donor, there are $4 \cdot 2 \cdot 2$, or 16, different classification categories, as shown.

| Blood type | | Rh | | Gender | |
|---|---|---|---|---|---|
| $\boxed{4}$ | $\cdot$ | $\boxed{2}$ | $\cdot$ | $\boxed{2}$ | $= 16$ |

A tree diagram for the events is shown in Figure 4–9.

When determining the number of different possibilities of a sequence of events, you must know whether repetitions are permissible.

## Example 4–41 Identification Cards

The manager of a department store chain wishes to make four-digit identification cards for her employees. How many different cards can be made if she uses the digits 1, 2, 3, 4, 5, and 6 and repetitions are permitted?

### Solution

Since there are 4 spaces to fill on each card and there are 6 choices for each space, the total number of cards that can be made is $6 \cdot 6 \cdot 6 \cdot 6 = 1296$.

Now, what if repetitions are not permitted? For Example 4–41, the first digit can be chosen in 6 ways. But the second digit can be chosen in only 5 ways, since there are only five digits left, etc. Thus, the solution is

$$6 \cdot 5 \cdot 4 \cdot 3 = 360$$

The same situation occurs when one is drawing balls from an urn or cards from a deck. If the ball or card is replaced before the next one is selected, then repetitions are permitted, since the same one can be selected again. But if the selected ball or card is not replaced, then repetitions are not permitted, since the same ball or card cannot be selected the second time.

These examples illustrate the fundamental counting rule. In summary: *If repetitions are permitted, then the numbers stay the same going from left to right. If repetitions are not permitted, then the numbers decrease by 1 for each place left to right.*

Two other rules that can be used to determine the total number of possibilities of a sequence of events are the permutation rule and the combination rule.

## Factorial Notation

**Historical Note**

In 1808 Christian Kramp first used the factorial notation.

These rules use *factorial notation.* The factorial notation uses the exclamation point.

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$
$$9! = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

To use the formulas in the permutation and combination rules, a special definition of 0! is needed. $0! = 1$.

**Factorial Formulas**

For any counting $n$

$$n! = n(n-1)(n-2) \cdots 1$$
$$0! = 1$$

## Permutations

A **permutation** is an arrangement of $n$ objects in a specific order.

Examples 4–42 and 4–43 illustrate permutations.

**Example 4–42**

### Business Location

Suppose a business owner has a choice of 5 locations in which to establish her business. She decides to rank each location according to certain criteria, such as price of the store and parking facilities. How many different ways can she rank the 5 locations?

**Solution**

There are

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$

different possible rankings. The reason is that she has 5 choices for the first location, 4 choices for the second location, 3 choices for the third location, etc.

In Example 4–42 all objects were used up. But what happens when not all objects are used up? The answer to this question is given in Example 4–43.

## Example 4–43 Business Location

Suppose the business owner in Example 4–42 wishes to rank only the top 3 of the 5 locations. How many different ways can she rank them?

**Solution**

Using the fundamental counting rule, she can select any one of the 5 for first choice, then any one of the remaining 4 locations for her second choice, and finally, any one of the remaining locations for her third choice, as shown.

| First choice | | Second choice | | Third choice | |
|---|---|---|---|---|---|
| 5 | · | 4 | · | 3 | = 60 |

The solutions in Examples 4–42 and 4–43 are permutations.

**Objective 6**

Find the number of ways that *r* objects can be selected from *n* objects, using the permutation rule.

### Permutation Rule

The arrangement of *n* objects in a specific order using *r* objects at a time is called a *permutation of n objects taking r objects at a time.* It is written as $_nP_r$, and the formula is

$$_nP_r = \frac{n!}{(n-r)!}$$

The notation $_nP_r$ is used for permutations.

$$_6P_4 \text{ means } \frac{6!}{(6-4)!} \quad \text{or} \quad \frac{6!}{2!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot \cancel{2} \cdot \cancel{1}}{\cancel{2} \cdot \cancel{1}} = 360$$

Although Examples 4–42 and 4–43 were solved by the multiplication rule, they can now be solved by the permutation rule.

In Example 4–42, 5 locations were taken and then arranged in order; hence,

$$_5P_5 = \frac{5!}{(5-5)!} = \frac{5!}{0!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1} = 120$$

(Recall that 0! = 1.)

In Example 4–43, 3 locations were selected from 5 locations, so $n = 5$ and $r = 3$; hence

$${}_5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5 \cdot 4 \cdot 3 \cdot \cancel{2} \cdot \cancel{1}}{\cancel{2} \cdot \cancel{1}} = 60$$

Examples 4–44 and 4–45 illustrate the permutation rule.

**Example 4–44**

### Television Ads

The advertising director for a television show has 7 ads to use on the program. If she selects 1 of them for the opening of the show, 1 for the middle of the show, and 1 for the ending of the show, how many possible ways can this be accomplished?

**Solution**

Since order is important, the solution is

$${}_7P_3 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = 210$$

Hence, there would be 210 ways to show 3 ads.

**Example 4–45**

### School Musical Plays

A school musical director can select 2 musical plays to present next year. One will be presented in the fall, and one will be presented in the spring. If she has 9 to pick from, how many different possibilities are there?

**Solution**

Order is important since one play can be presented in the fall and the other play in the spring.

$${}_9P_2 = \frac{9!}{(9-2)!} = \frac{9!}{7!} = \frac{9 \cdot 8 \cdot 7!}{7!} = 72$$

There are 72 different possibilities.

## Combinations

Suppose a dress designer wishes to select two colors of material to design a new dress, and she has on hand four colors. How many different possibilities can there be in this situation?

**Objective 7**

Find the number of ways that $r$ objects can be selected from $n$ objects without regard to order, using the combination rule.

This type of problem differs from previous ones in that the order of selection is not important. That is, if the designer selects yellow and red, this selection is the same as the selection red and yellow. This type of selection is called a *combination.* The difference between a permutation and a combination is that in a combination, the order or arrangement of the objects is not important; by contrast, order *is* important in a permutation. Example 4–46 illustrates this difference.

A selection of distinct objects without regard to order is called a **combination.**

## Example 4–46

### Letters

Given the letters A, B, C, and D, list the permutations and combinations for selecting two letters.

**Solution**

The permutations are

| | | | |
|---|---|---|---|
| AB | BA | CA | DA |
| AC | BC | CB | DB |
| AD | BD | CD | DC |

In permutations, AB is different from BA. But in combinations, AB is the same as BA since the order of the objects does not matter in combinations. Therefore, if duplicates are removed from a list of permutations, what is left is a list of combinations, as shown.

| | | | |
|---|---|---|---|
| AB | ~~BA~~ | ~~CA~~ | ~~DA~~ |
| AC | BC | ~~CB~~ | ~~DB~~ |
| AD | BD | CD | ~~DC~~ |

Hence the combinations of A, B, C, and D are AB, AC, AD, BC, BD, and CD. (Alternatively, BA could be listed and AB crossed out, etc.) The combinations have been listed alphabetically for convenience, but this is not a requirement.

**Interesting Fact**

The total number of hours spent mowing lawns in the United States each year: 2,220,000,000.

*Combinations are used when the order or arrangement is not important, as in the selecting process.* Suppose a committee of 5 students is to be selected from 25 students. The 5 selected students represent a combination, since it does not matter who is selected first, second, etc.

**Combination Rule**

The number of combinations of $r$ objects selected from $n$ objects is denoted by ${}_nC_r$ and is given by the formula

$${}_nC_r = \frac{n!}{(n-r)!r!}$$

**Example 4–47**

## Combinations

How many combinations of 4 objects are there, taken 2 at a time?

**Solution**

Since this is a combination problem, the answer is

$$_4C_2 = \frac{4!}{(4-2)!2!} = \frac{4!}{2!2!} = \frac{\overset{2}{\cancel{4}} \cdot 3 \cdot \cancel{2!}}{\cancel{2} \cdot 1 \cdot \cancel{2!}} = 6$$

This is the same result shown in Example 4–46.

Notice that the expression for $_nC_r$ is

$$\frac{n!}{(n-r)!r!}$$

which is the formula for permutations with $r!$ in the denominator. In other words,

$$_nC_r = \frac{_nP_r}{r!}$$

This $r!$ divides out the duplicates from the number of permutations, as shown in Example 4–46. For each two letters, there are two permutations but only one combination. Hence, dividing the number of permutations by $r!$ eliminates the duplicates. This result can be verified for other values of $n$ and $r$. *Note:* $_nC_n = 1$.

**Example 4–48**

## Book Reviews

A newspaper editor has received 8 books to review. He decides that he can use 3 reviews in his newspaper. How many different ways can these 3 reviews be selected?

**Solution**

$$_8C_3 = \frac{8!}{(8-3)!3!} = \frac{8!}{5!3!} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 56$$

There are 56 possibilities.

**Example 4–49**

## Committee Selection

In a club there are 7 women and 5 men. A committee of 3 women and 2 men is to be chosen. How many different possibilities are there?

**Solution**

Here, you must select 3 women from 7 women, which can be done in $_7C_3$, or 35, ways. Next, 2 men must be selected from 5 men, which can be done in $_5C_2$, or 10, ways. Finally, by the fundamental counting rule, the total number of different ways is $35 \cdot 10 = 350$, since you are choosing both men and women. Using the formula gives

$$_7C_3 \cdot {_5C_2} = \frac{7!}{(7-3)!3!} \cdot \frac{5!}{(5-2)!2!} = 350$$

Table 4–1 summarizes the counting rules.

**Table 4–1 Summary of Counting Rules**

| Rule | Definition | Formula |
|---|---|---|
| Fundamental counting rule | The number of ways a sequence of $n$ events can occur if the first event can occur in $k_1$ ways, the second event can occur in $k_2$ ways, etc. | $k_1 \cdot k_2 \cdot k_3 \cdots k_n$ |
| Permutation rule | The number of permutations of $n$ objects taking $r$ objects at a time (order is important) | ${}_nP_r = \frac{n!}{(n-r)!}$ |
| Combination rule | The number of combinations of $r$ objects taken from $n$ objects (order is not important) | ${}_nC_r = \frac{n!}{(n-r)!r!}$ |

## *Applying the Concepts* 4–4

### Garage Door Openers

Garage door openers originally had a series of four on/off switches so that homeowners could personalize the frequencies that opened their garage doors. If all garage door openers were set at the same frequency, anyone with a garage door opener could open anyone else's garage door.

1. Use a tree diagram to show how many different positions 4 consecutive on/off switches could be in.

After garage door openers became more popular, another set of 4 on/off switches was added to the systems.

2. Find a pattern of how many different positions are possible with the addition of each on/off switch.
3. How many different positions are possible with 8 consecutive on/off switches?
4. Is it reasonable to assume, if you owned a garage door opener with 8 switches, that someone could use his or her garage door opener to open your garage door by trying all the different possible positions?

In 1989 it was reported that the ignition keys for 1988 Dodge Caravans were made from a single blank that had five cuts on it. Each cut was made at one out of five possible levels. In 1988, assume there were 420,000 Dodge Caravans sold in the United States.

5. How many different possible keys can be made from the same key blank?
6. How many different 1988 Dodge Caravans could any one key start?

Look at the ignition key for your car and count the number of cuts on it. Assume that the cuts are made at one of any of five possible levels. Most car companies use one key blank for all their makes and models of cars.

7. Conjecture how many cars your car company sold over recent years, and then figure out how many other cars your car key could start. What would you do to decrease the odds of someone being able to open another vehicle with his or her key?

See page 250 for the answers.