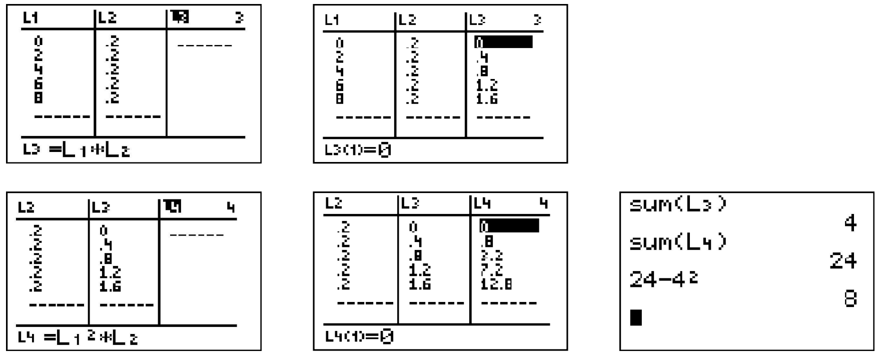
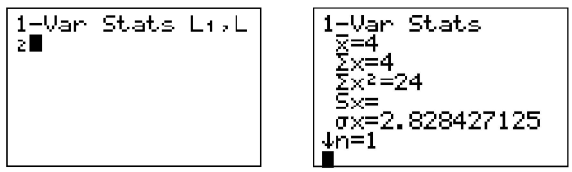
Using the data from Example TI5–1 gives the following:



To calculate the mean and standard deviation for a discrete random variable without using the formulas, modify the procedure to calculate the mean and standard deviation from grouped data (Chapter 3) by entering the x values into L_1 and the probabilities into L_2.



5–3 The Binomial Distribution

Many types of probability problems have only two outcomes or can be reduced to two outcomes. For example, when a coin is tossed, it can land heads or tails. When a baby is born, it will be either male or female. In a basketball game, a team either wins or loses. A true/false item can be answered in only two ways, true or false. Other situations can be

Objective 3

Find the exact probability for X successes in n trials of a binomial experiment.

Historical Note

In 1653, Blaise Pascal created a triangle of numbers called *Pascal's triangle* that can be used in the binomial distribution.

reduced to two outcomes. For example, a medical treatment can be classified as effective or ineffective, depending on the results. A person can be classified as having normal or abnormal blood pressure, depending on the measure of the blood pressure gauge. A multiple-choice question, even though there are four or five answer choices, can be classified as correct or incorrect. Situations like these are called *binomial experiments.*

A **binomial experiment** is a probability experiment that satisfies the following four requirements:

1. There must be a fixed number of trials.
2. Each trial can have only two outcomes or outcomes that can be reduced to two outcomes. These outcomes can be considered as either success or failure.
3. The outcomes of each trial must be independent of one another.
4. The probability of a success must remain the same for each trial.

A binomial experiment and its results give rise to a special probability distribution called the *binomial distribution.*

The outcomes of a binomial experiment and the corresponding probabilities of these outcomes are called a **binomial distribution.**

In binomial experiments, the outcomes are usually classified as successes or failures. For example, the correct answer to a multiple-choice item can be classified as a success, but any of the other choices would be incorrect and hence classified as a failure. The notation that is commonly used for binomial experiments and the binomial distribution is defined now.

Notation for the Binomial Distribution

$P(S)$ The symbol for the probability of success
$P(F)$ The symbol for the probability of failure
p The numerical probability of a success
q The numerical probability of a failure

$$P(S) = p \quad \text{and} \quad P(F) = 1 - p = q$$

n The number of trials
X The number of successes in n trials

Note that $0 \le X \le n$ and $X = 0, 1, 2, 3, \ldots, n$.

The probability of a success in a binomial experiment can be computed with this formula.

Binomial Probability Formula

In a binomial experiment, the probability of exactly X successes in n trials is

$$P(X) = \frac{n!}{(n - X)!X!} \cdot p^X \cdot q^{n-X}$$

An explanation of why the formula works is given following Example 5–15.

Example 5–15 Tossing Coins

A coin is tossed 3 times. Find the probability of getting exactly two heads.

Solution

This problem can be solved by looking at the sample space. There are three ways to get two heads.

HHH, <u>HHT, HTH, THH,</u> TTH, THT, HTT, TTT

The answer is $\frac{3}{8}$, or 0.375.

Looking at the problem in Example 5–15 from the standpoint of a binomial experiment, one can show that it meets the four requirements.

1. There are a fixed number of trials (three).
2. There are only two outcomes for each trial, heads or tails.
3. The outcomes are independent of one another (the outcome of one toss in no way affects the outcome of another toss).
4. The probability of a success (heads) is $\frac{1}{2}$ in each case.

In this case, $n = 3$, $X = 2$, $p = \frac{1}{2}$, and $q = \frac{1}{2}$. Hence, substituting in the formula gives

$$P(2 \text{ heads}) = \frac{3!}{(3-2)!2!} \cdot \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^1 = \frac{3}{8} = 0.375$$

which is the same answer obtained by using the sample space.

The same example can be used to explain the formula. First, note that there are three ways to get exactly two heads and one tail from a possible eight ways. They are HHT, HTH, and THH. In this case, then, the number of ways of obtaining two heads from three coin tosses is ${}_3C_2$, or 3, as shown in Chapter 4. In general, the number of ways to get X successes from n trials without regard to order is

$${}_nC_X = \frac{n!}{(n-X)!X!}$$

This is the first part of the binomial formula. (Some calculators can be used for this.)

Next, each success has a probability of $\frac{1}{2}$ and can occur twice. Likewise, each failure has a probability of $\frac{1}{2}$ and can occur once, giving the $(\frac{1}{2})^2(\frac{1}{2})^1$ part of the formula. To generalize, then, each success has a probability of p and can occur X times, and each failure has a probability of q and can occur $n - X$ times. Putting it all together yields the binomial probability formula.

Example 5–16 Survey on Doctor Visits

A survey found that one out of five Americans say he or she has visited a doctor in any given month. If 10 people are selected at random, find the probability that exactly 3 will have visited a doctor last month.

Source: *Reader's Digest.*

Solution

In this case, $n = 10$, $X = 3$, $p = \frac{1}{5}$, and $q = \frac{4}{5}$. Hence,

$$P(3) = \frac{10!}{(10-3)!3!}\left(\frac{1}{5}\right)^3\left(\frac{4}{5}\right)^7 = 0.201$$

Example 5–17 Survey on Employment

A survey from Teenage Research Unlimited (Northbrook, Illinois) found that 30% of teenage consumers receive their spending money from part-time jobs. If 5 teenagers are selected at random, find the probability that at least 3 of them will have part-time jobs.

Solution

To find the probability that at least 3 have part-time jobs, it is necessary to find the individual probabilities for 3, or 4, or 5 and then add them to get the total probability.

$$P(3) = \frac{5!}{(5-3)!3!}(0.3)^3(0.7)^2 = 0.132$$

$$P(4) = \frac{5!}{(5-4)!4!}(0.3)^4(0.7)^1 = 0.028$$

$$P(5) = \frac{5!}{(5-5)!5!}(0.3)^5(0.7)^0 = 0.002$$

Hence,

P(at least three teenagers have part-time jobs)
$= 0.132 + 0.028 + 0.002 = 0.162$

Computing probabilities by using the binomial probability formula can be quite tedious at times, so tables have been developed for selected values of n and p. Table B in Appendix C gives the probabilities for individual events. Example 5–18 shows how to use Table B to compute probabilities for binomial experiments.

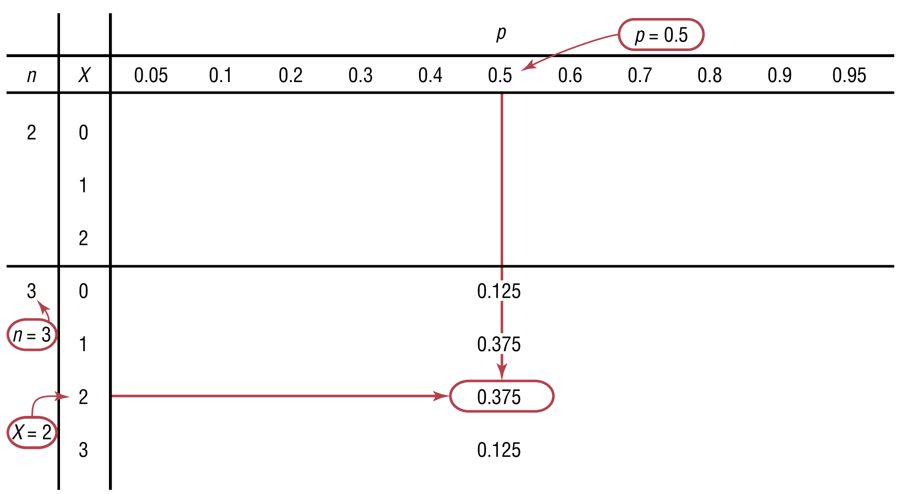
Example 5–18 Tossing Coins

Solve the problem in Example 5–15 by using Table B.

Solution

Since $n = 3$, $X = 2$, and $p = 0.5$, the value 0.375 is found as shown in Figure 5–3.

Figure 5–3
Using Table B for Example 5–18



Example 5–19

Survey on Fear of Being Home Alone at Night

Public Opinion reported that 5% of Americans are afraid of being alone in a house at night. If a random sample of 20 Americans is selected, find these probabilities by using the binomial table.

a. There are exactly 5 people in the sample who are afraid of being alone at night.
b. There are at most 3 people in the sample who are afraid of being alone at night.
c. There are at least 3 people in the sample who are afraid of being alone at night.

Source: *100% American* by Daniel Evan Weiss.

Solution

a. $n = 20$, $p = 0.05$, and $X = 5$. From the table, we get 0.002.
b. $n = 20$ and $p = 0.05$. "At most 3 people" means 0, or 1, or 2, or 3. Hence, the solution is

$$P(0) + P(1) + P(2) + P(3) = 0.358 + 0.377 + 0.189 + 0.060 = 0.984$$

c. $n = 20$ and $p = 0.05$. "At least 3 people" means 3, 4, 5, . . . , 20. This problem can best be solved by finding $P(0) + P(1) + P(2)$ and subtracting from 1.

$$P(0) + P(1) + P(2) = 0.358 + 0.377 + 0.189 = 0.924$$
$$1 - 0.924 = 0.076$$

Example 5–20

Driving While Intoxicated

A report from the Secretary of Health and Human Services stated that 70% of single-vehicle traffic fatalities that occur at night on weekends involve an intoxicated driver. If a sample of 15 single-vehicle traffic fatalities that occur at night on a weekend is selected, find the probability that exactly 12 involve a driver who is intoxicated.

Source: *100% American* by Daniel Evan Weiss.

Solution

Now, $n = 15$, $p = 0.70$, and $X = 12$. From Table B, $P(12) = 0.170$. Hence, the probability is 0.17.

Remember that in the use of the binomial distribution, the outcomes must be independent. For example, in the selection of components from a batch to be tested, each component must be replaced before the next one is selected. Otherwise, the outcomes are not independent. However, a dilemma arises because there is a chance that the same component could be selected again. This situation can be avoided by not replacing the component and using a distribution called the hypergeometric distribution to calculate the probabilities. The hypergeometric distribution is presented later in this chapter. Note that when the population is large and the sample is small, the binomial probabilities can be shown to be nearly the same as the corresponding hypergeometric probabilities.

Objective 4

Find the mean, variance, and standard deviation for the variable of a binomial distribution.

Mean, Variance, and Standard Deviation for the Binomial Distribution

The mean, variance, and standard deviation of a variable that has the *binomial distribution* can be found by using the following formulas.

Mean: $\mu = n \cdot p$ Variance: $\sigma^2 = n \cdot p \cdot q$ Standard deviation: $\sigma = \sqrt{n \cdot p \cdot q}$

These formulas are algebraically equivalent to the formulas for the mean, variance, and standard deviation of the variables for probability distributions, but because they are for variables of the binomial distribution, they have been simplified by using algebra. The algebraic derivation is omitted here, but their equivalence is shown in Example 5–21.

Example 5–21

Tossing a Coin

A coin is tossed 4 times. Find the mean, variance, and standard deviation of the number of heads that will be obtained.

Solution

With the formulas for the binomial distribution and $n = 4$, $p = \frac{1}{2}$, and $q = \frac{1}{2}$, the results are

$$\mu = n \cdot p = 4 \cdot \tfrac{1}{2} = 2$$
$$\sigma^2 = n \cdot p \cdot q = 4 \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} = 1$$
$$\sigma = \sqrt{1} = 1$$

From Example 5–21, when four coins are tossed many, many times, the average of the number of heads that appear is 2, and the standard deviation of the number of heads is 1. Note that these are theoretical values.

As stated previously, this problem can be solved by using the formulas for expected value. The distribution is shown.

No. of heads X	0	1	2	3	4
Probability $P(X)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

$$\mu = E(X) = \Sigma X \cdot P(X) = 0 \cdot \tfrac{1}{16} + 1 \cdot \tfrac{4}{16} + 2 \cdot \tfrac{6}{16} + 3 \cdot \tfrac{4}{16} + 4 \cdot \tfrac{1}{16} = \tfrac{32}{16} = 2$$
$$\sigma^2 = \Sigma X^2 \cdot P(X) - \mu^2$$
$$= 0^2 \cdot \tfrac{1}{16} + 1^2 \cdot \tfrac{4}{16} + 2^2 \cdot \tfrac{6}{16} + 3^2 \cdot \tfrac{4}{16} + 4^2 \cdot \tfrac{1}{16} - 2^2 = \tfrac{80}{16} - 4 = 1$$
$$\sigma = \sqrt{1} = 1$$

Hence, the simplified binomial formulas give the same results.

Example 5–22

Rolling a Die

A die is rolled 480 times. Find the mean, variance, and standard deviation of the number of 3s that will be rolled.

Solution

This is a binomial experiment since getting a 3 is a success and not getting a 3 is considered a failure.

Hence $n = 480$, $p = \frac{1}{6}$, and $q = \frac{5}{6}$.

$$\mu = n \cdot p = 480 \cdot \tfrac{1}{6} = 80$$
$$\sigma^2 = n \cdot p \cdot q = 480 \cdot \tfrac{1}{6} \cdot \tfrac{5}{6} = 66.67$$
$$\sigma = \sqrt{66.67} = 8.16$$

Example 5–23

Likelihood of Twins

The *Statistical Bulletin* published by Metropolitan Life Insurance Co. reported that 2% of all American births result in twins. If a random sample of 8000 births is taken, find the mean, variance, and standard deviation of the number of births that would result in twins.

Source: *100% American* by Daniel Evan Weiss.

Solution

This is a binomial situation, since a birth can result in either twins or not twins (i.e., two outcomes).

$$\mu = n \cdot p = (8000)(0.02) = 160$$
$$\sigma^2 = n \cdot p \cdot q = (8000)(0.02)(0.98) = 156.8$$
$$\sigma = \sqrt{n \cdot p \cdot q} = \sqrt{156.8} = 12.5$$

For the sample, the average number of births that would result in twins is 160, the variance is 156.8, or 157, and the standard deviation is 12.5, or 13 if rounded.

Applying the Concepts 5–3

Unsanitary Restaurants

Health officials routinely check sanitary conditions of restaurants. Assume you visit a popular tourist spot and read in the newspaper that in 3 out of every 7 restaurants checked, there were unsatisfactory health conditions found. Assuming you are planning to eat out 10 times while you are there on vacation, answer the following questions.

1. How likely is it that you will eat at three restaurants with unsanitary conditions?
2. How likely is it that you will eat at four or five restaurants with unsanitary conditions?
3. Explain how you would compute the probability of eating in at least one restaurant with unsanitary conditions. Could you use the complement to solve this problem?
4. What is the most likely number to occur in this experiment?
5. How variable will the data be around the most likely number?
6. How do you know that this is a binomial distribution?
7. If it is a binomial distribution, does that mean that the likelihood of a success is always 50% since there are only two possible outcomes?

Check your answers by using the following computer-generated table.

Mean = 4.29 Std. dev. = 1.56492

X	$P(X)$	Cum. Prob.
0	0.00371	0.00371
1	0.02784	0.03155
2	0.09396	0.12552
3	0.18793	0.31344
4	0.24665	0.56009
5	0.22199	0.78208
6	0.13874	0.92082
7	0.05946	0.98028
8	0.01672	0.99700
9	0.00279	0.99979
10	0.00021	1.00000

See page 298 for the answers.