8–2 *z* Test for a Mean

Objective 5

Test means when σ is known, using the *z* test.

In this chapter, two statistical tests will be explained: the *z* test is used when σ is known, and the *t* test is used when σ is unknown. This section explains the *z* test, and Section 8–3 explains the *t* test.

Many hypotheses are tested using a statistical test based on the following general formula:

$$\text{Test value} = \frac{(\text{observed value}) - (\text{expected value})}{\text{standard error}}$$

The observed value is the statistic (such as the sample mean) that is computed from the sample data. The expected value is the parameter (such as the population mean) that you would expect to obtain if the null hypothesis were true—in other words, the hypothesized value. The denominator is the standard error of the statistic being tested (in this case, the standard error of the mean).

The *z* test is defined formally as follows.

The ***z* test** is a statistical test for the mean of a population. It can be used when $n \geq 30$, or when the population is normally distributed and σ is known.

The formula for the *z* test is

$$z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}}$$

where

$\overline{X}$ = sample mean
μ = hypothesized population mean
σ = population standard deviation
n = sample size

For the *z* test, the observed value is the value of the sample mean. The expected value is the value of the population mean, assuming that the null hypothesis is true. The denominator $\sigma/\sqrt{n}$ is the standard error of the mean.

The formula for the *z* test is the same formula shown in Chapter 6 for the situation where you are using a distribution of sample means. Recall that the central limit theorem allows you to use the standard normal distribution to approximate the distribution of sample means when $n \geq 30$.

Note: Your first encounter with hypothesis testing can be somewhat challenging and confusing, since there are many new concepts being introduced at the same time. *To understand all the concepts, you must carefully follow each step in the examples and try each exercise that is assigned.* Only after careful study and patience will these concepts become clear.

Assumptions for the *z* Test for a Mean When σ Is Known

1. The sample is a random sample.
2. Either $n \geq 30$ or the population is normally distributed if $n < 30$.

As stated in Section 8–1, there are five steps for solving *hypothesis-testing* problems:

Step 1 State the hypotheses and identify the claim.

Step 2 Find the critical value(s).

Speaking of
Statistics

This study found that people who used pedometers reported having increased energy, mood improvement, and weight loss. State possible null and alternative hypotheses for the study. What would be a likely population? What is the sample size? Comment on the sample size.

RD HEALTH

Step to It

IT FITS in your hand, costs less than $30, and will make you feel great. Give up? A pedometer. Brenda Rooney, an epidemiologist at Gundersen Lutheran Medical Center in LaCrosse, Wis., gave 500 people pedometers and asked them to take 10,000 steps—about five miles—a day. (Office workers typically average about 4000 steps a day.) By the end of eight weeks, 56 percent reported having more energy, 47 percent improved their mood and 50 percent lost weight. The subjects reported that seeing their total step-count motivated them to take more.

— JENNIFER BRAUNSCHWEIGER

Source: Reprinted with permission from the April 2002 Reader's Digest.

Step 3 Compute the test value.

Step 4 Make the decision to reject or not reject the null hypothesis.

Step 5 Summarize the results.

Example 8–3 illustrates these five steps.

Example 8–3 Days on Dealers' Lots

A researcher wishes to see if the mean number of days that a basic, low-price, small automobile sits on a dealer's lot is 29. A sample of 30 automobile dealers has a mean of 30.1 days for basic, low-price, small automobiles. At $\alpha = 0.05$, test the claim that the mean time is greater than 29 days. The standard deviation of the population is 3.8 days.

Source: Based on information from *Power Information Network.*

Solution

Step 1 State the hypotheses and identify the claim.

H_0: $\mu = 29$ and H_1: $\mu > 29$ (claim)

Step 2 Find the critical value. Since $\alpha = 0.05$ and the test is a right-tailed test, the critical value is $z = +1.65$.

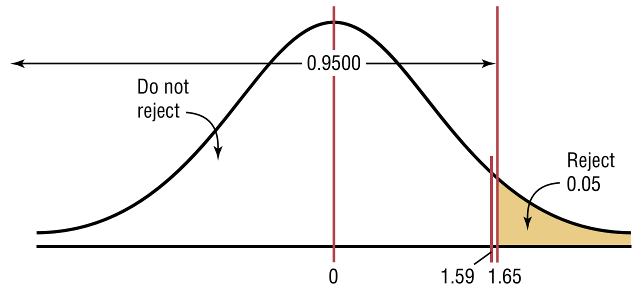
Step 3 Compute the test value.

$$z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} = \frac{30.1 - 29}{3.8/\sqrt{30}} = 1.59$$

Step 4 Make the decision. Since the test value, $+1.59$, is less than the critical value, $+1.65$, and is not in the critical region, the decision is to not reject the null hypothesis. This test is summarized in Figure 8–13.

Figure 8–13

Summary of the *z* Test of Example 8–3



Step 5 Summarize the results. There is not enough evidence to support the claim that the mean time is greater than 29 days.

Comment: Even though in Example 8–3 the sample mean of 30.1 is higher than the hypothesized population mean of 29, it is not *significantly* higher. Hence, the difference may be due to chance. When the null hypothesis is not rejected, there is still a probability of a type II error, i.e., of not rejecting the null hypothesis when it is false.

The probability of a type II error is not easily ascertained. Further explanation about the type II error is given in Section 8–6. For now, it is only necessary to realize that the probability of type II error exists when the decision is not to reject the null hypothesis.

Also note that when the null hypothesis is not rejected, it cannot be accepted as true. There is merely not enough evidence to say that it is false. This guideline may sound a little confusing, but the situation is analogous to a jury trial. The verdict is either guilty or not guilty and is based on the evidence presented. If a person is judged not guilty, it does not mean that the person is proved innocent; it only means that there was not enough evidence to reach the guilty verdict.

Example 8–4

Costs of Men's Athletic Shoes

A researcher claims that the average cost of men's athletic shoes is less than \$80. He selects a random sample of 36 pairs of shoes from a catalog and finds the following costs (in dollars). (The costs have been rounded to the nearest dollar.) Is there enough evidence to support the researcher's claim at $\alpha = 0.10$? Assume $\sigma = 19.2$.

60	70	75	55	80	55
50	40	80	70	50	95
120	90	75	85	80	60
110	65	80	85	85	45
75	60	90	90	60	95
110	85	45	90	70	70

Solution

Step 1 State the hypotheses and identify the claim

H_0: $\mu = \$80$ and H_1: $\mu < \$80$ (claim)

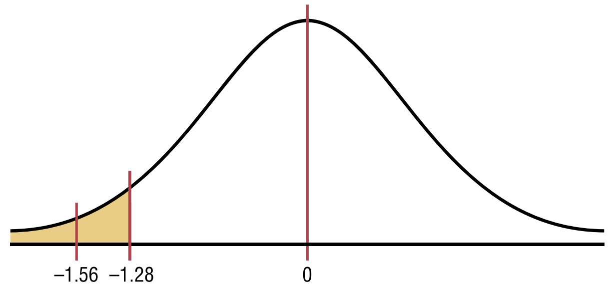
Step 2 Find the critical value. Since $\alpha = 0.10$ and the test is a left-tailed test, the critical value is -1.28.

Step 3 Compute the test value. Since the exercise gives raw data, it is necessary to find the mean of the data. Using the formulas in Chapter 3 or your calculator gives $\overline{X} = 75.0$ and $\sigma = 19.2$. Substitute in the formula

$$z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} = \frac{75 - 80}{19.2/\sqrt{36}} = -1.56$$

Step 4 Make the decision. Since the test value, −1.56, falls in the critical region, the decision is to reject the null hypothesis. See Figure 8–14.

Figure 8–14
Critical and Test Values for Example 8–4



Step 5 Summarize the results. There is enough evidence to support the claim that the average cost of men's athletic shoes is less than $80.

Comment: In Example 8–4, the difference is said to be significant. However, when the null hypothesis is rejected, there is always a chance of a type I error. In this case, the probability of a type I error is at most 0.10, or 10%.

Example 8–5

Cost of Rehabilitation

The Medical Rehabilitation Education Foundation reports that the average cost of rehabilitation for stroke victims is $24,672. To see if the average cost of rehabilitation is different at a particular hospital, a researcher selects a random sample of 35 stroke victims at the hospital and finds that the average cost of their rehabilitation is $26,343. The standard deviation of the population is $3251. At $\alpha = 0.01$, can it be concluded that the average cost of stroke rehabilitation at a particular hospital is different from $24,672?

Source: Snapshot, *USA TODAY.*

Solution

Step 1 State the hypotheses and identify the claim.

H_0: μ = $24,672 and H_1: $\mu \neq$ $24,672 (claim)

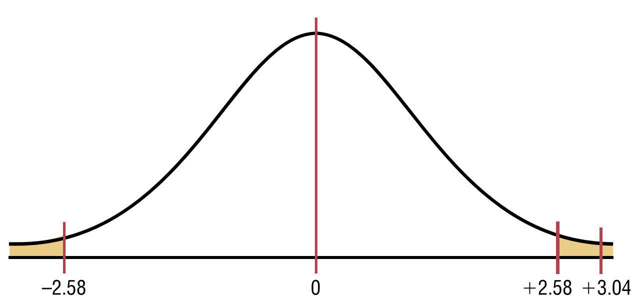
Step 2 Find the critical values. Since $\alpha = 0.01$ and the test is a two-tailed test, the critical values are +2.58 and −2.58.

Step 3 Compute the test value.

$$z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} = \frac{26{,}343 - 24{,}672}{3251/\sqrt{35}} = 3.04$$

Step 4 Make the decision. Reject the null hypothesis, since the test value falls in the critical region, as shown in Figure 8–15.

Figure 8–15
Critical and Test Values for Example 8–5



Step 5 Summarize the results. There is enough evidence to support the claim that the average cost of rehabilitation at the particular hospital is different from $24,672.

Students sometimes have difficulty summarizing the results of a hypothesis test. Figure 8–16 shows the four possible outcomes and the summary statement for each situation.

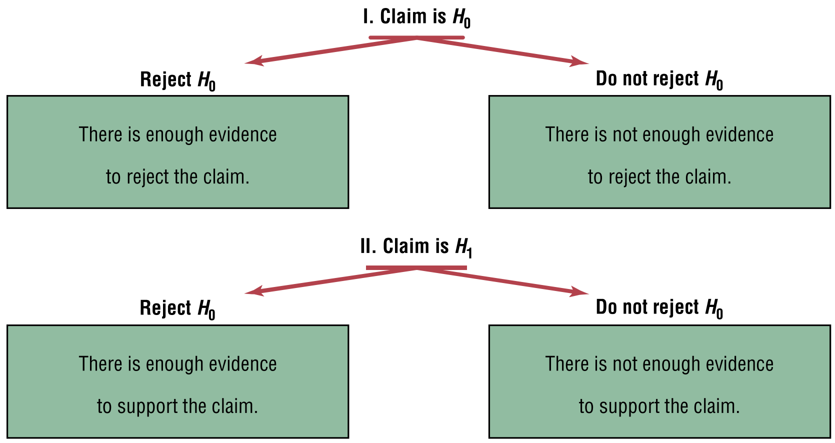


Figure 8–16
Outcomes of a Hypothesis-Testing Situation

First, the claim can be either the null or alternative hypothesis, and one should identify which it is. Second, after the study is completed, the null hypothesis is either rejected or not rejected. From these two facts, the decision can be identified in the appropriate block of Figure 8–16.

For example, suppose a researcher claims that the mean weight of an adult animal of a particular species is 42 pounds. In this case, the claim would be the null hypothesis, H_0: $\mu = 42$, since the researcher is asserting that the parameter is a specific value. If the null hypothesis is rejected, the conclusion would be that there is enough evidence to reject the claim that the mean weight of the adult animal is 42 pounds. See Figure 8–17(a).

On the other hand, suppose the researcher claims that the mean weight of the adult animals is not 42 pounds. The claim would be the alternative hypothesis H_1: $\mu \neq 42$. Furthermore, suppose that the null hypothesis is not rejected. The conclusion, then, would be that there is not enough evidence to support the claim that the mean weight of the adult animals is not 42 pounds. See Figure 8–17(b).

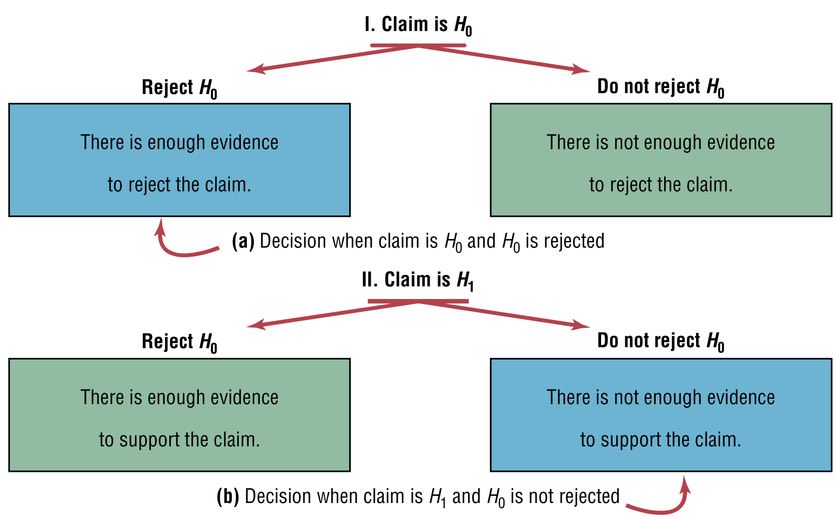


Figure 8–17
Outcomes of a Hypothesis-Testing Situation for Two Specific Cases

Again, remember that nothing is being proved true or false. The statistician is only stating that there is or is not enough evidence to say that a claim is *probably* true or false. As noted previously, the only way to prove something would be to use the entire population under study, and usually this cannot be done, especially when the population is large.

P-Value Method for Hypothesis Testing

Statisticians usually test hypotheses at the common α levels of 0.05 or 0.01 and sometimes at 0.10. Recall that the choice of the level depends on the seriousness of the type I error. Besides listing an α value, many computer statistical packages give a *P*-value for hypothesis tests.

> The ***P*-value** (or probability value) is the probability of getting a sample statistic (such as the mean) or a more extreme sample statistic in the direction of the alternative hypothesis when the null hypothesis is true.

In other words, the *P*-value is the actual area under the standard normal distribution curve (or other curve, depending on what statistical test is being used) representing the probability of a particular sample statistic or a more extreme sample statistic occurring if the null hypothesis is true.

For example, suppose that an alternative hypothesis is H_1: $\mu > 50$ and the mean of a sample is $\overline{X} = 52$. If the computer printed a *P*-value of 0.0356 for a statistical test, then the probability of getting a sample mean of 52 or greater is 0.0356 if the true population mean is 50 (for the given sample size and standard deviation). The relationship between the *P*-value and the α value can be explained in this manner. For $P = 0.0356$, the null hypothesis would be rejected at $\alpha = 0.05$ but not at $\alpha = 0.01$. See Figure 8–18.

When the hypothesis test is two-tailed, the area in one tail must be doubled. For a two-tailed test, if α is 0.05 and the area in one tail is 0.0356, the *P*-value will be $2(0.0356) = 0.0712$. That is, the null hypothesis should not be rejected at $\alpha = 0.05$, since 0.0712 is greater than 0.05. In summary, then, if the *P*-value is less than α, reject the null hypothesis. If the *P*-value is greater than α, do not reject the null hypothesis.

The *P*-values for the *z* test can be found by using Table E in Appendix C. First find the area under the standard normal distribution curve corresponding to the *z* test value. For a left-tailed test, use the area given in the table; for a right-tailed test, use 1.0000 minus the area given in the table. To get the *P*-value for a two-tailed test, double the area you found in the tail. This procedure is shown in step 3 of Examples 8–6 and 8–7.

The *P*-value method for testing hypotheses differs from the traditional method somewhat. The steps for the *P*-value method are summarized next.

Figure 8–18

Comparison of α Values and *P*-Values