THEOREM 1

Analyticity of the Logarithm

For every $n = 0, \pm 1, \pm 2, \cdots$ formula (3) defines a function, which is analytic, except at 0 and on the negative real axis, and has the derivative

(6) $$(\ln z)' = \frac{1}{z} \qquad (z \text{ not } 0 \text{ or negative real}).$$

PROOF We show that the Cauchy–Riemann equations are satisfied. From (1)–(3) we have

$$\ln z = \ln r + i(\theta + c) = \frac{1}{2} \ln (x^2 + y^2) + i\left(\arctan \frac{y}{x} + c\right)$$

where the constant c is a multiple of 2π. By differentiation,

$$u_x = \frac{x}{x^2 + y^2} = v_y = \frac{1}{1 + (y/x)^2} \cdot \frac{1}{x}$$

$$u_y = \frac{y}{x^2 + y^2} = -v_x = -\frac{1}{1 + (y/x)^2} \left(-\frac{y}{x^2}\right).$$

Hence the Cauchy–Riemann equations hold. [Confirm this by using these equations in polar form, which we did not use since we proved them only in the problems (to Sec. 13.4).] Formula (4) in Sec. 13.4 now gives (6),

$$(\ln z)' = u_x + iv_x = \frac{x}{x^2 + y^2} + i \frac{1}{1 + (y/x)^2} \left(-\frac{y}{x^2}\right) = \frac{x - iy}{x^2 + y^2} = \frac{1}{z}. \quad \blacksquare$$

Each of the infinitely many functions in (3) is called a **branch** of the logarithm. The negative real axis is known as a **branch cut** and is usually graphed as shown in Fig. 335. The branch for $n = 0$ is called the **principal branch** of $\ln z$.

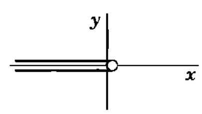

Fig. 335. Branch cut for ln z

General Powers

General powers of a complex number $z = x + iy$ are defined by the formula

(7) $$z^c = e^{c \ln z} \qquad (c \text{ complex}, z \neq 0).$$

Since $\ln z$ is infinitely many-valued, z^c will, in general, be multivalued. The particular value

$$z^c = e^{c \operatorname{Ln} z}$$

is called the **principal value** *of* z^c.

If $c = n = 1, 2, \cdots$, then z^n is single-valued and identical with the usual nth power of z. If $c = -1, -2, \cdots$, the situation is similar.

If $c = 1/n$, where $n = 2, 3, \cdots$, then

$$z^c = \sqrt[n]{z} = e^{(1/n)\ln z} \qquad (z \neq 0),$$

the exponent is determined up to multiples of $2\pi i/n$ and we obtain the n distinct values of the nth root, in agreement with the result in Sec. 13.2. If $c = p/q$, the quotient of two positive integers, the situation is similar, and z^c has only finitely many distinct values. However, if c is real irrational or genuinely complex, then z^c is infinitely many-valued.

EXAMPLE 3 General Power

$$i^i = e^{i \ln i} = \exp(i \ln i) = \exp\left[i\left(\frac{\pi}{2} i \pm 2n\pi i\right)\right] = e^{-(\pi/2)\mp 2n\pi}.$$

All these values are real, and the principal value ($n = 0$) is $e^{-\pi/2}$.

Similarly, by direct calculation and multiplying out in the exponent,

$$\begin{aligned}(1+i)^{2-i} &= \exp\left[(2-i)\ln(1+i)\right] = \exp\left[(2-i)\{\ln\sqrt{2} + \tfrac{1}{4}\pi i \pm 2n\pi i\}\right] \\ &= 2e^{\pi/4 \pm 2n\pi}\left[\sin\left(\tfrac{1}{2}\ln 2\right) + i\cos\left(\tfrac{1}{2}\ln 2\right)\right].\end{aligned}$$ ■

It is a ***convention*** that for real positive $z = x$ the expression z^c means $e^{c \ln x}$ where $\ln x$ is the elementary real natural logarithm (that is, the principal value $\operatorname{Ln} z$ $(z = x > 0)$ in the sense of our definition). Also, if $z = e$, the base of the natural logarithm, $z^c = e^c$ is *conventionally* regarded as the unique value obtained from (1) in Sec. 13.5.

From (7) we see that for any complex number a,

$$a^z = e^{z \ln a}. \tag{8}$$

We have now introduced the complex functions needed in practical work, some of them (e^z, $\cos z$, $\sin z$, $\cosh z$, $\sinh z$) entire (Sec. 13.5), some of them ($\tan z$, $\cot z$, $\tanh z$, $\coth z$) analytic except at certain points, and one of them ($\ln z$) splitting up into infinitely many functions, each analytic except at 0 and on the negative real axis.

For the **inverse trigonometric** and **hyperbolic functions** see the problem set.

[illegible]

1–9 Principal Value Ln z. Find $\operatorname{Ln} z$ when z equals:

1. -10
2. $2 + 2i$
3. $2 - 2i$
4. $-5 \pm 0.1i$
5. $-3 - 4i$
6. -100
7. $0.6 + 0.8i$
8. $-ei$
9. $1 - i$

10–16 All Values of ln z. Find all values and graph some of them in the complex plane.

10. $\ln 1$
11. $\ln(-1)$
12. $\ln e$
13. $\ln(-6)$
14. $\ln(4 + 3i)$
15. $\ln(-e^{-i})$
16. $\ln(e^{3i})$

17. Show that the set of values of $\ln(i^2)$ differs from the set of values of $2 \ln i$.

18–21 Equations. Solve for z:

18. $\ln z = (2 - \frac{1}{2}i)\pi$
19. $\ln z = 0.3 + 0.7i$
20. $\ln z = e - \pi i$
21. $\ln z = 2 + \frac{1}{4}\pi i$

22–28 **General Powers.** Showing the details of your work, find the principal value of:

22. i^{2i}, $(2i)^i$
23. 4^{3+i}
24. $(1-i)^{1+i}$
25. $(1+i)^{1-i}$
26. $(-1)^{1-2i}$
27. $i^{1/2}$
28. $(3-4i)^{1/3}$

29. How can you find the answer to Prob. 24 from the answer to Prob. 25?

30. **TEAM PROJECT. Inverse Trigonometric and Hyperbolic Functions.** By definition, the **inverse sine** $w = \arcsin z$ is the relation such that $\sin w = z$. The **inverse cosine** $w = \arccos z$ is the relation such that $\cos w = z$. The **inverse tangent, inverse cotangent, inverse hyperbolic sine,** etc., are defined and denoted in a similar fashion. (Note that all these relations are ***multivalued.***) Using $\sin w = (e^{iw} - e^{-iw})/(2i)$ and similar representations of $\cos w$, etc., show that

(a) $\arccos z = -i \ln(z + \sqrt{z^2 - 1})$

(b) $\arcsin z = -i \ln(iz + \sqrt{1 - z^2})$

(c) $\operatorname{arccosh} z = \ln(z + \sqrt{z^2 - 1})$

(d) $\operatorname{arcsinh} z = \ln(z + \sqrt{z^2 + 1})$

(e) $\arctan z = \dfrac{i}{2} \ln \dfrac{i+z}{i-z}$

(f) $\operatorname{arctanh} z = \dfrac{1}{2} \ln \dfrac{1+z}{1-z}$

(g) Show that $w = \arcsin z$ is infinitely many-valued, and if w_1 is one of these values, the others are of the form $w_1 \pm 2n\pi$ and $\pi - w_1 \pm 2n\pi$, $n = 0, 1, \cdots$. (The *principal value of* $w = u + iv = \arcsin z$ is defined to be the value for which $-\pi/2 \leqq u \leqq \pi/2$ if $v \geqq 0$ and $-\pi/2 < u < \pi/2$ if $v < 0$.)

CHAPTER 13 REVIEW QUESTIONS AND PROBLEMS

1. Add, subtract, multiply, and divide $26 - 7i$ and $3 + 4i$ as well as their complex conjugates.
2. Write the two given numbers in Prob. 1 in polar form. Find the principal value of their arguments.
3. What is the triangle inequality? Its geometric meaning? Its significance?
4. If you know the values of $\sqrt[6]{1}$, how do you get from them the values of $\sqrt[6]{z}$ for any z?
5. State the definition of the derivative from memory. It looks similar to that in calculus. But what is the big difference?
6. What is an analytic function? How would you test for analyticity?
7. Can a function be differentiable at a point without being analytic there? If yes, give an example.
8. Are $|z|$, $\bar{z}$, Re z, Im z analytic? Give reason.
9. State the definitions of e^z, $\cos z$, $\sin z$, $\cosh z$, $\sinh z$ and the relations between these functions. Do these relations have analogs in real?
10. What properties of e^z are similar to those of e^x ? Which one is different?
11. What is the fundamental region of e^z ? Its significance?
12. What is an entire function? Give examples.
13. Why is $\ln z$ much more complicated than $\ln x$? Explain from memory.
14. What is the principal value of $\ln z$?
15. How is the general power z^c defined? Give examples.

16–21 **Complex Numbers.** Find, in the form $x + iy$, showing the details:

16. $(1+i)^{12}$
17. $(-2+6i)^2$
18. $1/(3-7i)$
19. $(1-i)/(1+i)^2$
20. $\sqrt{-5-12i}$
21. $(43-19i)/(8+i)$

22–26 **Polar Form.** Represent in polar form, with the principal argument:

22. $1 - 3i$
23. $-6 + 6i$
24. $\sqrt{20}/(4+2i)$
25. $-12i$
26. $2 + 2i$

27–30 **Roots.** Find and graph all values of

27. $\sqrt{8i}$
28. $\sqrt[4]{256}$
29. $\sqrt[4]{-1}$
30. $\sqrt{32-24i}$

31–35 **Analytic Functions.** Find $f(z) = u(x, y) + iv(x, y)$ with u or v as given. Check for analyticity.

31. $u = x/(x^2 + y^2)$
32. $v = e^{-3x} \sin 3y$
33. $u = x^2 - 2xy - y^2$
34. $u = \cos 2x \cosh 2y$
35. $v = e^{x^2-y^2} \sin 2xy$

36–39 **Harmonic Functions.** Are the following functions harmonic? If so, find a harmonic conjugate.

36. x^2y^2
37. xy
38. $e^{-x/2} \cos \frac{1}{2}y$
39. $x^2 + y^2$

40–45 **Special Function Values.** Find the values of

40. $\sin(3 + 4\pi i)$
41. $\sinh 4\pi i$
42. $\cos(5\pi + 2i)$
43. $\operatorname{Ln}(0.8 + 0.6i)$
44. $\tan(1 + i)$
45. $\cosh(1 + \pi i)$

Complex Numbers and Functions

For arithmetic operations with **complex numbers**

(1) $$z = x + iy = re^{i\theta} = r(\cos\theta + i\sin\theta),$$

$r = |z| = \sqrt{x^2 + y^2}$, $\theta = \arctan(y/x)$, and for their representation in the complex plane, see Secs. 13.1 and 13.2.

A complex function $f(z) = u(x, y) + iv(x, y)$ is **analytic** in a domain D if it has a **derivative** (Sec. 13.3)

(2) $$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

everywhere in D. Also, $f(z)$ is *analytic at a point* $z = z_0$ if it has a derivative in a neighborhood of z_0 (not merely at z_0 itself).

If $f(z)$ is analytic in D, then $u(x, y)$ and $v(x, y)$ satisfy the (very important!) **Cauchy–Riemann equations** (Sec. 13.4)

(3) $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

everywhere in D. Then u and v also satisfy **Laplace's equation**

(4) $$u_{xx} + u_{yy} = 0, \qquad v_{xx} + v_{yy} = 0$$

everywhere in D. If $u(x, y)$ and $v(x, y)$ are continuous and have *continuous* partial derivatives in D that satisfy (3) in D, then $f(z) = u(x, y) + iv(x, y)$ is analytic in D. See Sec. 13.4. (More on Laplace's equation and complex analysis follows in Chap. 18.)

The complex **exponential function** (Sec. 13.5)

(5) $$e^z = \exp z = e^x(\cos y + i \sin y)$$

reduces to e^x if $z = x$ ($y = 0$). It is periodic with $2\pi i$ and has the derivative e^z.

The **trigonometric functions** are (Sec. 13.6)

(6) $$\begin{aligned} \cos z &= \frac{1}{2}(e^{iz} + e^{-iz}) = \cos x \cosh y - i \sin x \sinh y \\ \sin z &= \frac{1}{2i}(e^{iz} - e^{-iz}) = \sin x \cosh y + i \cos x \sinh y \end{aligned}$$

and, furthermore,

$$\tan z = (\sin z)/\cos z, \qquad \cot z = 1/\tan z, \quad \text{etc.}$$

The **hyperbolic functions** are (Sec. 13.6)

(7) $\cosh z = \frac{1}{2}(e^z + e^{-z}) = \cos iz, \qquad \sinh z = \frac{1}{2}(e^z - e^{-z}) = -i \sin iz$

etc. The functions (5)–(7) are **entire,** that is, analytic everywhere in the complex plane.

The **natural logarithm** is (Sec. 13.7)

(8) $$\ln z = \ln |z| + i \arg z = \ln |z| + i \operatorname{Arg} z \pm 2n\pi i$$

where $z \neq 0$ and $n = 0, 1, \cdots$. Arg z is the **principal value** of arg z, that is, $-\pi < \operatorname{Arg} z \leqq \pi$. We see that $\ln z$ is infinitely many-valued. Taking $n = 0$ gives the **principal value** Ln z of ln z; thus $\operatorname{Ln} z = \ln |z| + i \operatorname{Arg} z$.

General powers are defined by (Sec. 13.7)

(9) $$z^c = e^{c \ln z} \qquad (c \text{ complex}, z \neq 0).$$