CHAPTER 5

Transportation Model and Its Variants

Chapter Guide. The transportation model is a special class of linear programs that deals with shipping a commodity from *sources* (e.g., factories) to *destinations* (e.g., warehouses). The objective is to determine the shipping schedule that minimizes the total shipping cost while satisfying supply and demand limits. The application of the transportation model can be extended to other areas of operation, including inventory control, employment scheduling, and personnel assignment.

As you study the material in this chapter, keep in mind that the steps of the transportation algorithm are precisely those of the simplex method. Another point is that the transportation algorithm was developed in the early days of OR to enhance hand computations. Now, with the tremendous power of the computer, such shortcuts may not be warranted and, indeed, are never used in commercial codes in the strict manner presented in this chapter. Nevertheless, the presentation shows that the special transportation tableau is useful in modeling a class of problems in a concise manner (as opposed to the familiar LP model with explicit objective function and constraints). In particular, the transportation tableau format simplifies the solution of the problem by Excel Solver. The representation also provides interesting ideas about how the basic theory of linear programming is exploited to produce shortcuts in computations.

You will find TORA's tutorial module helpful in understanding the details of the transportation algorithm. The module allows you to make the decisions regarding the logic of the computations with immediate feedback.

This chapter includes a summary of 1 real-life application, 12 solved examples, 1 Solver model, 4 AMPL models, 46 end-of-section problems, and 5 cases. The cases are in Appendix E on the CD. The AMPL/Excel/Solver/TORA programs are in folder ch5Files.

Real-life Application—Scheduling Appointments at Australian Trade Events

The Australian Tourist Commission (ATC) organizes trade events around the world to provide a forum for Australian sellers to meet international buyers of tourism products, including accommodation, tours, and transport. During these events, sellers are

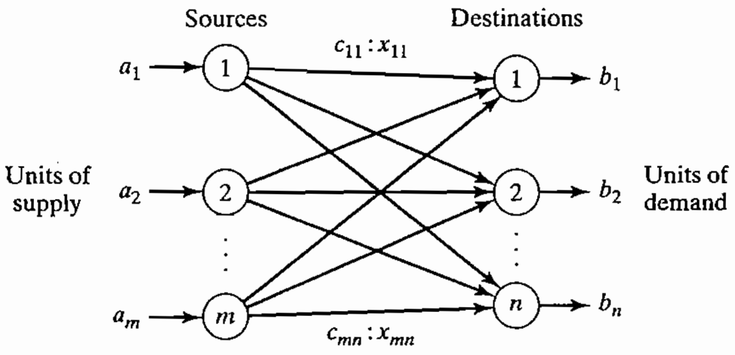

FIGURE 5.1

Representation of the transportation model with nodes and arcs

stationed in booths and are visited by buyers according to scheduled appointments. Because of the limited number of time slots available in each event and the fact that the number of buyers and sellers can be quite large (one such event held in Melbourne in 1997 attracted 620 sellers and 700 buyers), ATC attempts to schedule the seller-buyer appointments in advance of the event in a manner that maximizes preferences. The model has resulted in greater satisfaction for both the buyers and sellers. Case 3 in Chapter 24 on the CD provides the details of the study.

5.1 DEFINITION OF THE TRANSPORTATION MODEL

The general problem is represented by the network in Figure 5.1. There are m sources and n destinations, each represented by a **node**. The **arcs** represent the routes linking the sources and the destinations. Arc (i, j) joining source i to destination j carries two pieces of information: the transportation cost per unit, c_{ij}, and the amount shipped, x_{ij}. The amount of supply at source i is a_i and the amount of demand at destination j is b_j. The objective of the model is to determine the unknowns x_{ij} that will minimize the total transportation cost while satisfying all the supply and demand restrictions.

Example 5.1-1

MG Auto has three plants in Los Angeles, Detroit, and New Orleans, and two major distribution centers in Denver and Miami. The capacities of the three plants during the next quarter are 1000, 1500, and 1200 cars. The quarterly demands at the two distribution centers are 2300 and 1400 cars. The mileage chart between the plants and the distribution centers is given in Table 5.1.

The trucking company in charge of transporting the cars charges 8 cents per mile per car. The transportation costs per car on the different routes, rounded to the closest dollar, are given in Table 5.2.

The LP model of the problem is given as

$$\text{Minimize } z = 80x_{11} + 215x_{12} + 100x_{21} + 108x_{22} + 102x_{31} + 68x_{32}$$

TABLE 5.1 Mileage Chart

	Denver	Miami
Los Angeles	1000	2690
Detroit	1250	1350
New Orleans	1275	850

TABLE 5.2 Transportation Cost per Car

	Denver (1)	Miami (2)
Los Angeles (1)	$80	$215
Detroit (2)	$100	$108
New Orleans (3)	$102	$68

subject to

$$\begin{array}{lllllll}
x_{11} + x_{12} & & & & & = 1000 & \text{(Los Angeles)} \\
& x_{21} + x_{22} & & & & = 1500 & \text{(Detroit)} \\
& & + x_{31} + x_{32} & & & = 1200 & \text{(New Oreleans)} \\
x_{11} & + x_{21} + & x_{31} & & & = 2300 & \text{(Denver)} \\
x_{12} + & x_{22} & + x_{32} & & & = 1400 & \text{(Miami)} \\
\end{array}$$

$$x_{ij} \geq 0, i = 1, 2, 3, j = 1, 2$$

These constraints are all equations because the total supply from the three sources (= 1000 + 1500 + 1200 = 3700 cars) equals the total demand at the two destinations (= 2300 + 1400 = 3700 cars).

The LP model can be solved by the simplex method. However, with the special structure of the constraints we can solve the problem more conveniently using the **transportation tableau** shown in Table 5.3.

TABLE 5.3 MG Transportation Model

	Denver	Miami	Supply
Los Angeles	80 x_{11}	215 x_{12}	**1000**
Detroit	100 x_{21}	108 x_{22}	**1500**
New Orleans	102 x_{31}	68 x_{32}	**1200**
Demand	**2300**	**1400**	

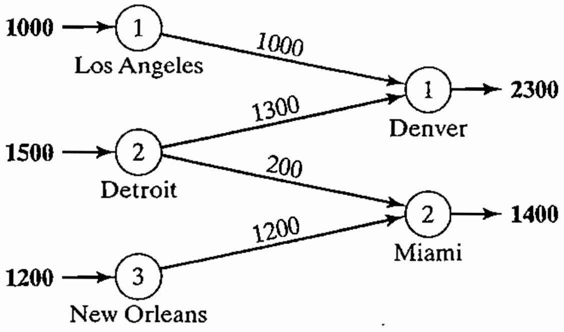

FIGURE 5.2
Optimal solution of MG Auto model

The optimal solution in Figure 5.2 (obtained by TORA[1]) calls for shipping 1000 cars from Los Angeles to Denver, 1300 from Detroit to Denver, 200 from Detroit to Miami, and 1200 from New Orleans to Miami. The associated minimum transportation cost is computed as 1000 × \$80 + 1300 × \$100 + 200 × \$108 + 1200 × \$68 = \$313,200.

Balancing the Transportation Model. The transportation algorithm is based on the assumption that the model is balanced, meaning that the total demand equals the total supply. If the model is unbalanced, we can always add a dummy source or a dummy destination to restore balance.

Example 5.1-2

In the MG model, suppose that the Detroit plant capacity is 1300 cars (instead of 1500). The total supply (= 3500 cars) is less than the total demand (= 3700 cars), meaning that part of the demand at Denver and Miami will not be satisfied.

Because the demand exceeds the supply, a dummy source (plant) with a capacity of 200 cars (= 3700 − 3500) is added to balance the transportation model. The unit transportation costs from the dummy plant to the two destinations are zero because the plant does not exist.

Table 5.4 gives the balanced model together with its optimum solution. The solution shows that the dummy plant ships 200 cars to Miami, which means that Miami will be 200 cars short of satisfying its demand of 1400 cars.

We can make sure that a specific destination does not experience shortage by assigning a very high unit transportation cost from the dummy source to that destination. For example, a penalty of \$1000 in the dummy-Miami cell will prevent shortage at Miami. Of course, we cannot use this "trick" with all the destinations, because shortage must occur somewhere in the system.

The case where the supply exceeds the demand can be demonstrated by assuming that the demand at Denver is 1900 cars only. In this case, we need to add a dummy distribution center to "receive" the surplus supply. Again, the unit transportation costs to the dummy distribution center are zero, unless we require a factory to "ship out" completely. In this case, we must assign a high unit transportation cost from the designated factory to the dummy destination.

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[1] To use TORA, from Main Menu select Transportation Model. From the SOLVE/MODIFY menu, select Solve ⇒ Final solution to obtain a summary of the optimum solution. A detailed description of the iterative solution of the transportation model is given in Section 5.3.3.

TABLE 5.4 MG Model with Dummy Plant

	Denver	Miami	Supply
Los Angeles	80 **1000**	215	**1000**
Detroit	100 **1300**	108	**1300**
New Orleans	102	68 **1200**	**1200**
Dummy Plant	0	0 **200**	**200**
Demand	**2300**	**1400**	

TABLE 5.5 MG Model with Dummy Destination

	Denver	Miami	Dummy	
Los Angeles	80 **1000**	215	0	**1000**
Detroit	100 **900**	108 **200**	0 **400**	**1500**
New Orleans	102	68 **1200**	0	**1200**
Demand	**1900**	**1400**	**400**	

Table 5.5 gives the new model and its optimal solution (obtained by TORA). The solution shows that the Detroit plant will have a surplus of 400 cars.

PROBLEM SET 5.1A[2]

1. True or False?
   - **(a)** To balance a transportation model, it may be necessary to add both a dummy source and a dummy destination.
   - **(b)** The amounts shipped to a dummy destination represent surplus at the shipping source.
   - **(c)** The amounts shipped from a dummy source represent shortages at the receiving destinations.

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[2] In this set, you may use TORA to find the optimum solution. AMPL and Solver models for the transportation problem will be introduced at the end of Section 5.3.2.

2. In each of the following cases, determine whether a dummy source or a dummy destination must be added to balance the model.

(a) Supply: $a_1 = 10, a_2 = 5, a_3 = 4, a_4 = 6$
Demand: $b_1 = 10, b_2 = 5, b_3 = 7, b_4 = 9$

(b) Supply: $a_1 = 30, a_2 = 44$
Demand: $b_1 = 25, b_2 = 30, b_3 = 10$

3. In Table 5.4 of Example 5.1-2, where a dummy plant is added, what does the solution mean when the dummy plant "ships" 150 cars to Denver and 50 cars to Miami?

***4.** In Table 5.5 of Example 5.1-2, where a dummy destination is added, suppose that the Detroit plant must ship out *all* its production. How can this restriction be implemented in the model?

5. In Example 5.1-2, suppose that for the case where the demand exceeds the supply (Table 5.4), a penalty is levied at the rate of \$200 and \$300 for each undelivered car at Denver and Miami, respectively. Additionally, no deliveries are made from the Los Angeles plant to the Miami distribution center. Set up the model, and determine the optimal shipping schedule for the problem.

***6.** Three electric power plants with capacities of 25, 40, and 30 million kWh supply electricity to three cities. The maximum demands at the three cities are estimated at 30, 35, and 25 million kWh. The price per million kWh at the three cities is given in Table 5.6.

During the month of August, there is a 20% increase in demand at each of the three cities, which can be met by purchasing electricity from another network at a premium rate of \$1000 per million kWh. The network is not linked to city 3, however. The utility company wishes to determine the most economical plan for the distribution and purchase of additional energy.

(a) Formulate the problem as a transportation model.

(b) Determine an optimal distribution plan for the utility company.

(c) Determine the cost of the additional power purchased by each of the three cities.

7. Solve Problem 6, assuming that there is a 10% power transmission loss through the network.

8. Three refineries with daily capacities of 6, 5, and 8 million gallons, respectively, supply three distribution areas with daily demands of 4, 8, and 7 million gallons, respectively. Gasoline is transported to the three distribution areas through a network of pipelines. The transportation cost is 10 cents per 1000 gallons per pipeline mile. Table 5.7 gives the mileage between the refineries and the distribution areas. Refinery 1 is not connected to distribution area 3.

(a) Construct the associated transportation model.

(b) Determine the optimum shipping schedule in the network.

TABLE 5.6 Price/Million kWh for Problem 6

		City	
	1	2	3
Plant 1	\$600	\$700	\$400
Plant 2	\$320	\$300	\$350
Plant 3	\$500	\$480	\$450

TABLE 5.7 Mileage Chart for Problem 8

	Distribution area 1	2	3
Refinery 1	120	180	—
2	300	100	80
3	200	250	120

***9.** In Problem 8, suppose that the capacity of refinery 3 is 6 million gallons only and that distribution area 1 must receive all its demand. Additionally, any shortages at areas 2 and 3 will incur a penalty of 5 cents per gallon.

(a) Formulate the problem as a transportation model.

(b) Determine the optimum shipping schedule.

10. In Problem 8, suppose that the daily demand at area 3 drops to 4 million gallons. Surplus production at refineries 1 and 2 is diverted to other distribution areas by truck. The transportation cost per 100 gallons is $1.50 from refinery 1 and $2.20 from refinery 2. Refinery 3 can divert its surplus production to other chemical processes within the plant.

(a) Formulate the problem as a transportation model.

(b) Determine the optimum shipping schedule.

11. Three orchards supply crates of oranges to four retailers. The daily demand amounts at the four retailers are 150, 150, 400, and 100 crates, respectively. Supplies at the three orchards are dictated by available regular labor and are estimated at 150, 200, and 250 crates daily. However, both orchards 1 and 2 have indicated that they could supply more crates, if necessary, by using overtime labor. Orchard 3 does not offer this option. The transportation costs per crate from the orchards to the retailers are given in Table 5.8.

(a) Formulate the problem as a transportation model.

(b) Solve the problem.

(c) How many crates should orchards 1 and 2 supply using overtime labor?

12. Cars are shipped from three distribution centers to five dealers. The shipping cost is based on the mileage between the sources and the destinations, and is independent of whether the truck makes the trip with partial or full loads. Table 5.9 summarizes the mileage between the distribution centers and the dealers together with the monthly supply and demand figures given in *number* of cars. A full truckload includes 18 cars. The transportation cost per truck mile is $25.

(a) Formulate the associated transportation model.

(b) Determine the optimal shipping schedule.

TABLE 5.8 Transportation Cost/Crate for Problem 11

	Retailer 1	2	3	4
Orchard 1	$1	$2	$3	$2
2	$2	$4	$1	$2
3	$1	$3	$5	$3

TABLE 5.9 Mileage Chart and Supply and Demand for Problem 12

		Dealer				
	1	2	3	4	5	Supply
Center 1	100	150	200	140	35	**400**
Center 2	50	70	60	65	80	**200**
Center 3	40	90	100	150	130	**150**
Demand	**100**	**200**	**150**	**160**	**140**	

13. MG Auto, of Example 5.1-1, produces four car models: $M1, M2, M3$, and $M4$. The Detroit plant produces models $M1, M2$, and $M4$. Models $M1$ and $M2$ are also produced in New Orleans. The Los Angeles plant manufactures models $M3$ and $M4$. The capacities of the various plants and the demands at the distribution centers are given in Table 5.10.

The mileage chart is the same as given in Example 5.1-1, and the transportation rate remains at 8 cents per car mile for all models. Additionally, it is possible to satisfy a percentage of the demand for some models from the supply of others according to the specifications in Table 5.11.

(a) Formulate the corresponding transportation model.

(b) Determine the optimum shipping schedule.

(Hint: Add four new destinations corresponding to the new combinations $[M1, M2]$, $[M3, M4]$, $[M1, M2]$, and $[M2, M4]$. The demands at the new destinations are determined from the given percentages.)

TABLE 5.10 Capacities and Demands for Problem 13

	Model				
	$M1$	$M2$	$M3$	$M4$	Totals
Plant					
Los Angeles	—	—	700	300	1000
Detroit	500	600	—	400	1500
New Orleans	800	400	—	—	1200
Distribution center					
Denver	700	500	500	600	2300
Miami	600	500	200	100	1400

TABLE 5.11 Interchangeable Models in Problem 13

Distribution center	Percentage of demand	Interchangeable models
Denver	10	$M1, M2$
	20	$M3, M4$
Miami	10	$M1, M2$
	5	$M2, M4$

5.2 NONTRADITIONAL TRANSPORTATION MODELS

The application of the transportation model is not limited to *transporting* commodities between geographical sources and destinations. This section presents two applications in the areas of production-inventory control and tool sharpening service.

Example 5.2-1 (Production-Inventory Control)

Boralis manufactures backpacks for serious hikers. The demand for its product occurs during March to June of each year. Boralis estimates the demand for the four months to be 100, 200, 180, and 300 units, respectively. The company uses part-time labor to manufacture the backpacks and, accordingly, its production capacity varies monthly. It is estimated that Boralis can produce 50, 180, 280, and 270 units in March through June. Because the production capacity and demand for the different months do not match, a current month's demand may be satisfied in one of three ways.

1. Current month's production.
2. Surplus production in an earlier month.
3. Surplus production in a later month (backordering).

In the first case, the production cost per backpack is \$40. The second case incurs an additional holding cost of \$.50 per backpack per month. In the third case, an additional penalty cost of \$2.00 per backpack is incurred for each month delay. Boralis wishes to determine the optimal production schedule for the four months.

The situation can be modeled as a transportation model by recognizing the following parallels between the elements of the production-inventory problem and the transportation model:

Transportation	Production-inventory
1. Source i	1. Production period i
2. Destination j	2. Demand period j
3. Supply amount at source i	3. Production capacity of period i
4. Demand at destination j	4. Demand for period j
5. Unit transportation cost from source i to destination j	5. Unit cost (production + inventory + penalty) in period i for period j

The resulting transportation model is given in Table 5.12.

TABLE 5.12 Transportation Model for Example 5.2-1

	1	2	3	4	Capacity
1	\$40.00	\$40.50	\$41.00	\$41.50	**50**
2	\$42.00	\$40.00	\$40.50	\$41.00	**180**
3	\$44.00	\$42.00	\$40.00	\$40.50	**280**
4	\$46.00	\$44.00	\$42.00	\$40.00	**270**
Demand	**100**	**200**	**180**	**300**	

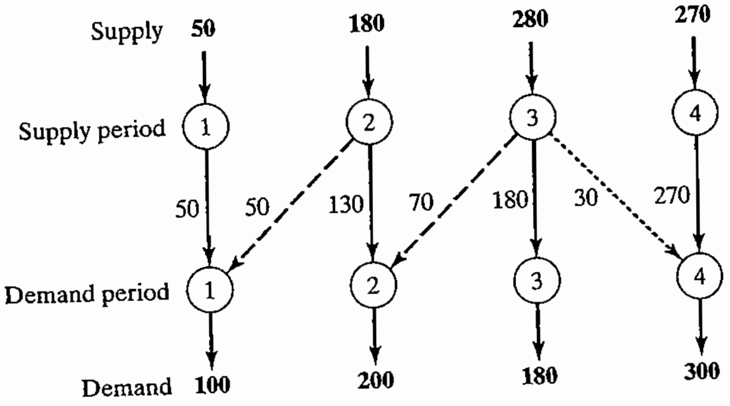

FIGURE 5.3
Optimal solution of the production–inventory model

The unit "transportation" cost from period i to period j is computed as

$$c_{ij} = \begin{cases} \text{Production cost in } i, i = j \\ \text{Production cost in } i + \text{holding cost from } i \text{ to } j, i < j \\ \text{Production cost in } i + \text{penaty cost from } i \text{ to } j, i > j \end{cases}$$

For example,

$$c_{11} = \$40.00$$

$$c_{24} = \$40.00 + (\$.50 + \$.50) = \$41.00$$

$$c_{41} = \$40.00 + (\$2.00 + \$2.00 + \$2.00) = \$46.00$$

The optimal solution is summarized in Figure 5.3. The dashed lines indicate back-ordering, the dotted lines indicate production for a future period, and the solid lines show production in a period for itself. The total cost is $31,455.

Example 5.2-2 (Tool Sharpening)

Arkansas Pacific operates a medium-sized saw mill. The mill prepares different types of wood that range from soft pine to hard oak according to a weekly schedule. Depending on the type of wood being milled, the demand for sharp blades varies from day to day according to the following 1-week (7-day) data:

Day	Mon.	Tue.	Wed.	Thu.	Fri.	Sat.	Sun.
Demand (blades)	24	12	14	20	18	14	22

The mill can satisfy the daily demand in the following manner:

1. Buy new blades at the cost of $12 a blade.
2. Use an overnight sharpening service at the cost of $6 a blade.
3. Use a slow 2-day sharpening service at the cost of $3 a blade.

The situation can be represented as a transportation model with eight sources and seven destinations. The destinations represent the 7 days of the week. The sources of the model are defined as follows: Source 1 corresponds to buying new blades, which, in the extreme case, can provide sufficient supply to cover the demand for all 7 days (= 24 + 12 + 14 + 20 + 18 + 14 + 22 = 124). Sources 2 to 8 correspond to the 7 days of the week. The amount of supply for each of these sources equals the number of used blades at the end of the associated day. For example, source 2 (i.e., Monday) will have a supply of used blades equal to the demand for Monday. The unit "transportation cost" for the model is $12, $6, or $3, depending on whether the blade is supplied from new blades, overnight sharpening, or 2-day sharpening. Notice that the overnight service means that used blades sent at the *end* of day i will be available for use at the *start* of day $i + 1$ or day $i + 2$, because the slow 2-day service will not be available until the *start* of day $i + 3$. The "disposal" column is a dummy destination needed to balance the model. The complete model and its solution are given in Table 5.13.

TABLE 5.13 Tool Sharpening Problem Expressed as a Transportation Model

	1 Mon.	2 Tue.	3 Wed.	4 Thu.	5 Fri.	6 Sat.	7 Sun.	8 Disposal	
1-New	$12 24	$12 2	$12	$12	$12	$12	$12	$0 98	124
2-Mon.	*M*	$6 10	$6 8	$3 6	$3	$3	$3	$0	24
3-Tue.	*M*	*M*	$6 6	$6	$3 6	$3	$3	$0	12
4-Wed.	*M*	*M*	*M*	$6 14	$6	$3	$3	$0	14
5-Thu.	*M*	*M*	*M*	*M*	$6 12	$6	$3 8	$0	20
6-Fri.	*M*	*M*	*M*	*M*	*M*	$6 14	$6	$0 4	18
7-Sat.	*M*	*M*	*M*	*M*	*M*	*M*	$6 14	$0	14
8-Sun.	*M*	*M*	*M*	*M*	*M*	*M*	*M*	$0 22	22
	24	12	14	20	18	14	22	124	

The problem has alternative optima at a cost of $840 (file toraEx5.2-2.txt). The following table summarizes one such solution.

	Number of sharp blades (Target day)			
Period	*New*	*Overnight*	*2-day*	Disposal
Mon.	24 (Mon.)	10 (Tue.) + 8 (Wed.)	6 (Thu.)	0
Tues.	2 (Tue.)	6 (Wed.)	6 (Fri.)	0
Wed.	0	14 (Thu.)	0	0
Thu.	0	12 (Fri.)	8 (Sun.)	0
Fri.	0	14 (Sat.)	0	4
Sat.	0	14 (Sun.)	0	0
Sun.	0	0	0	22

Remarks. The model in Table 5.13 is suitable only for the first week of operation because it does not take into account the *rotational* nature of the days of the week, in the sense that this week's days can act as sources for next week's demand. One way to handle this situation is to assume that the very first week of operation starts with all new blades for each day. From then on, we use a model consisting of exactly 7 sources and 7 destinations corresponding to the days of the week. The new model will be similar to Table 5.13 less source "New" and destination "Disposal." Also, only diagonal cells will be blocked (unit cost $= M$). The remaining cells will have a unit cost of either $3.00 or $6.00. For example, the unit cost for cell (Sat., Mon.) is $6.00 and that for cells (Sat., Tue.), (Sat., Wed.), (Sat., Thu.), and (Sat., Fri.) is $3.00. The table below gives the solution costing $372. As expected, the optimum solution will always use the 2-day service only. The problem has alternative optima (see file toraEx5.2-2a.txt).

	Week $i+1$							
Week i	*Mon.*	*Tue.*	*Wed.*	*Thu.*	*Fri.*	*Sat.*	*Sun.*	Total
Mon.				6			18	24
Tue.					8		4	12
Wed.	12					2		14
Thu.	8	12						20
Fri.	4		14					18
Sat.				14				14
Sun.					10	12		22
Total	24	12	14	20	18	14	22	

PROBLEM SET 5.2A[3]

1. In Example 5.2-1, suppose that the holding cost per unit is period-dependent and is given by 40, 30, and 70 cents for periods 1, 2, and 3, respectively. The penalty and production costs remain as given in the example. Determine the optimum solution and interpret the results.

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[3] In this set, you may use TORA to find the optimum solution. AMPL and Solver models for the transportation problem will be introduced at the end of Section 5.3.2.

***2.** In Example 5.2-2, suppose that the sharpening service offers 3-day service for $1 a blade on Monday and Tuesday (days 1 and 2). Reformulate the problem, and interpret the optimum solution.

3. In Example 5.2-2, if a blade is not used the day it is sharpened, a holding cost of 50 cents per blade per day is incurred. Reformulate the model, and interpret the optimum solution.

4. JoShop wants to assign four different categories of machines to five types of tasks. The numbers of machines available in the four categories are 25, 30, 20, and 30. The numbers of jobs in the five tasks are 20, 20, 30, 10, and 25. Machine category 4 cannot be assigned to task type 4. Table 5.14 provides the unit cost (in dollars) of assigning a machine category to a task type. The objective of the problem is to determine the optimum number of machines in each category to be assigned to each task type. Solve the problem and interpret the solution.

***5.** The demand for a perishable item over the next four months is 400, 300, 420, and 380 tons, respectively. The supply capacities for the same months are 500, 600, 200, and 300 tons. The purchase price per ton varies from month to month and is estimated at $100, $140, $120, and $150, respectively. Because the item is perishable, a current month's supply must be consumed within 3 months (starting with current month). The storage cost per ton per month is $3. The nature of the item does not allow back-ordering. Solve the problem as a transportation model and determine the optimum delivery schedule for the item over the next 4 months.

6. The demand for a special small engine over the next five quarters is 200, 150, 300, 250, and 400 units. The manufacturer supplying the engine has different production capacities estimated at 180, 230, 430, 300, and 300 for the five quarters. Back-ordering is not allowed, but the manufacturer may use overtime to fill the immediate demand, if necessary. The overtime capacity for each period is half the regular capacity. The production costs per unit for the five periods are $100, $96, $116, $102, and $106, respectively. The overtime production cost per engine is 50% higher than the regular production cost. If an engine is produced now for use in later periods, an additional storage cost of $4 per engine per period is incurred. Formulate the problem as a transportation model. Determine the optimum number of engines to be produced during regular time and overtime of each period.

7. Periodic preventive maintenance is carried out on aircraft engines, where an important component must be replaced. The numbers of aircraft scheduled for such maintenance over the next six months are estimated at 200, 180, 300, 198, 230, and 290, respectively. All maintenance work is done during the first day of the month, where a used component may be replaced with a new or an overhauled component. The overhauling of used components may be done in a local repair facility, where they will be ready for use at the beginning of next month, or they may be sent to a central repair shop, where a delay of

TABLE 5.14 Unit Costs for Problem 4

		Task type				
		1	2	3	4	5
Machine category	1	10	2	3	15	9
	2	5	10	15	2	4
	3	15	5	14	7	15
	4	20	15	13	—	8

TABLE 5.15 Bids per Acre for Problem 8

			Location	
		1	2	3
Bidder	1	$520	$210	$570
	2	—	$510	$495
	3	$650	—	$240
	4	$180	$430	$710

3 months (including the month in which maintenance occurs) is expected. The repair cost in the local shop is $120 per component. At the central facility, the cost is only $35 per component. An overhauled component used in a later month will incur an additional storage cost of $1.50 per unit per month. New components may be purchased at $200 each in month 1, with a 5% price increase every 2 months. Formulate the problem as a transportation model, and determine the optimal schedule for satisfying the demand for the component over the next six months.

8. The National Parks Service is receiving four bids for logging at three pine forests in Arkansas. The three locations include 10,000, 20,000, and 30,000 acres. A single bidder can bid for at most 50% of the total acreage available. The bids per acre at the three locations are given in Table 5.15. Bidder 2 does not wish to bid on location 1, and bidder 3 cannot bid on location 2.

(a) In the present situation, we need to *maximize* the total bidding revenue for the Parks Service. Show how the problem can be formulated as a transportation model.

(b) Determine the acreage that should be assigned to each of the four bidders.

5.3 THE TRANSPORTATION ALGORITHM

The transportation algorithm follows the *exact steps* of the simplex method (Chapter 3). However, instead of using the regular simplex tableau, we take advantage of the special structure of the transportation model to organize the computations in a more convenient form.

The special transportation algorithm was developed early on when hand computations were the norm and the shortcuts were warranted. Today, we have powerful computer codes that can solve a transportation model of any size as a regular LP.[4] Nevertheless, the transportation algorithm, aside from its historical significance, does provide insight into the use of the theoretical primal-dual relationships (introduced in Section 4.2) to achieve a practical end result, that of improving hand computations. The exercise is theoretically intriguing.

The details of the algorithm are explained using the following numeric example.

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[4] In fact, TORA handles all necessary computations in the background using the regular simplex method and uses the transportation model format only as a screen "veneer."

TABLE 5.16 SunRay Transportation Model

	Mill				
	1	2	3	4	Supply
1	10 x_{11}	2 x_{12}	20 x_{13}	11 x_{14}	**15**
Silo 2	12 x_{21}	7 x_{22}	9 x_{23}	20 x_{24}	**25**
3	4 x_{31}	14 x_{32}	16 x_{33}	18 x_{34}	**10**
Demand	**5**	**15**	**15**	**15**	

Example 5.3-1 (SunRay Transport)

SunRay Transport Company ships truckloads of grain from three silos to four mills. The supply (in truckloads) and the demand (also in truckloads) together with the unit transportation costs per truckload on the different routes are summarized in the transportation model in Table 5.16. The unit transportation costs, c_{ij}, (shown in the northeast corner of each box) are in hundreds of dollars. The model seeks the minimum-cost shipping schedule x_{ij} between silo i and mill j $(i = 1, 2, 3; j = 1, 2, 3, 4)$.

Summary of the Transportation Algorithm. The steps of the transportation algorithm are exact parallels of the simplex algorithm.

Step 1. Determine a *starting* basic feasible solution, and go to step 2.

Step 2. Use the optimality condition of the simplex method to determine the *entering variable* from among all the nonbasic variables. If the optimality condition is satisfied, stop. Otherwise, go to step 3.

Step 3. Use the feasibility condition of the simplex method to determine the *leaving variable* from among all the current basic variables, and find the new basic solution. Return to step 2.

5.3.1 Determination of the Starting Solution

A general transportation model with m sources and n destinations has $m + n$ constraint equations, one for each source and each destination. However, because the transportation model is always balanced (sum of the supply = sum of the demand), one of these equations is redundant. Thus, the model has $m + n - 1$ independent constraint equations, which means that the starting basic solution consists of $m + n - 1$ basic variables. Thus, in Example 5.3-1, the starting solution has $3 + 4 - 1 = 6$ basic variables.

The special structure of the transportation problem allows securing a nonartificial starting basic solution using one of three methods:[5]

1. Northwest-corner method
2. Least-cost method
3. Vogel approximation method

The three methods differ in the "quality" of the starting basic solution they produce, in the sense that a better starting solution yields a smaller objective value. In general, though not always, the Vogel method yields the best starting basic solution, and the northwest-corner method yields the worst. The tradeoff is that the northwest-corner method involves the least amount of computations.

Northwest-Corner Method. The method starts at the northwest-corner cell (route) of the tableau (variable x_{11}).

Step 1. Allocate as much as possible to the selected cell, and adjust the associated amounts of supply and demand by subtracting the allocated amount.

Step 2. Cross out the row or column with zero supply or demand to indicate that no further assignments can be made in that row or column. If both a row and a column net to zero simultaneously, *cross out one only*, and leave a zero supply (demand) in the uncrossed-out row (column).

Step 3. If *exactly one* row or column is left uncrossed out, stop. Otherwise, move to the cell to the right if a column has just been crossed out or below if a row has been crossed out. Go to step 1.

Example 5.3-2

The application of the procedure to the model of Example 5.3-1 gives the starting basic solution in Table 5.17. The arrows show the order in which the allocated amounts are generated.

The starting basic solution is

$$x_{11} = 5, x_{12} = 10$$

$$x_{22} = 5, x_{23} = 15, x_{24} = 5$$

$$x_{34} = 10$$

The associated cost of the schedule is

$$z = 5 \times 10 + 10 \times 2 + 5 \times 7 + 15 \times 9 + 5 \times 20 + 10 \times 18 = \$520$$

Least-Cost Method. The least-cost method finds a better starting solution by concentrating on the cheapest routes. The method assigns as much as possible to the cell with the smallest unit cost (ties are broken arbitrarily). Next, the satisfied row or column is crossed out and the amounts of supply and demand are adjusted accordingly.

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[5] All three methods are featured in TORA's tutorial module. See the end of Section 5.3.3.

TABLE 5.17 Northwest-Corner Starting Solution

	1	2	3	4	Supply
1	10 5	2 10	20	11	**15**
2	12	7 5	9 15	20 5	**25**
3	4	14	16	18 10	**10**
Demand	**5**	**15**	**15**	**15**	

If both a row and a column are satisfied simultaneously, *only one is crossed out*, the same as in the northwest-corner method. Next, look for the uncrossed-out cell with the smallest unit cost and repeat the process until exactly one row or column is left uncrossed out.

Example 5.3-3

The least-cost method is applied to Example 5.3-1 in the following manner:

1. Cell (1, 2) has the least unit cost in the tableau (= \$2). The most that can be shipped through (1, 2) is $x_{12} = 15$ truckloads, which happens to satisfy both row 1 and column 2 simultaneously. We arbitrarily cross out column 2 and adjust the supply in row 1 to 0.
2. Cell (3, 1) has the smallest uncrossed-out unit cost (= \$4). Assign $x_{31} = 5$, and cross out column 1 because it is satisfied, and adjust the demand of row 3 to 10 − 5 = 5 truckloads.
3. Continuing in the same manner, we successively assign 15 truckloads to cell (2, 3), 0 truckloads to cell (1, 4), 5 truckloads to cell (3, 4), and 10 truckloads to cell (2, 4) (verify!).

The resulting starting solution is summarized in Table 5.18. The arrows show the order in which the allocations are made. The starting solution (consisting of 6 basic variables) is $x_{12} = 15, x_{14} = 0, x_{23} = 15, x_{24} = 10, x_{31} = 5, x_{34} = 5$. The associated objective value is

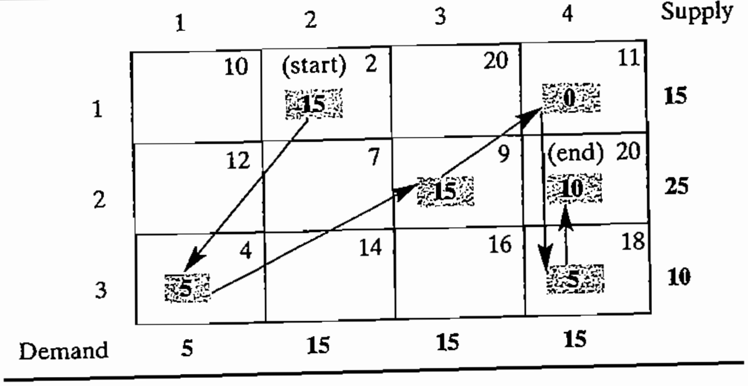
$$z = 15 \times 2 + 0 \times 11 + 15 \times 9 + 10 \times 20 + 5 \times 4 + 5 \times 18 = \$475$$

The quality of the least-cost starting solution is better than that of the northwest-corner method (Example 5.3-2) because it yields a smaller value of z (\$475 versus \$520 in the northwest-corner method).

Vogel Approximation Method (VAM). VAM is an improved version of the least-cost method that generally, but not always, produces better starting solutions.

Step 1. For each row (column), determine a penalty measure by subtracting the *smallest* unit cost element in the row (column) from the *next smallest* unit cost element in the same row (column).

TABLE 5.18 Least-Cost Starting Solution



	1	2	3	4	Supply
1	10	(start) 2 15	20	11 0	**15**
2	12	7	9 15	(end) 20 10	**25**
3	4 5	14	16	18 5	**10**
Demand	**5**	**15**	**15**	**15**	

Step 2. Identify the row or column with the largest penalty. Break ties arbitrarily. Allocate as much as possible to the variable with the least unit cost in the selected row or column. Adjust the supply and demand, and cross out the satisfied row *or* column. If a row and a column are satisfied simultaneously, only one of the two is crossed out, and the remaining row (column) is assigned zero supply (demand).

Step 3. **(a)** If exactly one row or column with zero supply or demand remains uncrossed out, stop.

(b) If one row (column) with *positive* supply (demand) remains uncrossed out, determine the basic variables in the row (column) by the least-cost method. Stop.

(c) If all the uncrossed out rows and columns have (remaining) zero supply and demand, determine the *zero* basic variables by the least-cost method. Stop.

(d) Otherwise, go to step 1.

Example 5.3-4

VAM is applied to Example 5.3-1. Table 5.19 computes the first set of penalties.

Because row 3 has the largest penalty (= 10) and cell (3, 1) has the smallest unit cost in that row, the amount 5 is assigned to x_{31}. Column 1 is now satisfied and must be crossed out. Next, new penalties are recomputed as in Table 5.20.

Table 5.20 shows that row 1 has the highest penalty (= 9). Hence, we assign the maximum amount possible to cell (1, 2), which yields $x_{12} = 15$ and simultaneously satisfies both row 1 and column 2. We arbitrarily cross out column 2 and adjust the supply in row 1 to zero.

Continuing in the same manner, row 2 will produce the highest penalty (= 11), and we assign $x_{23} = 15$, which crosses out column 3 and leaves 10 units in row 2. Only column 4 is left, and it has a positive supply of 15 units. Applying the least-cost method to that column, we successively assign $x_{14} = 0$, $x_{34} = 5$, and $x_{24} = 10$ (verify!). The associated objective value for this solution is

$$z = 15 \times 2 + 0 \times 11 + 15 \times 9 + 10 \times 20 + 5 \times 4 + 5 \times 18 = \$475$$

This solution happens to have the same objective value as in the least-cost method.

TABLE 5.19 Row and Column Penalties in VAM

	1	2	3	4		Row penalty
1	10	2	20	11	**15**	10 − 2 = 8
2	12	7	9	20	**25**	9 − 7 = 2
3	4 **5**	14	16	18	**10**	14 − 4 = 10
	5	**15**	**15**	**15**		
Column penalty	10 − 4 = 6	7 − 2 = 5	16 − 9 = 7	18 − 11 = 7		

TABLE 5.20 First Assignment in VAM ($x_{31} = 5$)

	1	2	3	4		Row penalty
1	10	2	20	11	**15**	9
2	12	7	9	20	**25**	2
3	4 **5**	14	16	18	**10**	2
	5	**15**	**15**	**15**		
Column penalty	—	5	7	7		

PROBLEM SET 5.3A

1. Compare the starting solutions obtained by the northwest-corner, least-cost, and Vogel methods for each of the following models:

***(a)**

0	2	1	**6**
2	1	5	**7**
2	4	3	**7**
5	**5**	**10**	

(b)

1	2	6	**7**
0	4	2	**12**
3	1	5	**11**
10	**10**	**10**	

(c)

5	1	8	**12**
2	4	0	**14**
3	6	7	**4**
9	**10**	**11**	

5.3.2 Iterative Computations of the Transportation Algorithm

After determining the starting solution (using any of the three methods in Section 5.3.1), we use the following algorithm to determine the optimum solution:

Step 1. Use the simplex *optimality condition* to determine the *entering variable* as the current nonbasic variable that can improve the solution. If the optimality condition is satisfied, stop. Otherwise, go to step 2.

Step 2. Determine the *leaving variable* using the simplex *feasibility condition*. Change the basis, and return to step 1.

The optimality and feasibility conditions do not involve the familiar row operations used in the simplex method. Instead, the special structure of the transportation model allows simpler computations.

Example 5.3-5

Solve the transportation model of Example 5.3-1, starting with the northwest-corner solution.

Table 5.21 gives the northwest-corner starting solution as determined in Table 5.17, Example 5.3-2.

The determination of the entering variable from among the current nonbasic variables (those that are not part of the starting basic solution) is done by computing the nonbasic coefficients in the z-row, using the **method of multipliers** (which, as we show in Section 5.3.4, is rooted in LP duality theory).

In the method of multipliers, we associate the multipliers u_i and v_j with row i and column j of the transportation tableau. For each current *basic* variable x_{ij}, these multipliers are shown in Section 5.3.4 to satisfy the following equations:

$$u_i + v_j = c_{ij}, \text{ for each } \textit{basic } x_{ij}$$

As Table 5.21 shows, the starting solution has 6 basic variables, which leads to 6 equations in 7 unknowns. To solve these equations, the method of multipliers calls for arbitrarily setting any $u_i = 0$, and then solving for the remaining variables as shown below.

Basic variable	(u, v) Equation	Solution
x_{11}	$u_1 + v_1 = 10$	Set $u_1 = 0 \rightarrow v_1 = 10$
x_{12}	$u_1 + v_2 = 2$	$u_1 = 0 \rightarrow v_2 = 2$
x_{22}	$u_2 + v_2 = 7$	$v_2 = 2 \rightarrow u_2 = 5$
x_{23}	$u_2 + v_3 = 9$	$u_2 = 5 \rightarrow v_3 = 4$
x_{24}	$u_2 + v_4 = 20$	$u_2 = 5 \rightarrow v_4 = 15$
x_{34}	$u_3 + v_4 = 18$	$v_4 = 15 \rightarrow u_3 = 3$

To summarize, we have

$$u_1 = 0, u_2 = 5, u_3 = 3$$

$$v_1 = 10, v_2 = 2, v_3 = 4, v_4 = 15$$

Next, we use u_i and v_j to evaluate the nonbasic variables by computing

$$u_i + v_j - c_{ij}, \text{ for each } \textit{nonbasic } x_{ij}$$

TABLE 5.21 Starting Iteration

	1	2	3	4	Supply
1	10 **5**	2 **10**	20	11	**15**
2	12	7 **5**	9 **15**	20 **5**	**25**
3	4	14	16	18 **10**	**10**
Demand	**5**	**15**	**15**	**15**	

The results of these evaluations are shown in the following table:

Nonbasic variable	$u_i + v_j - c_{ij}$
x_{13}	$u_1 + v_3 - c_{13} = 0 + 4 - 20 = -16$
x_{14}	$u_1 + v_4 - c_{14} = 0 + 15 - 11 = 4$
x_{21}	$u_2 + v_1 - c_{21} = 5 + 10 - 12 = 3$
x_{31}	$u_3 + v_1 - c_{31} = 3 + 10 - 4 = \mathbf{9}$
x_{32}	$u_3 + v_2 - c_{32} = 3 + 2 - 14 = -9$
x_{33}	$u_3 + v_3 - c_{33} = 3 + 4 - 16 = -9$

The preceding information, together with the fact that $u_i + v_j - c_{ij} = 0$ for each basic x_{ij}, is actually equivalent to computing the z-row of the simplex tableau, as the following summary shows.

Basic	x_{11}	x_{12}	x_{13}	x_{14}	x_{21}	x_{22}	x_{23}	x_{24}	↓ x_{31}	x_{32}	x_{33}	x_{34}
z	0	0	−16	4	3	0	0	0	9	−9	−9	0

Because the transportation model seeks to *minimize* cost, the entering variable is the one having the *most positive* coefficient in the z-row. Thus, x_{31} is the entering variable.

The preceding computations are usually done directly on the transportation tableau as shown in Table 5.22, meaning that it is not necessary really to write the (u, v)-equations explicitly. Instead, we start by setting $u_1 = 0$.[6] Then we can compute the v-values of all the columns that have *basic* variables in row 1—namely, v_1 and v_2. Next, we compute u_2 based on the (u, v)-equation of basic x_{22}. Now, given u_2, we can compute v_3 and v_4. Finally, we determine u_3 using the basic equation of x_{33}. Once all the u's and v's have been determined, we can evaluate the nonbasic variables by computing $u_i + v_j - c_{ij}$ for each nonbasic x_{ij}. These evaluations are shown in Table 5.22 in the boxed southeast corner of each cell.

Having identified x_{31} as the entering variable, we need to determine the leaving variable. Remember that if x_{31} enters the solution to become basic, one of the current basic variables must leave as nonbasic (at zero level).

TABLE 5.22 Iteration 1 Calculations

	$v_1 = 10$	$v_2 = 2$	$v_3 = 4$	$v_4 = 15$	Supply
$u_1 \equiv 0$	10 **5**	2 **10**	20 $\boxed{-16}$	11 $\boxed{4}$	**15**
$u_2 = 5$	12 $\boxed{3}$	7 **5**	9 **15**	20 **5**	**25**
$u_3 = 3$	4 $\boxed{9}$	14 $\boxed{-9}$	16 $\boxed{-9}$	18 **10**	**10**
Demand	**5**	**15**	**15**	**15**	

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[6] The tutorial module of TORA is designed to demonstrate that assigning a zero initial value to any u or v does not affect the optimization results. See TORA Moment on page 216.

The selection of x_{31} as the entering variable means that we want to ship through this route because it reduces the total shipping cost. What is the most that we can ship through the new route? Observe in Table 5.22 that if route (3, 1) ships θ units (i.e., $x_{31} = \theta$), then the maximum value of θ is determined based on two conditions.

1. Supply limits and demand requirements remain satisfied.
2. Shipments through all routes remain nonnegative.

These two conditions determine the maximum value of θ and the leaving variable in the following manner. First, construct a *closed loop* that starts and ends at the entering variable cell, (3, 1). The loop consists of *connected horizontal* and *vertical* segments only (no diagonals are allowed).[7] Except for the entering variable cell, each corner of the closed loop must coincide with a basic variable. Table 5.23 shows the loop for x_{31}. Exactly one loop exists for a given entering variable.

Next, we assign the amount θ to the entering variable cell (3, 1). For the supply and demand limits to remain satisfied, we must alternate between subtracting and adding the amount θ at the successive *corners* of the loop as shown in Table 5.23 (it is immaterial whether the loop is traced in a clockwise or counterclockwise direction). For $\theta \geq 0$, the new values of the variables then remain nonnegative if

$$x_{11} = 5 - \theta \geq 0$$

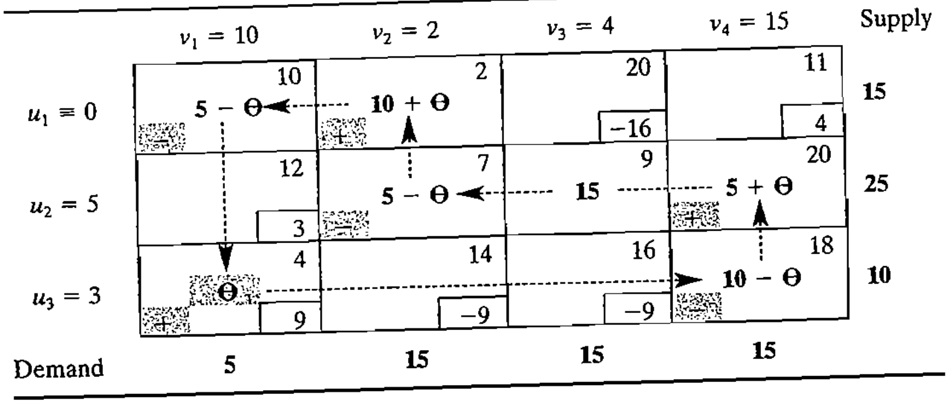
$$x_{22} = 5 - \theta \geq 0$$

$$x_{34} = 10 - \theta \geq 0$$

The corresponding maximum value of θ is 5, which occurs when both x_{11} and x_{22} reach zero level. Because only one current basic variable must leave the basic solution, we can choose either x_{11} or x_{22} as the leaving variable. We arbitrarily choose x_{11} to leave the solution.

The selection of x_{31} (= 5) as the entering variable and x_{11} as the leaving variable requires adjusting the values of the basic variables at the corners of the closed loop as Table 5.24 shows. Because each unit shipped through route (3, 1) reduces the shipping cost by \$9 ($= u_3 + v_1 - c_{31}$), the total cost associated with the new schedule is \$9 × 5 = \$45 less than in the previous schedule. Thus, the new cost is \$520 − \$45 = \$475.

TABLE 5.23 Determination of Closed Loop for x_{31}



	$v_1 = 10$	$v_2 = 2$	$v_3 = 4$	$v_4 = 15$	Supply
$u_1 = 0$	10 5 − θ −	2 10 + θ +	20 −16	11 4	15
$u_2 = 5$	12 3	7 5 − θ −	9 15	20 5 + θ +	25
$u_3 = 3$	4 θ + 9	14 −9	16 −9	18 10 − θ −	10
Demand	5	15	15	15	

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[7] TORA's tutorial module allows you to determine the cells of the *closed loop* interactively with immediate feedback regarding the validity of your selections. See TORA Moment on page 216.

TABLE 5.24 Iteration 2 Calculations

	$v_1 = 1$	$v_2 = 2$	$v_3 = 4$	$v_4 = 15$	Supply
$u_1 \equiv 0$	10 −9	2 **15 − Θ** −	20 −16	11 **Θ** + 4	**15**
$u_2 = 5$	12 −6	7 **0 + Θ** +	9 **15**	20 **10 − Θ** −	**25**
$u_3 = 3$	4 **5**	14 −9	16 −9	18 **5**	**10**
Demand	**5**	**15**	**15**	**15**	

TABLE 5.25 Iteration 3 Calculations (Optimal)

	$v_1 = -3$	$v_2 = 2$	$v_3 = 4$	$v_4 = 11$	Supply
$u_1 \equiv 0$	10 −13	2 **5**	20 −16	11 **10**	**15**
$u_2 = 5$	12 −10	7 **10**	9 **15**	20 −4	**25**
$u_3 = 7$	4 **5**	14 −5	16 −5	18 **5**	**10**
Demand	**5**	**15**	**15**	**15**	

Given the new basic solution, we repeat the computation of the multipliers u and v, as Table 5.24 shows. The entering variable is x_{14}. The closed loop shows that $x_{14} = 10$ and that the leaving variable is x_{24}.

The new solution, shown in Table 5.25, costs \$4 × 10 = \$40 less than the preceding one, thus yielding the new cost \$475 − \$40 = \$435. The new $u_i + v_j - c_{ij}$ are now negative for all nonbasic x_{ij}. Thus, the solution in Table 5.25 is optimal.

The following table summarizes the optimum solution.

From silo	To mill	Number of truckloads
1	2	5
1	4	10
2	2	10
2	3	15
3	1	5
3	4	5
	Optimal cost = \$435	