3.5 SPECIAL CASES IN THE SIMPLEX METHOD

This section considers four special cases that arise in the use of the simplex method.

1. Degeneracy
2. Alternative optima
3. Unbounded solutions
4. Nonexisting (or infeasible) solutions

Our interest in studying these special cases is twofold: (1) to present a *theoretical* explanation of these situations and (2) to provide a *practical* interpretation of what these special results could mean in a real-life problem.

3.5.1 Degeneracy

In the application of the feasibility condition of the simplex method, a tie for the minimum ratio may occur and can be broken arbitrarily. When this happens, at least one *basic* variable will be zero in the next iteration and the new solution is said to be **degenerate**.

There is nothing alarming about a degenerate solution, with the exception of a small theoretical inconvenience, called **cycling** or **circling**, which we shall discuss shortly. From the practical standpoint, the condition reveals that the model has at least one *redundant* constraint. To provide more insight into the practical and theoretical impacts of degeneracy, a numeric example is used.

Example 3.5-1 (Degenerate Optimal Solution)

$$\text{Maximize } z = 3x_1 + 9x_2$$

subject to

$$x_1 + 4x_2 \leq 8$$
$$x_1 + 2x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

Given the slack variables x_3 and x_4, the following tableaus provide the simplex iterations of the problem:

Iteration	Basic	x_1	x_2	x_3	x_4	Solution
0	z	-3	-9	0	0	0
x_2 enters	x_3	1	4	1	0	8
x_3 leaves	x_4	1	2	0	1	4
1	z	$-\frac{3}{4}$	0	$\frac{9}{4}$	0	18
x_1 enters	x_2	$\frac{1}{4}$	1	$\frac{1}{4}$	0	2
x_4 leaves	x_4	$\frac{1}{2}$	0	$-\frac{1}{2}$	1	0
2	z	0	0	$\frac{3}{2}$	$\frac{3}{2}$	18
(optimum)	x_2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	2
	x_1	1	0	-1	2	0

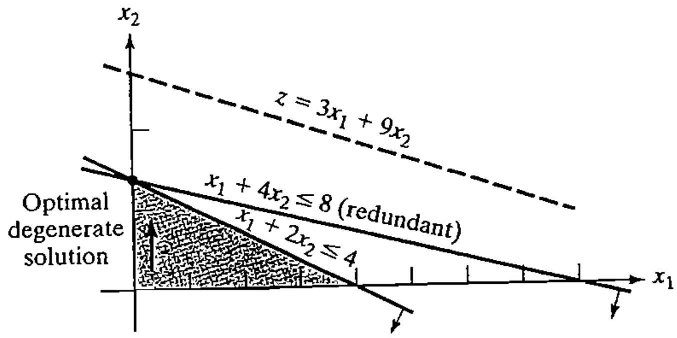


FIGURE 3.7

LP degeneracy in Example 3.5-1

In iteration 0, x_3 and x_4 tie for the leaving variable, leading to degeneracy in iteration 1 because the basic variable x_4 assumes a zero value. The optimum is reached in one additional iteration.

What is the practical implication of degeneracy? Look at the graphical solution in Figure 3.7. Three lines pass through the optimum point ($x_1 = 0, x_2 = 2$). Because this is a two-dimensional problem, the point is *overdetermined* and one of the constraints is redundant.[2] In practice, the mere knowledge that some resources are superfluous can be valuable during the implementation of the solution. The information may also lead to discovering irregularities in the construction of the model. Unfortunately, there are no efficient computational techniques for identifying the redundant constraints directly from the tableau.

From the theoretical standpoint, degeneracy has two implications. The first is the phenomenon of **cycling** or **circling**. Looking at simplex iterations 1 and 2, you will notice that the objective value does not improve ($z = 18$). It is thus possible for the simplex method to enter a repetitive sequence of iterations, never improving the objective value and never satisfying the optimality condition (see Problem 4, Set 3.5a). Although there are methods for eliminating cycling, these methods lead to drastic slowdown in computations. For this reason, most LP codes do not include provisions for cycling, relying on the fact that it is a rare occurrence in practice.

The second theoretical point arises in the examination of iterations 1 and 2. Both iterations, though differing in the basic-nonbasic categorization of the variables, yield identical values for all the variables and objective value—namely,

$$x_1 = 0, x_2 = 2, x_3 = 0, x_4 = 0, z = 18$$

Is it possible then to stop the computations at iteration 1 (when degeneracy first appears), even though it is not optimum? The answer is no, because the solution may be *temporarily* degenerate as Problem 2, Set 3.5a demonstrates.

---

[2]Redundancy generally implies that constraints can be removed without affecting the feasible solution space. A sometimes quoted counterexample is $x + y \leq 1, x \geq 1, y \geq 0$. Here, the removal of any one constraint will change the feasible space from a single point to a region. Suffice it to say, however, that this condition is true only if the solution space consists of a single feasible point, a highly unlikely occurrence in real-life LPs.

PROBLEM SET 3.5A

*1. Consider the graphical solution space in Figure 3.8. Suppose that the simplex iterations start at A and that the optimum solution occurs at D. Further, assume that the objective function is defined such that at A, x_1 enters the solution first.
   - **(a)** Identify (on the graph) the corner points that define the simplex method path to the optimum point.
   - **(b)** Determine the maximum possible number of simplex iterations needed to reach the optimum solution, assuming no cycling.

2. Consider the following LP:

$$\text{Maximize } z = 3x_1 + 2x_2$$

subject to

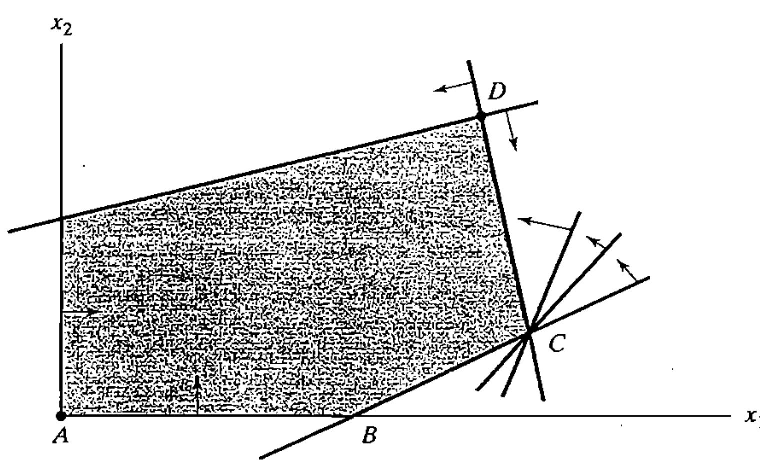
$$\begin{aligned} 4x_1 - x_2 &\leq 8 \\ 4x_1 + 3x_2 &\leq 12 \\ 4x_1 + x_2 &\leq 8 \\ x_1, x_2 &\geq 0 \end{aligned}$$

   - **(a)** Show that the associated simplex iterations are temporarily degenerate (you may use TORA for convenience).
   - **(b)** Verify the result by solving the problem graphically (TORA's Graphic module can be used here).

3. *TORA experiment.* Consider the LP in Problem 2.
   - **(a)** Use TORA to generate the simplex iterations. How many iterations are needed to reach the optimum?
   - **(b)** Interchange constraints (1) and (3) and re-solve the problem with TORA. How many iterations are needed to solve the problem?
   - **(c)** Explain why the numbers of iterations in (a) and (b) are different.

FIGURE 3.8

Solution space of Problem 1, Set 3.5a



4. *TORA Experiment.* Consider the following LP (authored by E.M. Beale to demonstrate cycling):

$$\text{Maximize } z = \tfrac{3}{4}x_1 - 20x_2 + \tfrac{1}{2}x_3 - 6x_4$$

subject to

$$\begin{aligned} \tfrac{1}{4}x_1 - 8x_2 - x_3 + 9x_4 &\le 0 \\ \tfrac{1}{2}x_1 - 12x_2 - \tfrac{1}{2}x_3 + 3x_4 &\le 0 \\ x_3 &\le 1 \\ x_1, x_2, x_3, x_4 &\ge 0 \end{aligned}$$

From TORA's SOLVE/MODIFY menu, select Solve ⇒ Algebraic ⇒ Iterations ⇒ All-slack. Next, "thumb" through the successive simplex iterations using the command Next iteration (do not use All iterations, because the simplex method will then cycle indefinitely). You will notice that the starting all-slack basic feasible solution at iteration 0 will reappear identically in iteration 6. This example illustrates the occurrence of cycling in the simplex iterations and the possibility that the algorithm may never converge to the optimum solution.

It is interesting that cycling will not occur in this example if all the coefficients in this LP are converted to integer values by using proper multiples (try it!).

3.5.2 Alternative Optima

When the objective function is parallel to a nonredundant **binding constraint** (i.e., a constraint that is satisfied as an equation at the optimal solution), the objective function can assume the same optimal value at more than one solution point, thus giving rise to alternative optima. The next example shows that there is an *infinite* number of such solutions. It also demonstrates the practical significance of encountering such solutions.

Example 3.5-2 (Infinite Number of Solutions)

$$\text{Maximize } z = 2x_1 + 4x_2$$

subject to

$$\begin{aligned} x_1 + 2x_2 &\le 5 \\ x_1 + x_2 &\le 4 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Figure 3.9 demonstrates how alternative optima can arise in the LP model when the objective function is parallel to a binding constraint. Any point on the *line segment BC* represents an alternative optimum with the same objective value $z = 10$.

The iterations of the model are given by the following tableaus.

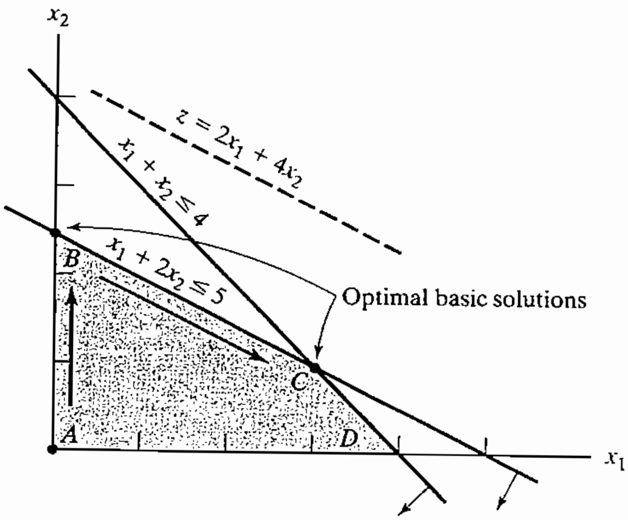


FIGURE 3.9
LP alternative optima in Example 3.5-2

Iteration	Basic	x_1	x_2	x_3	x_4	Solution
0	z	-2	-4	0	0	0
x_2 enters	x_3	1	2	1	0	5
x_3 leaves	x_4	1	1	0	1	4
1 (optimum)	z	0	0	2	0	10
x_1 enters	x_2	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{5}{2}$
x_4 leaves	x_4	$\frac{1}{2}$	0	$-\frac{1}{2}$	1	$\frac{3}{2}$
2	z	0	0	2	0	10
(alternative optimum)	x_2	0	1	1	-1	1
	x_1	1	0	-1	2	3

Iteration 1 gives the optimum solution $x_1 = 0, x_2 = \frac{5}{2}$, and $z = 10$, which coincides with point B in Figure 3.9. How do we know from this tableau that alternative optima exist? Look at the z-equation coefficients of the *non*basic variables in iteration 1. The coefficient of nonbasic x_1 is zero, indicating that x_1 can enter the basic solution without changing the value of z, but causing a change in the values of the variables. Iteration 2 does just that—letting x_1 enter the basic solution and forcing x_4 to leave. The new solution point occurs at $C(x_1 = 3, x_2 = 1, z = 10)$. (TORA's Iterations option allows determining one alternative optimum at a time.)

The simplex method determines only the two corner points B and C. Mathematically, we can determine all the points (x_1, x_2) on the line segment BC as a nonnegative weighted average of points B and C. Thus, given

$$B: x_1 = 0, x_2 = \frac{5}{2}$$

$$C: x_1 = 3, x_2 = 1$$

then all the points on the line segment BC are given by

$$\left.\begin{aligned}\hat{x}_1 &= \alpha(0) + (1-\alpha)(3) = 3 - 3\alpha\\ \hat{x}_2 &= \alpha\left(\tfrac{5}{2}\right) + (1-\alpha)(1) = 1 + \tfrac{3}{2}\alpha\end{aligned}\right\}, 0 \le \alpha \le 1$$

When $\alpha = 0$, $(\hat{x}_1, \hat{x}_2) = (3,1)$, which is point C. When $\alpha = 1$, $(\hat{x}_1, \hat{x}_2) = \left(0, \frac{5}{2}\right)$, which is point B. For values of α between 0 and 1, $(\hat{x}_1, \hat{x}_2)$ lies between B and C.

Remarks. In practice, alternative optima are useful because we can choose from many solutions without experiencing deterioration in the objective value. For instance, in the present example, the solution at B shows that activity 2 only is at a positive level, whereas at C both activities are positive. If the example represents a product-mix situation, there may be advantages in producing two products rather than one to meet market competition. In this case, the solution at C may be more appealing.

PROBLEM SET 3.5B

*1. For the following LP, identify three alternative optimal basic solutions, and then write a general expression for all the nonbasic alternative optima comprising these three basic solutions.

$$\text{Maximize } z = x_1 + 2x_2 + 3x_3$$

subject to

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &\le 10\\ x_1 + x_2 &\le 5\\ x_1 &\le 1\\ x_1, x_2, x_3 &\ge 0\end{aligned}$$

Note: Although the problem has more than three alternative basic solution optima, you are only required to identify three of them. You may use TORA for convenience.

2. Solve the following LP:

$$\text{Maximize } z = 2x_1 - x_2 + 3x_3$$

subject to

$$\begin{aligned} x_1 - x_2 + 5x_3 &\le 10\\ 2x_1 - x_2 + 3x_3 &\le 40\\ x_1, x_2, x_3 &\ge 0\end{aligned}$$

From the optimal tableau, show that all the alternative optima are not corner points (i.e., nonbasic). Give a two-dimensional graphical demonstration of the type of solution space and objective function that will produce this result. (You may use TORA for convenience.)

3. For the following LP, show that the optimal solution is degenerate and that none of the alternative solutions are corner points (you may use TORA for convenience).

$$\text{Maximize } z = 3x_1 + x_2$$

subject to

$$\begin{aligned} x_1 + 2x_2 \quad &\leq 5 \\ x_1 + x_2 - x_3 &\leq 2 \\ 7x_1 + 3x_2 - 5x_3 &\leq 20 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

3.5.3 Unbounded Solution

In some LP models, the values of the variables may be increased indefinitely without violating any of the constraints—meaning that the solution space is *unbounded* in at least one variable. As a result, the objective value may increase (maximization case) or decrease (minimization case) indefinitely. In this case, both the solution space and the optimum objective value are unbounded.

Unboundedness points to the possibility that the model is poorly constructed. The most likely irregularity in such models is that one or more nonredundant constraints have not been accounted for, and the parameters (constants) of some constraints may not have been estimated correctly.

The following examples show how unboundedness, in both the solution space and the objective value, can be recognized in the simplex tableau.

Example 3.5-3 (Unbounded Objective Value)

$$\text{Maximize } z = 2x_1 + x_2$$

subject to

$$\begin{aligned} x_1 - x_2 &\leq 10 \\ 2x_1 \quad &\leq 40 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Starting Iteration

Basic	x_1	x_2	x_3	x_4	Solution
z	-2	-1	0	0	0
x_3	1	-1	1	0	10
x_4	2	0	0	1	40

In the starting tableau, both x_1 and x_2 have negative z-equation coefficients. Hence either one can improve the solution. Because x_1 has the most negative coefficient, it is normally selected as the entering variable. However, *all* the *constraint* coefficients under x_2 (i.e., the denominators of the ratios of the feasibility condition) are *negative* or *zero*. This means that there is no leaving variable and that x_2 can be increased indefinitely without violating any of the constraints (compare with the graphical interpretation of the minimum ratio in Figure 3.5). Because each unit increase in x_2 will increase z by 1, an infinite increase in x_2 leads to an infinite increase in z. Thus, the problem has no bounded solution. This result can be seen in Figure 3.10. The solution space is unbounded in the direction of x_2, and the value of z can be increased indefinitely.

Remarks. What would have happened if we had applied the strict optimality condition that calls for x_1 to enter the solution? The answer is that a succeeding tableau would eventually have led to an entering variable with the same characteristics as x_2. See Problem 1, Set3.5c.

3.5

PROBLEM SET 3.5C

1. *TORA Experiment.* Solve Example 3.5-3 using TORA's Iterations option and show that even though the solution starts with x_1 as the entering variable (per the optimality condition), the simplex algorithm will point eventually to an unbounded solution.

***2.** Consider the LP:

$$\text{Maximize } z = 20x_1 + 10x_2 + x_3$$

subject to

$$\begin{aligned} 3x_1 - 3x_2 + 5x_3 &\le 50 \\ x_1 \qquad + x_3 &\le 10 \\ x_1 - x_2 + 4x_3 &\le 20 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

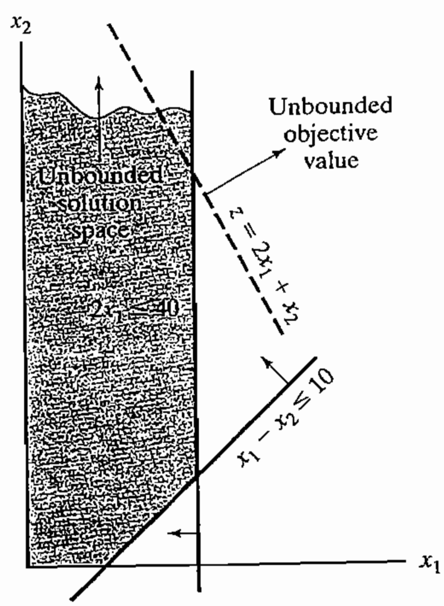


FIGURE 3.10
LP unbounded solution in Example 3.5-3

(a) By inspecting the constraints, determine the direction (x_1, x_2, or x_3) in which the solution space is unbounded.

(b) Without further computations, what can you conclude regarding the optimum objective value?

3. In some ill-constructed LP models, the solution space may be unbounded even though the problem may have a bounded objective value. Such an occurrence can point only to irregularities in the construction of the model. In large problems, it may be difficult to detect unboundedness by inspection. Devise a procedure for determining whether or not a solution space is unbounded.

3.5.4 Infeasible Solution

LP models with inconsistent constraints have no feasible solution. This situation can never occur if *all* the constraints are of the type $\leq$ with nonnegative right-hand sides because the slacks provide a feasible solution. For other types of constraints, we use artificial variables. Although the artificial variables are penalized in the objective function to force them to zero at the optimum, this can occur only if the model has a feasible space. Otherwise, at least one artificial variable will be *positive* in the optimum iteration. From the practical standpoint, an infeasible space points to the possibility that the model is not formulated correctly.

Example 3.5-4 (Infeasible Solution Space)

Consider the following LP:

$$\text{Maximize } z = 3x_1 + 2x_2$$

subject to

$$2x_1 + x_2 \leq 2$$

$$3x_1 + 4x_2 \geq 12$$

$$x_1, x_2 \geq 0$$

Using the penalty $M = 100$ for the artificial variable R, the following tableaux provide the simplex iterations of the model.

Iteration	Basic	x_1	x_2	x_4	x_3	R	Solution
0	z	−303	−402	100	0	0	−1200
x_2 enters	x_3	2	1	0	1	0	2
x_3 leaves	R	3	4	−1	0	1	12
1	z	501	0	100	402	0	−396
(pseudo-optimum)	x_2	2	1	0	1	0	2
	R	−5	0	−1	−4	1	4

Optimum iteration 1 shows that the artificial variable R is *positive* (= 4), which indicates that the problem is infeasible. Figure 3.11 demonstrates the infeasible solution space. By allowing

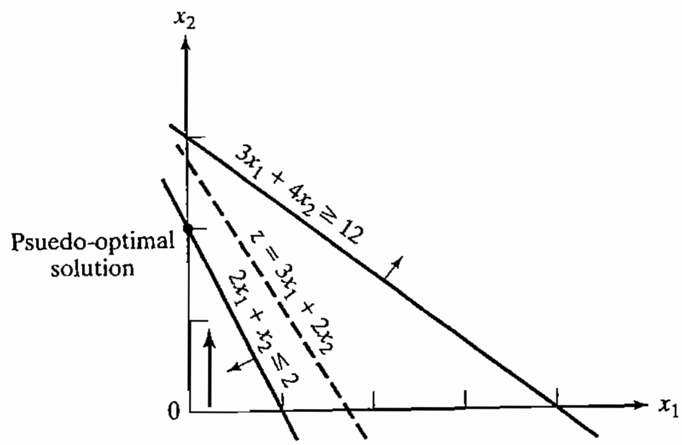


FIGURE 3.11
Infeasible solution of Example 3.5-4

the artificial variable to be positive, the simplex method, in essence, has reversed the direction of the inequality from $3x_1 + 4x_2 \geq 12$ to $3x_1 + 4x_2 \leq 12$ (can you explain how?). The result is what we may call a **pseudo-optimal** solution.

PROBLEM SET 3.5D

*1. Toolco produces three types of tools, *T1*, *T2*, and *T3*. The tools use two raw materials, *M1* and *M2*, according to the data in the following table:

	Number of units of raw materials per tool		
Raw material	*T1*	*T2*	*T3*
M1	3	5	6
M2	5	3	4

The available daily quantities of raw materials *M1* and *M2* are 1000 units and 1200 units, respectively. The marketing department informed the production manager that according to their research, the daily demand for all three tools must be at least 500 units. Will the manufacturing department be able to satisfy the demand? If not, what is the most Toolco can provide of the three tools?

2. *TORA Experiment.* Consider the LP model

$$\text{Maximize } z = 3x_1 + 2x_1 + 3x_3$$

subject to

$$2x_1 + x_2 + x_3 \leq 2$$
$$3x_1 + 4x_2 + 2x_3 \geq 8$$
$$x_1, x_2, x_3 \geq 0$$

Use TORA's Iterations ⇒ M-Method to show that the optimal solution includes an artificial basic variable, but at zero level. Does the problem have a *feasible* optimal solution?

3.6 SENSITIVITY ANALYSIS

In LP, the parameters (input data) of the model can change within certain limits without causing the optimum solution to change. This is referred to as *sensitivity analysis*, and will be the subject matter of this section. Later, in Chapter 4, we will study *post-optimal analysis* which deals with determining the new optimum solution resulting from making targeted changes in the input data.

In LP models, the parameters are usually not exact. With sensitivity analysis, we can ascertain the impact of this uncertainty on the quality of the optimum solution. For example, for an estimated unit profit of a product, if sensitivity analysis reveals that the optimum remains the same for a ±10% change in the unit profit, we can conclude that the solution is more robust than in the case where the indifference range is only ±1%.

We will start with the more concrete graphical solution to explain the basics of sensitivity analysis. These basics will then be extended to the general LP problem using the simplex tableau results.

3.6.1 Graphical Sensitivity Analysis

This section demonstrates the general idea of sensitivity analysis. Two cases will be considered:

1. Sensitivity of the optimum solution to changes in the availability of the resources (right-hand side of the constraints).
2. Sensitivity of the optimum solution to changes in unit profit or unit cost (coefficients of the objective function).

We will consider the two cases separately, using examples of two-variable graphical LPs.

Example 3.6-1 (Changes in the Right-Hand Side)

JOBCO produces two products on two machines. A unit of product 1 requires 2 hours on machine 1 and 1 hour on machine 2. For product 2, a unit requires 1 hour on machine 1 and 3 hours on machine 2. The revenues per unit of products 1 and 2 are \$30 and \$20, respectively. The total daily processing time available for each machine is 8 hours.

Letting x_1 and x_2 represent the daily number of units of products 1 and 2, respectively, the LP model is given as

$$\text{Maximize } z = 30x_1 + 20x_2$$

subject to

$$2x_1 + x_2 \le 8 \quad \text{(Machine 1)}$$

$$x_1 + 3x_2 \le 8 \quad \text{(Machine 2)}$$

$$x_1, x_2 \ge 0$$

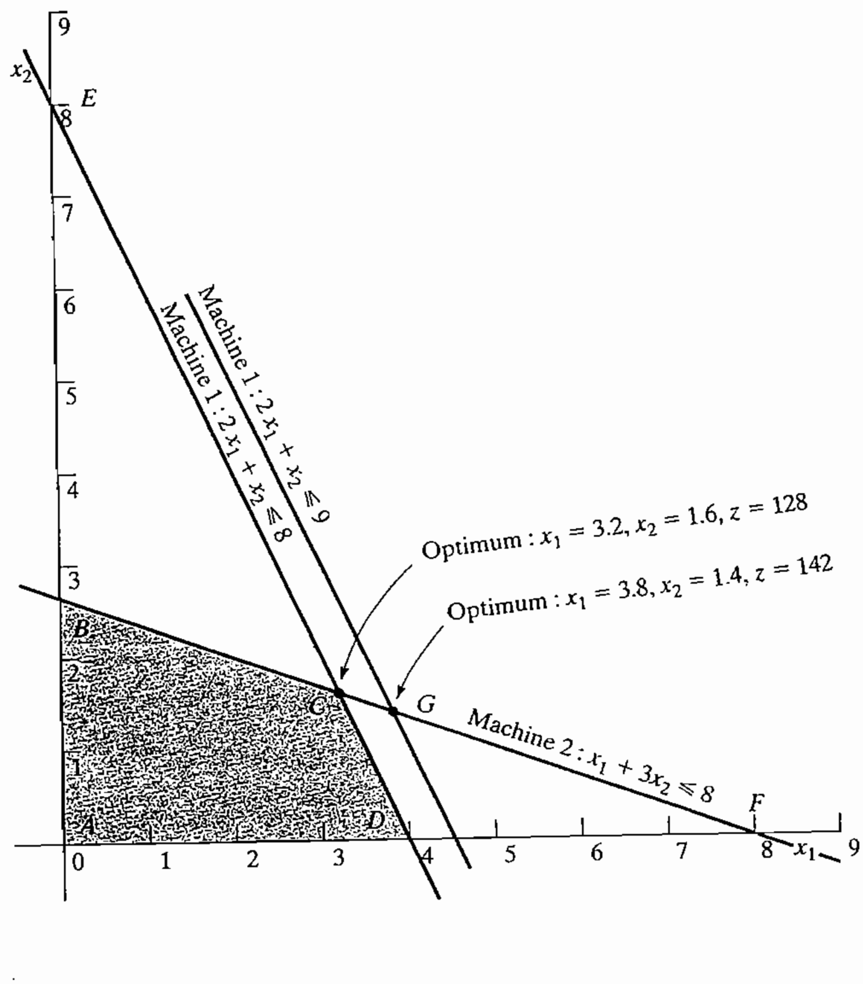
Figure 3.12 illustrates the change in the optimum solution when changes are made in the capacity of machine 1. If the daily capacity is increased from 8 hours to 9 hours, the new optimum will occur at point G. The rate of change in optimum z resulting from changing machine 1 capacity from 8 hours to 9 hours can be computed as follows:

$$\left(\begin{array}{c}\text{Rate of revenue change}\\ \text{resulting from increasing}\\ \text{machine 1 capacity by 1 hr}\\ (\text{point } C \text{ to point } G)\end{array}\right) = \frac{z_G - z_C}{(\text{Capacity change})} = \frac{142 - 128}{9 - 8} = \$14.00/\text{hr}$$

The computed rate provides a *direct link* between the model input (resources) and its output (total revenue) that represents the **unit worth of a resource** (in \$/hr)—that is, the change in the optimal objective value per unit change in the availability of the resource (machine capacity). This means that a unit increase (decrease) in machine 1 capacity will increase (decrease) revenue by \$14.00. Although *unit worth of a resource* is an apt description of the rate of change of the objective function, the technical name **dual** or **shadow price** is now standard in the LP literature and all software packages and, hence, will be used throughout the book.

FIGURE 3.12

Graphical sensitivity of optimal solution to changes in the availability of resources (right-hand side of the constraints)



Looking at Figure 3.12, we can see that the dual price of \$14.00/hr remains valid for changes (increases or decreases) in machine 1 capacity that move its constraint parallel to itself to any point on the line segment BF. This means that the range of applicability of the given dual price can be computed as follows:

$$\text{Minimum machine 1 capacity [at } B = (0, 2.67)] = 2 \times 0 + 1 \times 2.67 = 2.67 \text{ hr}$$

$$\text{Maximum machine 1 capacity [at } F = (8, 0)] = 2 \times 8 + 1 \times 0 = 16 \text{ hr}$$

We can thus conclude that the dual price of \$14.00/hr will remain valid for the range

$$2.67 \text{ hrs} \leq \text{Machine 1 capacity} \leq 16 \text{ hrs}$$

Changes outside this range will produce a different dual price (worth per unit).

Using similar computations, you can verify that the dual price for machine 2 capacity is \$2.00/hr and it remains valid for changes (increases or decreases) that move its constraint parallel to itself to any point on the line segment DE, which yields the following limits:

$$\text{Minimum machine 2 capacity [at } D = (4, 0)] = 1 \times 4 + 3 \times 0 = 4 \text{ hr}$$

$$\text{Maximum machine 2 capacity [at } E = (8, 0)] = 1 \times 0 + 3 \times 8 = 24 \text{ hr}$$

The conclusion is that the dual price of \$2.00/hr for machine 2 will remain applicable for the range

$$4 \text{ hr} \leq \text{Machine 2 capacity} \leq 24 \text{ hr}$$

The computed limits for machine 1 and 2 are referred to as the **feasibility ranges.** All software packages provide information about the dual prices and their feasibility ranges. Section 3.6.4 shows how AMPL, Solver, and TORA generate this information.

The dual prices allow making economic decisions about the LP problem, as the following questions demonstrate:

Question 1. If JOBCO can increase the capacity of both machines, which machine should receive higher priority?

The dual prices for machines 1 and 2 are \$14.00/hr and \$2.00/hr. This means that each additional hour of machine 1 will increase revenue by \$14.00, as opposed to only \$2.00 for machine 2. Thus, priority should be given to machine 1.

Question 2. A suggestion is made to increase the capacities of machines 1 and 2 at the additional cost of \$10/hr. Is this advisable?

For machine 1, the additional net revenue per hour is $14.00 - 10.00 = \$4.00$ and for machine 2, the net is $\$2.00 - \$10.00 = -\$8.00$. Hence, only the capacity of machine 1 should be increased.

Question 3. If the capacity of machine 1 is increased from the present 8 hours to 13 hours, how will this increase impact the optimum revenue?

The dual price for machine 1 is \$14.00 and is applicable in the range (2.67, 16) hr. The proposed increase to 13 hours falls within the feasibility range. Hence, the increase in revenue is $\$14.00(13 - 8) = \$70.00$, which means that the total revenue will be increased to (current revenue + change in revenue) $= 128 + 70 = \$198.00$.

Question 4. Suppose that the capacity of machine 1 is increased to 20 hours, how will this increase impact the optimum revenue?

The proposed change is outside the range (2.67, 16) hr for which the dual price of \$14.00 remains applicable. Thus, we can only make an immediate conclusion regarding an increase up to 16 hours. Beyond that, further calculations are needed to find the answer (see Chapter 4). Remember that falling outside the feasibility range does *not* mean that the problem has no solution. It only means that we do not have sufficient information to make an *immediate* decision.

Question 5. We know that the change in the optimum objective value equals (dual price $\times$ change in resource) so long as the change in the resource is within the feasibility range. What about the associated optimum values of the variables?

The optimum values of the variables will definitely change. However, the level of information we have from the graphical solution is not sufficient to determine the new values. Section 3.6.2, which treats the sensitivity problem algebraically, provides this detail.

PROBLEM SET 3.6A

1. A company produces two products, A and B. The unit revenues are \$2 and \$3, respectively. Two raw materials, $M1$ and $M2$, used in the manufacture of the two products have respective daily availabilities of 8 and 18 units. One unit of A uses 2 units of $M1$ and 2 units of $M2$, and 1 unit of B uses 3 units of $M1$ and 6 units of $M2$.
   (a) Determine the dual prices of $M1$ and $M2$ and their feasibility ranges.
   (b) Suppose that 4 additional units of $M1$ can be acquired at the cost of 30 cents per unit. Would you recommend the additional purchase?
   (c) What is the most the company should pay per unit of $M2$?
   (d) If $M2$ availability is increased by 5 units, determine the associated optimum revenue.

*2. Wild West produces two types of cowboy hats. A Type 1 hat requires twice as much labor time as a Type 2. If all the available labor time is dedicated to Type 2 alone, the company can produce a total of 400 Type 2 hats a day. The respective market limits for the two types are 150 and 200 hats per day. The revenue is \$8 per Type 1 hat and \$5 per Type 2 hat.
   (a) Use the graphical solution to determine the number of hats of each type that maximizes revenue.
   (b) Determine the dual price of the production capacity (in terms of the Type 2 hat) and the range for which it is applicable.
   (c) If the daily demand limit on the Type 1 hat is decreased to 120, use the dual price to determine the corresponding effect on the optimal revenue.
   (d) What is the dual price of the market share of the Type 2 hat? By how much can the market share be increased while yielding the computed worth per unit?

Example 3.6-2 (Changes in the Objective Coefficients)

Figure 3.13 shows the graphical solution space of the JOBCO problem presented in Example 3.6-1. The optimum occurs at point C ($x_1 = 3.2, x_2 = 1.6, z = 128$). Changes in revenue units (i.e., objective-function coefficients) will change the slope of z. However, as can be seen from the figure, the optimum solution will remain at point C so long as the objective function lies between lines BF and DE, the two constraints that define the optimum point. This means that there is a range for the coefficients of the objective function that will keep the optimum solution unchanged at C.

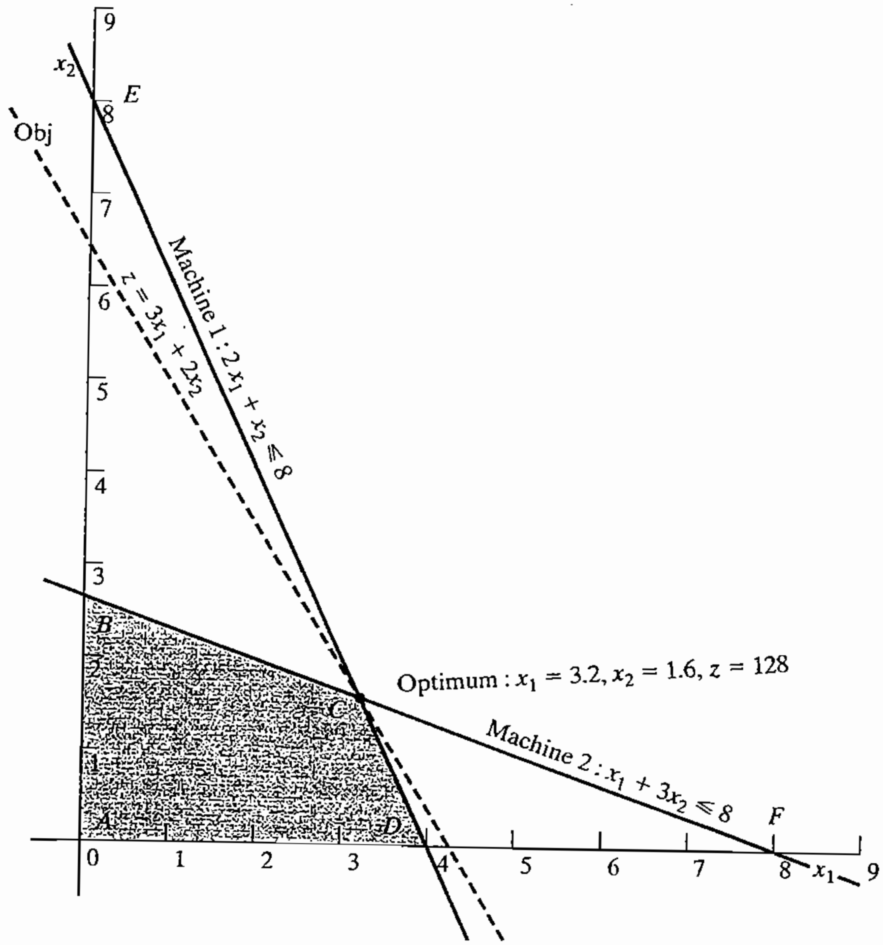


FIGURE 3.13

Graphical sensitivity of optimal solution to changes in the revenue units (coefficients of the objective function)

We can write the objective function in the general format

$$\text{Maximize } z = c_1x_1 + c_2x_2$$

Imagine now that the line z is pivoted at C and that it can rotate clockwise and counterclockwise. The optimum solution will remain at point C so long as $z = c_1x_1 + c_2x_2$ lies between the two lines $x_1 + 3x_2 = 8$ and $2x_1 + x_2 = 8$. This means that the ratio $\frac{c_1}{c_2}$ can vary between $\frac{1}{3}$ and $\frac{2}{1}$, which yields the following condition:

$$\frac{1}{3} \leq \frac{c_1}{c_2} \leq \frac{2}{1} \quad \text{or} \quad .333 \leq \frac{c_1}{c_2} \leq 2$$

This information can provide immediate answers regarding the optimum solution as the following questions demonstrate:

Question 1. Suppose that the unit revenues for products 1 and 2 are changed to \$35 and \$25, respectively. Will the current optimum remain the same?

The new objective function is

$$\text{Maximize } z = 35x_1 + 25x_2$$

The solution at C will remain optimal because $\frac{c_1}{c_2} = \frac{35}{25} = 1.4$ remains within the optimality range (.333, 2). When the ratio falls outside this range, additional calculations are needed to find the new optimum (see Chapter 4). Notice that although the values of the variables at the optimum point C remain unchanged, the optimum value of z changes to $35 \times (3.2) + 25 \times (1.6) = \$152.00$.

Question 2. Suppose that the unit revenue of product 2 is fixed at its current value of $c_2 = \$20.00$. What is the associated range for c_1, the unit revenue for product 1 that will keep the optimum unchanged?

Substituting $c_2 = 20$ in the condition $\frac{1}{3} \leq \frac{c_1}{c_2} \leq 2$, we get

$$\frac{1}{3} \times 20 \leq c_1 \leq 2 \times 20$$

Or

$$6.67 \leq c_1 \leq 40$$

This range is referred to as the **optimality range** for c_1, and it implicitly assumes that c_2 is fixed at \$20.00.

We can similarly determine the *optimality range* for c_2 by fixing the value of c_1 at \$30.00. Thus,

$$c_2 \leq 30 \times 3 \text{ and } c_2 \geq \frac{30}{2}$$

Or

$$15 \leq c_2 \leq 90$$

As in the case of the right-hand side, all software packages provide the optimality ranges. Section 3.6.4 shows how AMPL, Solver, and TORA generate these results.

Remark. Although the material in this section has dealt only with two variables, the results lay the foundation for the development of sensitivity analysis for the general LP problem in Sections 3.6.2 and 3.6.3.

PROBLEM SET 3.6B

1. Consider Problem 1, Set 3.6a.
   - **(a)** Determine the optimality condition for $\frac{c_A}{c_B}$ that will keep the optimum unchanged.
   - **(b)** Determine the optimality ranges for c_A and c_B, assuming that the other coefficient is kept constant at its present value.
   - **(c)** If the unit revenues c_A and c_B are changed simultaneously to \$5 and \$4, respectively, determine the new optimum solution.
   - **(d)** If the changes in (c) are made one at a time, what can be said about the optimum solution?
2. In the Reddy Mikks model of Example 2.2-1;
   - **(a)** Determine the range for the ratio of the unit revenue of exterior paint to the unit revenue of interior paint.