CHAPTER 4

Duality and Post-Optimal Analysis

Chapter Guide. Chapter 3 dealt with the sensitivity of the optimal solution by determining the ranges for the model parameters that will keep the optimum basic solution unchanged. A natural sequel to sensitivity analysis is *post-optimal analysis*, where the goal is to determine the new optimum that results from making targeted changes in the model parameters. Although post-optimal analysis can be carried out using the simplex tableau computations in Section 3.6, this chapter is based entirely on the dual problem.

At a minimum, you will need to study the dual problem and its economic interpretation (Sections 4.1, 4.2, and 4.3). The mathematical definition of the dual problem in Section 4.1 is purely abstract. Yet, when you study Section 4.3, you will see that the dual problem leads to intriguing economic interpretations of the LP model, including *dual prices* and *reduced costs*. It also provides the foundation for the development of the new *dual simplex algorithm*, a prerequisite for post-optimal analysis. The dual simplex algorithm is also needed for integer programming in Chapter 9.

The *generalized simplex algorithm* in Section 4.4.2 is intended to show that the simplex method is not rigid, in the sense that you can modify the rules to handle problems that start both infeasible and nonoptimal. However, this material may be skipped without loss of continuity.

You may use TORA's interactive mode to reinforce your understanding of the computational details of the dual simplex method.

This chapter includes 14 solved examples, 56 end-of-section problems, and 2 cases. The cases are in Appendix E on the CD.

4.1 DEFINITION OF THE DUAL PROBLEM

The **dual** problem is an LP defined directly and systematically from the **primal** (or original) LP model. The two problems are so closely related that the optimal solution of one problem automatically provides the optimal solution to the other.

In most LP treatments, the dual is defined for various forms of the primal depending on the sense of optimization (maximization or minimization), types of constraints

($\leq$, $\geq$, or $=$), and orientation of the variables (nonnegative or unrestricted). This type of treatment is somewhat confusing, and for this reason we offer a *single* definition that automatically subsumes *all* forms of the primal.

Our definition of the dual problem requires expressing the primal problem in the *equation form* presented in Section 3.1 (all the constraints are equations with nonnegative right-hand side and all the variables are nonnegative). This requirement is consistent with the format of the simplex starting tableau. Hence, any results obtained from the primal optimal solution will apply directly to the associated dual problem.

To show how the dual problem is constructed, define the primal in *equation form* as follows:

$$\text{Maximize or minimize } z = \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, i = 1, 2, \ldots, m$$

$$x_j \geq 0, j = 1, 2, \ldots, n$$

The variables x_j, $j = 1, 2, \ldots, n$, include the surplus, slack, and artificial variables, if any.

Table 4.1 shows how the dual problem is constructed from the primal. Effectively, we have

1. A dual variable is defined for each primal (constraint) equation.
2. A dual constraint is defined for each primal variable.
3. The constraint (column) coefficients of a primal variable define the left-hand-side coefficients of the dual constraint and its objective coefficient define the right-hand side.
4. The objective coefficients of the dual equal the right-hand side of the primal constraint equations.

TABLE 4.1 Construction of the Dual from the Primal

	Primal variables						
	x_1	x_2	$\cdots$	x_j	$\cdots$	x_n	
Dual variables	c_1	c_2	$\cdots$	c_j	$\cdots$	c_n	Right-hand side
y_1	a_{11}	a_{12}	$\cdots$	a_{1j}	$\cdots$	a_{1n}	b_1
y_2	a_{21}	a_{22}	$\cdots$	a_{2j}	$\cdots$	a_{2n}	b_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	
y_m	a_{m1}	a_{m2}	$\cdots$	a_{mj}	$\cdots$	a_{mn}	b_m
				↑ *j*th dual constraint			↑ Dual objective coefficients

TABLE 4.2 Rules for Constructing the Dual Problem

Primal problem objective[a]	Dual problem Objective	Constraints type	Variables sign
Maximization	Minimization	$\geq$	Unrestricted
Minimization	Maximization	$\leq$	Unrestricted

[a] All primal constraints are equations with nonnegative right-hand side and all the variables are nonnegative.

The rules for determining the sense of optimization (maximization or minimization), the type of the constraint ($\leq$, $\geq$, or $=$), and the sign of the dual variables are summarized in Table 4.2. Note that the sense of optimization in the dual is always opposite to that of the primal. An easy way to remember the constraint type in the dual (i.e., $\leq$ or $\geq$) is that if the dual objective is *minimization* (i.e., pointing *down*), then the constraints are all of the type $\geq$ (i.e., pointing *up*). The opposite is true when the dual objective is maximization.

The following examples demonstrate the use of the rules in Table 4.2 and also show that our definition incorporates all forms of the primal automatically.

Example 4.1-1

Primal	Primal in equation form	Dual variables
Maximize $z = 5x_1 + 12x_2 + 4x_3$	Maximize $z = 5x_1 + 12x_2 + 4x_3 + 0x_4$	
subject to	subject to	
$x_1 + 2x_2 + x_3 \leq 10$	$x_1 + 2x_2 + x_3 + x_4 = 10$	y_1
$2x_1 - x_2 + 3x_3 = 8$	$2x_1 - x_2 + 3x_3 + 0x_4 = 8$	y_2
$x_1, x_2, x_3 \geq 0$	$x_1, x_2, x_3, x_4 \geq 0$	

Dual Problem

$$\text{Minimize } w = 10y_1 + 8y_2$$

subject to

$$\begin{aligned} y_1 + 2y_2 &\geq 5 \\ 2y_1 - y_2 &\geq 12 \\ y_1 + 3y_2 &\geq 4 \\ \left.\begin{aligned} y_1 + 0y_2 &\geq 0 \\ y_1, y_2 &\text{ unrestricted} \end{aligned}\right\} &\Rightarrow (y_1 \geq 0,\ y_2 \text{ unrestricted}) \end{aligned}$$

Example 4.1-2

Primal	Primal in equation form	Dual variables
Minimize $z = 15x_1 + 12x_2$	Minimize $z = 15x_1 + 12x_2 + 0x_3 + 0x_4$	
subject to	subject to	
$x_1 + 2x_2 \geq 3$	$x_1 + 2x_2 - x_3 + 0x_4 = 3$	y_1
$2x_1 - 4x_2 \leq 5$	$2x_1 - 4x_2 + 0x_3 + x_4 = 5$	y_2
$x_1, x_2 \geq 0$	$x_1, x_2, x_3, x_4 \geq 0$	

Dual Problem

$$\text{Maximize } w = 3y_1 + 5y_2$$

subject to

$$\begin{aligned} y_1 + 2y_2 &\le 15 \\ 2y_1 - 4y_2 &\le 12 \\ \left.\begin{aligned} -y_1 \quad &\le 0 \\ y_2 &\le 0 \\ y_1, y_2 \text{ unrestricted} \end{aligned}\right\} &\Rightarrow (y_1 \ge 0, y_2 \le 0) \end{aligned}$$

Example 4.1-3

Primal	Primal in equation form	Dual variables
	Substitute $x_1 = x_1^+ - x_1^-$	
Maximize $z = 5x_1 + 6x_2$	Maximize $z = 5x_1^+ - 5x_1^- + 6x_2$	
subject to	subject to	
$x_1 + 2x_2 = 5$	$x_1^- - x_1^+ + 2x_2 = 5$	y_1
$-x_1 + 5x_2 \ge 3$	$-x_1^- + x_1^+ + 5x_2 - x_3 = 3$	y_2
$4x_1 + 7x_2 \le 8$	$4x_1^- - 4x_1^+ + 7x_2 + x_4 = 8$	y_3
x_1 unrestricted, $x_2 \ge 0$	$x_1^-, x_1^+, x_2, x_3, x_4 \ge 0$	

Dual Problem

$$\text{Minimize } z = 5y_1 + 3y_2 + 8y_3$$

subject to

$$\left.\begin{aligned} y_1 - y_2 + 4y_3 &\ge 5 \\ -y_1 + y_2 - 4y_3 &\ge -5 \end{aligned}\right\} \Rightarrow (y_1 - y_2 + 4y_3 = 5)$$

$$2y_1 + 5y_2 + 7y_3 \ge 6$$

$$\left.\begin{aligned} -y_2 \quad &\ge 0 \\ y_3 &\ge 0 \\ y_1, y_2, y_3 \text{ unrestricted} \end{aligned}\right\} \Rightarrow (y_1 \text{ unrestricted}, y_2 \le 0, y_3 \ge 0)$$

The first and second constraints are replaced by an equation. The general rule in this case is that an unrestricted primal variable always corresponds to an equality dual constraint. Conversely, a primal equation produces an unrestricted dual variable, as the first primal constraint demonstrates.

Summary of the Rules for Constructing the Dual. The general conclusion from the preceding examples is that the variables and constraints in the primal and dual problems are defined by the rules in Table 4.3. It is a good exercise to verify that these explicit rules are subsumed by the general rules in Table 4.2.

TABLE 4.3 Rules for Constructing the Dual Problem

Maximization problem		Minimization problem
Constraints		*Variables*
$\geq$	$\Leftrightarrow$	≤ 0
$\leq$	$\Leftrightarrow$	≥ 0
$=$	$\Leftrightarrow$	Unrestricted
Variables		*Constraints*
≥ 0	$\Leftrightarrow$	$\geq$
≤ 0	$\Leftrightarrow$	$\leq$
Unrestricted	$\Leftrightarrow$	$=$

Note that the table does not use the designation primal and dual. What matters here is the sense of optimization. If the primal is maximization, then the dual is minimization, and vice versa.

PROBLEM SET 4.1A

1. In Example 4.1-1, derive the associated dual problem if the sense of optimization in the primal problem is changed to minimization.

*2. In Example 4.1-2, derive the associated dual problem given that the primal problem is augmented with a third constraint, $3x_1 + x_2 = 4$.

3. In Example 4.1-3, show that even if the sense of optimization in the primal is changed to minimization, an unrestricted primal variable always corresponds to an equality dual constraint.

4. Write the dual for each of the following primal problems:

(a) Maximize $z = -5x_1 + 2x_2$
subject to

$$-x_1 + x_2 \leq -2$$
$$2x_1 + 3x_2 \leq 5$$
$$x_1, x_2 \geq 0$$

(b) Minimize $z = 6x_1 + 3x_2$
subject to

$$6x_1 - 3x_2 + x_3 \geq 2$$
$$3x_1 + 4x_2 + x_3 \geq 5$$
$$x_1, x_2, x_3 \geq 0$$

***(c)** Maximize $z = x_1 + x_2$
subject to

$$2x_1 + x_2 = 5$$
$$3x_1 - x_2 = 6$$
$$x_1, x_2 \text{ unrestricted}$$

*5. Consider Example 4.1-1. The application of the simplex method to the primal requires the use of an artificial variable in the second constraint of the standard primal to secure a starting basic solution. Show that the presence of an artificial primal in equation form variable does not affect the definition of the dual because it leads to a redundant dual constraint.

6. True or False?
   (a) The dual of the dual problem yields the original primal.
   (b) If the primal constraint is originally in equation form, the corresponding dual variable is necessarily unrestricted.
   (c) If the primal constraint is of the type $\leq$, the corresponding dual variable will be nonnegative (nonpositive) if the primal objective is maximization (minimization).
   (d) If the primal constraint is of the type $\geq$, the corresponding dual variable will be nonnegative (nonpositive) if the primal objective is minimization (maximization).
   (e) An unrestricted primal variable will result in an equality dual constraint.

4.2 PRIMAL-DUAL RELATIONSHIPS

Changes made in the original LP model will change the elements of the current optimal tableau, which in turn may affect the optimality and/or the feasibility of the current solution. This section introduces a number of primal-dual relationships that can be used to recompute the elements of the optimal simplex tableau. These relationships will form the basis for the economic interpretation of the LP model as well as for post-optimality analysis.

This section starts with a brief review of matrices, a convenient tool for carrying out the simplex tableau computations.

4.2.1 Review of Simple Matrix Operations

The simplex tableau computations use only three elementary matrix operations: (row vector) $\times$ (matrix), (matrix) $\times$ (column vector), and (scalar) $\times$ (matrix). These operations are summarized here for convenience. First, we introduce some matrix definitions:[1]

1. A *matrix*, **A**, of size $(m \times n)$ is a rectangular array of elements with m rows and n columns.
2. A *row vector*, **V**, of size m is a $(1 \times m)$ matrix.
3. A *column vector*, **P**, of size n is an $(n \times 1)$ matrix.

These definitions can be represented mathematically as

$$\mathbf{V} = (v_1, v_2, \ldots, v_m),\ \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \vdots & a_{1n} \\ a_{21} & a_{22} & \vdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \vdots & a_{mn} \end{pmatrix},\ \mathbf{P} = \begin{pmatrix} p_1 \\ p_2 \\ \cdots \\ p_n \end{pmatrix}$$

[1]Appendix D on the CD provides a more complete review of matrices.

1. (Row vector × matrix, VA). The operation is defined only if the size of the row vector **V** equals the number of rows of **A**. In this case,

$$\mathbf{VA} = \left(\sum_{i=1}^{m} v_i a_{i1}, \sum_{i=1}^{m} v_i a_{i2}, \ldots, \sum_{i=1}^{m} v_i a_{in}\right)$$

For example,

$$(11, 22, 33)\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} = (1 \times 11 + 3 \times 22 + 5 \times 33, 2 \times 11 + 4 \times 22 + 6 \times 33)$$

$$= (242, 308)$$

2. (Matrix × column vector, AP). The operation is defined only if the number of columns of **A** equals the size of column vector **P**. In this case,

$$\mathbf{AP} = \begin{pmatrix} \sum_{j=1}^{n} a_{1j} p_j \\ \sum_{j=1}^{n} a_{2j} p_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj} p_j \end{pmatrix}$$

As an illustration, we have

$$\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}\begin{pmatrix} 11 \\ 22 \\ 33 \end{pmatrix} = \begin{pmatrix} 1 \times 11 + 3 \times 22 + 5 \times 33 \\ 2 \times 11 + 4 \times 22 + 6 \times 33 \end{pmatrix} = \begin{pmatrix} 242 \\ 308 \end{pmatrix}$$

3. (Scalar × matrix, αA). Given the scalar (or constant) quantity α, the multiplication operation $\alpha\mathbf{A}$ will result in a matrix of the same size as **A** whose (i, j)th element equals αa_{ij}. For example, given $\alpha = 10$,

$$(10)\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 10 & 20 & 30 \\ 40 & 50 & 60 \end{pmatrix}$$

In general, $\alpha\mathbf{A} = \mathbf{A}\alpha$. The same operation is extended equally to the multiplication of vectors by scalars. For example, $\alpha\mathbf{V} = \mathbf{V}\alpha$ and $\alpha\mathbf{P} = \mathbf{P}\alpha$.

PROBLEM SET 4.2A

1. Consider the following matrices:

$$\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}, \mathbf{P}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{P}_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\mathbf{V}_1 = (11, 22), \mathbf{V}_2 = (-1, -2, -3)$$

In each of the following cases, indicate whether the given matrix operation is legitimate, and, if so, calculate the result.

*(a) $\mathbf{AV}_1$

(b) $\mathbf{AP}_1$

(c) $\mathbf{AP}_2$

(d) $\mathbf{V}_1\mathbf{A}$

*(e) $\mathbf{V}_2\mathbf{A}$

(f) $\mathbf{P}_1\mathbf{P}_2$

(g) $\mathbf{V}_1\mathbf{P}_1$

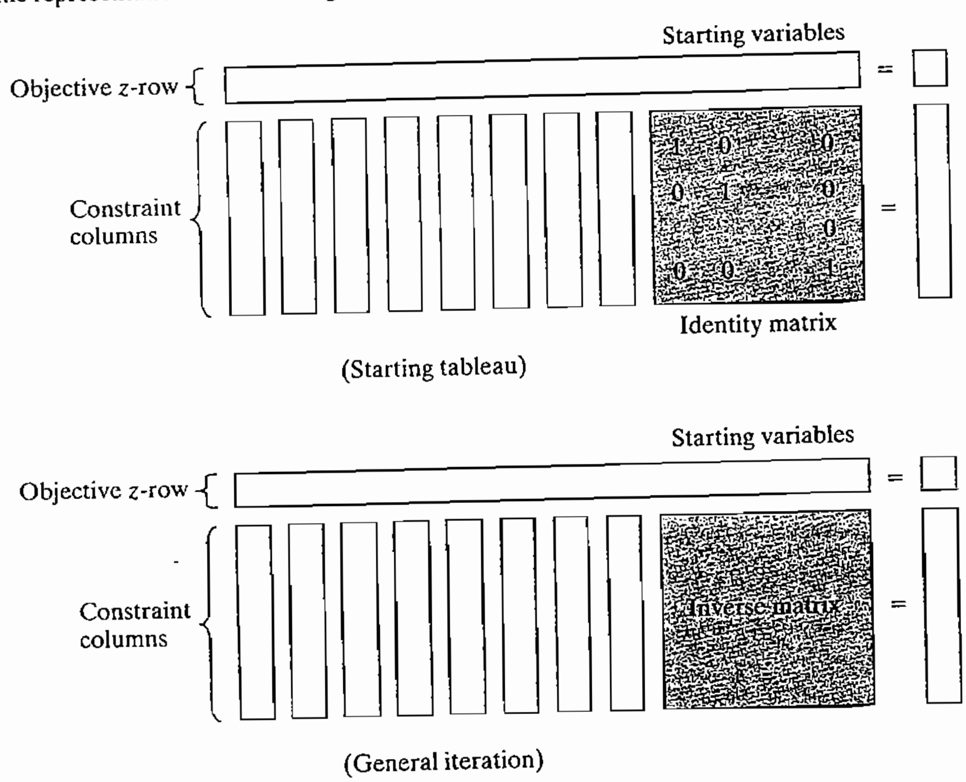
4.2.2 Simplex Tableau Layout

In Chapter 3, we followed a specific format for setting up the simplex tableau. This format is the basis for the development in this chapter.

Figure 4.1 gives a schematic representation of the *starting* and *general* simplex tableaus. In the starting tableau, the constraint coefficients under the starting variables form an **identity matrix** (all main-diagonal elements equal 1 and all off-diagonal elements equal zero). With this arrangement, subsequent iterations of the simplex tableau generated by the Gauss-Jordan row operations (see Chapter 3) will modify the elements of the identity matrix to produce what is known as the **inverse matrix**. As we will see in the remainder of this chapter, the inverse matrix is key to computing all the elements of the associated simplex tableau.

FIGURE 4.1

Schematic representation of the starting and general simplex tableaus



PROBLEM SET 4.2B

1. Consider the optimal tableau of Example 3.3-1.
   ***(a)** Identify the optimal inverse matrix.
   (b) Show that the right-hand side equals the inverse multiplied by the original right-hand side vector of the original constraints.
2. Repeat Problem 1 for the last tableau of Example 3.4-1.

4.2.3 Optimal Dual Solution

The primal and dual solutions are so closely related that the optimal solution of either problem directly yields (with little additional computation) the optimal solution to the other. Thus, in an LP model in which the number of variables is considerably smaller than the number of constraints, computational savings may be realized by solving the dual, from which the primal solution is determined automatically. This result follows because the amount of simplex computation depends largely (though not totally) on the number of constraints (see Problem 2, Set 4.2c).

This section provides two methods for determining the dual values. Note that the dual of the dual is itself the primal, which means that the dual solution can also be used to yield the optimal primal solution automatically.

Method 1.

$$\begin{pmatrix}\text{Optimal value of} \\ \textit{dual} \text{ variable } y_i\end{pmatrix} = \begin{pmatrix}\text{Optimal primal } z\text{-coefficient of } \textit{starting} \text{ variable } x_i \\ + \\ \textit{Original} \text{ objective coefficient of } x_i\end{pmatrix}$$

Method 2.

$$\begin{pmatrix}\text{Optimal values} \\ \text{of } \textit{dual} \text{ variables}\end{pmatrix} = \begin{pmatrix}\text{Row vector of} \\ \text{original objective coefficients} \\ \text{of optimal } \textit{primal} \text{ basic variables}\end{pmatrix} \times \begin{pmatrix}\text{Optimal } \textit{primal} \\ \text{inverse}\end{pmatrix}$$

The elements of the row vector must appear in the same order in which the basic variables are listed in the *Basic* column of the simplex tableau.

Example 4.2-1

Consider the following LP:

$$\text{Maximize } z = 5x_1 + 12x_2 + 4x_3$$

subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 &\le 10 \\ 2x_1 - x_2 + 3x_3 &= 8 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

To prepare the problem for solution by the simplex method, we add a slack x_4 in the first constraint and an artificial R in the second. The resulting primal and the associated dual problems are thus defined as follows:

Primal	Dual
Maximize $z = 5x_1 + 12x_2 + 4x_3 - MR$	Minimize $w = 10y_1 + 8y_2$
subject to	subject to
$x_1 + 2x_2 + x_3 + x_4 = 10$	$y_1 + 2y_2 \geq 5$
$2x_1 - x_2 + 3x_3 + R = 8$	$2y_1 - y_2 \geq 12$
$x_1, x_2, x_3, x_4, R \geq 0$	$y_1 + 3y_2 \geq 4$
	$y_1 \geq 0$
	$y_2 \geq -M$ ($\Rightarrow y_2$ unrestricted)

Table 4.4 provides the optimal primal tableau.

We now show how the optimal dual values are determined using the two methods described at the start of this section.

Method 1. In Table 4.4, the starting primal variables x_4 and R uniquely correspond to the dual variables y_1 and y_2, respectively. Thus, we determine the optimum dual solution as follows:

Starting primal basic variables	x_4	R
z-equation coefficients	$\frac{29}{5}$	$-\frac{2}{5} + M$
Original objective coefficient	0	$-M$
Dual variables	y_1	y_2
Optimal dual values	$\frac{29}{5} + 0 = \frac{29}{5}$	$-\frac{2}{5} + M + (-M) = -\frac{2}{5}$

Method 2. The optimal inverse matrix, highlighted under the starting variables x_4 and R, is given in Table 4.4 as

$$\text{Optimal inverse} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix}$$

First, we note that the optimal primal variables are listed in the tableau in *row order* as x_2 and then x_1. This means that the elements of the original objective coefficients for the two variables must appear in the same order—namely,

$$\begin{aligned}(\text{Original objective coefficients}) &= (\text{Coefficient of } x_2, \text{coefficient of } x_1) \\ &= (12, 5)\end{aligned}$$

TABLE 4.4 Optimal Tableau of the Primal of Example 4.2-1

Basic	x_1	x_2	x_3	x_4	R	Solution
z	0	0	$\frac{3}{5}$	$\frac{29}{5}$	$-\frac{2}{5} + M$	$54\frac{4}{5}$
x_2	0	1	$-\frac{1}{5}$	$\frac{2}{5}$	$-\frac{1}{5}$	$\frac{12}{5}$
x_1	1	0	$\frac{7}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{26}{5}$

Thus, the optimal dual values are computed as

$$(y_1, y_2) = \begin{pmatrix}\text{Original objective} \\ \text{coefficients of } x_2, x_1\end{pmatrix} \times (\text{Optimal inverse})$$

$$= (12, 5)\begin{pmatrix}\frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5}\end{pmatrix}$$

$$= \left(\tfrac{29}{5}, -\tfrac{2}{5}\right)$$

Primal-dual objective values. Having shown how the optimal dual values are determined, next we present the relationship between the primal and dual objective values. For any pair of *feasible* primal and dual solutions,

$$\begin{pmatrix}\text{Objective value in the} \\ \textit{maximization} \text{ problem}\end{pmatrix} \leq \begin{pmatrix}\text{Objective value in the} \\ \textit{minimization} \text{ problem}\end{pmatrix}$$

At the optimum, the relationship holds as a strict equation. The relationship does not specify which problem is primal and which is dual. Only the sense of optimization (maximization or minimization) is important in this case.

The optimum cannot occur with z strictly less than w (i.e., $z < w$) because, no matter how close z and w are, there is always room for improvement, which contradicts optimality as Figure 4.2 demonstrates.

Example 4.2-2

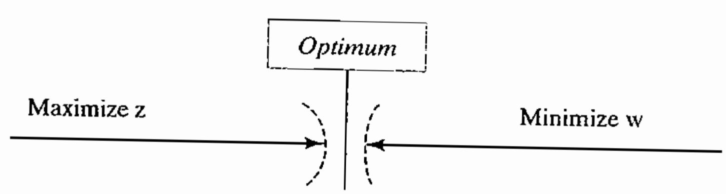
In Example 4.2-1, $\left(x_1 = 0, x_2 = 0, x_3 = \frac{8}{3}\right)$ and $(y_1 = 6, y_2 = 0)$ are feasible primal and dual solutions. The associated values of the objective functions are

$$z = 5x_1 + 12x_2 + 4x_3 = 5(0) + 12(0) + 4\left(\tfrac{8}{3}\right) = 10\tfrac{2}{3}$$
$$w = 10y_1 + 8y_2 = 10(6) + 8(0) = 60$$

Thus, $z \left(= 10\frac{2}{3}\right)$ for the maximization problem (primal) is less than w $(= 60)$ for the minimization problem (dual). The optimum value of $z \left(= 54\frac{4}{5}\right)$ falls within the range $\left(10\frac{2}{3}, 60\right)$.

FIGURE 4.2

Relationship between maximum z and minimum w



PROBLEM SET 4.2C

1. Find the optimal value of the objective function for the following problem by inspecting only its dual. (Do not solve the dual by the simplex method.)

$$\text{Minimize } z = 10x_1 + 4x_2 + 5x_3$$

subject to

$$5x_1 - 7x_2 + 3x_3 \geq 50$$
$$x_1, x_2, x_3 \geq 0$$

2. Solve the dual of the following problem, then find its optimal solution from the solution of the dual. Does the solution of the dual offer computational advantages over solving the primal directly?

$$\text{Minimize } z = 5x_1 + 6x_2 + 3x_3$$

subject to

$$\begin{aligned}
5x_1 + 5x_2 + 3x_3 &\geq 50\\
x_1 + x_2 - x_3 &\geq 20\\
7x_1 + 6x_2 - 9x_3 &\geq 30\\
5x_1 + 5x_2 + 5x_3 &\geq 35\\
2x_1 + 4x_2 - 15x_3 &\geq 10\\
12x_1 + 10x_2 &\geq 90\\
x_2 - 10x_3 &\geq 20\\
x_1, x_2, x_3 &\geq 0
\end{aligned}$$

*3. Consider the following LP:

$$\text{Maximize } z = 5x_1 + 2x_2 + 3x_3$$

subject to

$$\begin{aligned}
x_1 + 5x_2 + 2x_3 &= 30\\
x_1 - 5x_2 - 6x_3 &\leq 40\\
x_1, x_2, x_3 &\geq 0
\end{aligned}$$

Given that the artificial variable x_4 and the slack variable x_5 form the starting basic variables and that M was set equal to 100 when solving the problem, the *optimal* tableau is given as

Basic	x_1	x_2	x_3	x_4	x_5	Solution
z	0	23	7	105	0	150
x_1	1	5	2	1	0	30
x_5	0	−10	−8	−1	1	10

Write the associated dual problem and determine its optimal solution in two ways.

4. Consider the following LP:

$$\text{Minimize } z = 4x_1 + x_2$$

subject to

$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

The starting solution consists of artificial x_4 and x_5 for the first and second constraints and slack x_6 for the third constraint. Using $M = 100$ for the artificial variables, the optimal tableau is given as

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	0	0	0	−98.6	−100	−.2	3.4
x_1	1	0	0	.4	0	−.2	.4
x_2	0	1	0	.2	0	.6	1.8
x_3	0	0	1	1	−1	1	1.0

Write the associated dual problem and determine its optimal solution in two ways.

5. Consider the following LP:

$$\text{Maximize } z = 2x_1 + 4x_2 + 4x_3 - 3x_4$$

subject to

$$x_1 + x_2 + x_3 = 4$$
$$x_1 + 4x_2 + x_4 = 8$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Using x_3 and x_4 as starting variables, the optimal tableau is given as

Basic	x_1	x_2	x_3	x_4	Solution
z	2	0	0	3	16
x_3	.75	0	1	−.25	2
x_2	.25	1	0	.25	2

Write the associated dual problem and determine its optimal solution in two ways.

***6.** Consider the following LP:

$$\text{Maximize } z = x_1 + 5x_2 + 3x_3$$

subject to

$$x_1 + 2x_2 + x_3 = 3$$
$$2x_1 - x_2 = 4$$
$$x_1, x_2, x_3 \geq 0$$

The starting solution consists of x_3 in the first constraint and an artificial x_4 in the second constraint with $M = 100$. The optimal tableau is given as

Basic	x_1	x_2	x_3	x_4	Solution
z	0	2	0	99	5
x_3	1	2.5	1	−.5	1
x_1	0	−.5	0	.5	2

Write the associated dual problem and determine its optimal solution in two ways.

7. Consider the following set of inequalities:

$$\begin{aligned} 2x_1 + 3x_2 &\le 12 \\ -3x_1 + 2x_2 &\le -4 \\ 3x_1 - 5x_2 &\le 2 \\ x_1 &\text{ unrestricted} \\ x_2 &\ge 0 \end{aligned}$$

A feasible solution can be found by augmenting the trivial objective function Maximize $z = x_1 + x_2$ and then solving the problem. Another way is to solve the dual; from which a solution for the set of inequalities can be found. Apply the two methods.

8. Estimate a range for the optimal objective value for the following LPs:

*(a) Minimize $z = 5x_1 + 2x_2$
subject to

$$\begin{aligned} x_1 - x_2 &\ge 3 \\ 2x_1 + 3x_2 &\ge 5 \\ x_1, x_2 &\ge 0 \end{aligned}$$

(b) Maximize $z = x_1 + 5x_2 + 3x_3$
subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ 2x_1 - x_2 \qquad &= 4 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

(c) Maximize $z = 2x_1 + x_2$
subject to

$$\begin{aligned} x_1 - x_2 &\le 10 \\ 2x_1 \qquad &\le 40 \\ x_1, x_2 &\ge 0 \end{aligned}$$

(d) Maximize $z = 3x_1 + 2x_2$
subject to

$$\begin{aligned} 2x_1 + x_2 &\le 3 \\ 3x_1 + 4x_2 &\le 12 \\ x_1, x_2 &\ge 0 \end{aligned}$$

4.2

9. In Problem 7(a), let y_1 and y_2 be the dual variables. Determine whether the following pairs of primal-dual solutions are optimal:
   *(a) $(x_1 = 3, x_2 = 1; y_1 = 4, y_2 = 1)$
   (b) $(x_1 = 4, x_2 = 1; y_1 = 1, y_2 = 0)$
   (c) $(x_1 = 3, x_2 = 0; y_1 = 5, y_2 = 0)$

4.2.4 Simplex Tableau Computations

This section shows how *any iteration* of the entire simplex tableau can be generated from the *original* data of the problem, the *inverse* associated with the iteration, and the dual problem. Using the layout of the simplex tableau in Figure 4.1, we can divide the computations into two types:

1. Constraint columns (left- and right-hand sides).
2. Objective z-row.

Formula 1: Constraint Column Computations. In any simplex iteration, a left-hand or a right-hand side column is computed as follows:

$$\begin{pmatrix}\text{Constraint column}\\ \text{in iteration } i\end{pmatrix} = \begin{pmatrix}\text{Inverse in}\\ \text{iteration } i\end{pmatrix} \times \begin{pmatrix}\text{Original}\\ \text{constraint column}\end{pmatrix}$$

Formula 2: Objective z-row Computations. In any simplex iteration, the objective equation coefficient (reduced cost) of x_j is computed as follows:

$$\begin{pmatrix}\textit{Primal } z\text{-equation}\\ \text{coefficient of variable } x_j\end{pmatrix} = \begin{pmatrix}\text{Left-hand side of}\\ j\text{th } \textit{dual} \text{ constraint}\end{pmatrix} - \begin{pmatrix}\text{Right-hand side of}\\ j\text{th } \textit{dual} \text{ constraint}\end{pmatrix}$$

Example 4.2-3

We use the LP in Example 4.2-1 to illustrate the application of Formulas 1 and 2. From the optimal tableau in Table 4.4, we have

$$\text{Optimal inverse} = \begin{pmatrix}\frac{2}{5} & -\frac{1}{5}\\ \frac{1}{5} & \frac{2}{5}\end{pmatrix}$$

The use of Formula 1 is illustrated by computing all the left- and right-hand side columns of the optimal tableau:

$$\begin{pmatrix}x_1\text{-column in}\\ \text{optimal iteration}\end{pmatrix} = \begin{pmatrix}\text{Inverse in}\\ \text{optimal iteration}\end{pmatrix} \times \begin{pmatrix}\text{original}\\ x_1\text{-column}\end{pmatrix}$$

$$= \begin{pmatrix}\frac{2}{5} & -\frac{1}{5}\\ \frac{1}{5} & \frac{2}{5}\end{pmatrix} \times \begin{pmatrix}1\\ 2\end{pmatrix} = \begin{pmatrix}0\\ 1\end{pmatrix}$$

In a similar manner, we compute the remaining constraint columns; namely,

$$\begin{pmatrix} x_2\text{-column in} \\ \text{optimal iteration} \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_3\text{-column in} \\ \text{optimal iteration} \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{5} \\ \frac{7}{5} \end{pmatrix}$$

$$\begin{pmatrix} x_4\text{-column in} \\ \text{optimal iteration} \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{2}{5} \\ \frac{1}{5} \end{pmatrix}$$

$$\begin{pmatrix} R\text{-column in} \\ \text{optimal iteration} \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{5} \\ \frac{2}{5} \end{pmatrix}$$

$$\begin{pmatrix} \text{Right-hand side} \\ \text{column in} \\ \text{optimal iteration} \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix} \times \begin{pmatrix} 10 \\ 8 \end{pmatrix} = \begin{pmatrix} \frac{12}{5} \\ \frac{26}{5} \end{pmatrix}$$

Next, we demonstrate how the objective row computations are carried out using Formula 2. The optimal values of the dual variables, $(y_1, y_2) = \left(\frac{29}{5}, -\frac{2}{5}\right)$, were computed in Example 4.2-1 using two different methods. These values are used in Formula 2 to determine the associated z-coefficients; namely,

$$\begin{aligned}
z\text{-cofficient of } x_1 &= y_1 + 2y_2 - 5 &&= \tfrac{29}{5} + 2 \times -\tfrac{2}{5} - 5 &&= 0 \\
z\text{-cofficient of } x_2 &= 2y_1 - y_2 - 12 &&= 2 \times \tfrac{29}{5} - \left(-\tfrac{2}{5}\right) - 12 &&= 0 \\
z\text{-cofficient of } x_3 &= y_1 + 3y_2 - 4 &&= \tfrac{29}{5} + 3 \times -\tfrac{2}{5} - 4 &&= \tfrac{3}{5} \\
z\text{-cofficient of } x_4 &= y_1 - 0 &&= \tfrac{29}{5} - 0 &&= \tfrac{29}{5} \\
z\text{-cofficient of } R &= y_2 - (-M) &&= -\tfrac{2}{5} - (-M) &&= -\tfrac{2}{5} + M
\end{aligned}$$

Notice that Formula 1 and Formula 2 calculations can be applied at any iteration of either the primal or the dual problems. All we need is the inverse associated with the (primal or dual) iteration and the original LP data.

PROBLEM SET 4.2D

1. Generate the first simplex iteration of Example 4.2-1 (you may use TORA's Iterations $\Rightarrow$ M-method for convenience), then use Formulas 1 and 2 to verify all the elements of the resulting tableau.
2. Consider the following LP model:

$$\text{Maximize } z = 4x_1 + 14x_2$$

subject to

$$\begin{aligned}
2x_1 + 7x_2 + x_3 \quad\quad &= 21 \\
7x_1 + 2x_2 \quad\quad + x_4 &= 21 \\
x_1, x_2, x_3, x_4 &\geq 0
\end{aligned}$$

Check the optimality and feasibility of each of the following basic solutions.

***(a)** Basic variables $= (x_2, x_4)$, Inverse $= \begin{pmatrix} \frac{1}{7} & 0 \\ -\frac{2}{7} & 1 \end{pmatrix}$

(b) Basic variables $= (x_2, x_3)$, Inverse $= \begin{pmatrix} 0 & \frac{1}{2} \\ 1 & -\frac{7}{2} \end{pmatrix}$

(c) Basic variables $= (x_2, x_1)$, Inverse $= \begin{pmatrix} \frac{7}{45} & -\frac{2}{45} \\ -\frac{2}{45} & \frac{7}{45} \end{pmatrix}$

(d) Basic variables $= (x_1, x_4)$, Inverse $= \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{7}{2} & 1 \end{pmatrix}$

3. Consider the following LP model:

$$\text{Maximize } z = 3x_1 + 2x_2 + 5x_3$$

subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 + x_4 &= 30 \\ 3x_1 + 2x_3 + x_5 &= 60 \\ x_1 + 4x_2 + x_6 &= 20 \\ x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0 \end{aligned}$$

Check the optimality and feasibility of the following basic solutions:

(a) Basic variables $= (x_4, x_3, x_6)$, Inverse $= \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(b) Basic variables $= (x_2, x_3, x_1)$, Inverse $= \begin{pmatrix} \frac{1}{4} & -\frac{1}{8} & \frac{1}{8} \\ \frac{3}{2} & -\frac{1}{4} & -\frac{3}{4} \\ -1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

(c) Basic variables $= (x_2, x_3, x_6)$, Inverse $= \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 1 & 1 \end{pmatrix}$

***4.** Consider the following LP model:

$$\text{Minimize } z = 2x_1 + x_2$$

subject to

$$\begin{aligned} 3x_1 + x_2 - x_3 &= 3 \\ 4x_1 + 3x_2 - x_4 &= 6 \\ x_1 + 2x_2 + x_5 &= 3 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

Compute the entire simplex tableau associated with the following basic solution and check it for optimality and feasibility.

$$\text{Basic variables} = (x_1, x_2, x_5), \text{ Inverse} = \begin{pmatrix} \frac{3}{5} & -\frac{1}{5} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

5. Consider the following LP model:

$$\text{Maximize } z = 5x_1 + 12x_2 + 4x_3$$

subject to

$$x_1 + 2x_2 + x_3 + x_4 = 10$$
$$2x_1 - x_2 + 3x_3 = 2$$
$$x_1, x_2, x_3, x_4 \geq 0$$

(a) Identify the best solution from among the following basic feasible solutions:

(i) Basic variables $= (x_4, x_3)$, Inverse $= \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 & \frac{1}{3} \end{pmatrix}$

(ii) Basic variables $= (x_2, x_1)$, Inverse $= \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix}$

(iii) Basic variables $= (x_2, x_3)$, Inverse $= \begin{pmatrix} \frac{3}{7} & -\frac{1}{7} \\ \frac{1}{7} & \frac{2}{7} \end{pmatrix}$

(b) Is the solution obtained in (a) optimum for the LP model?

6. Consider the following LP model:

$$\text{Maximize } z = 5x_1 + 2x_2 + 3x_3$$

subject to

$$x_1 + 5x_2 + 2x_3 \leq b_1$$
$$x_1 - 5x_2 - 6x_3 \leq b_2$$
$$x_1, x_2, x_3 \geq 0$$

The following optimal tableau corresponds to specific values of b_1 and b_2:

Basic	x_1	x_2	x_3	x_4	x_5	Solution
z	0	a	7	d	e	150
x_1	1	b	2	1	0	30
x_5	0	c	-8	-1	1	10

Determine the following:

(a) The right-hand-side values, b_1 and b_2.

(b) The optimal dual solution.

(c) The elements a, b, c, d, e.