CHAPTER 3

The Simplex Method and Sensitivity Analysis

Chapter Guide. This chapter details the simplex method for solving the general LP problem. It also explains how simplex-based sensitivity analysis is used to provide important economic interpretations about the optimum solution, including the *dual prices* and the *reduced cost.*

The simplex method computations are particularly tedious, repetitive, and, above all, boring. As you do these computations, you should not lose track of the big picture; namely, the simplex method attempts to move from one corner point of the solution space to a better corner point until the optimum is found. To assist you in this regard, TORA's interactive *user-guided* module (with instant feedback) allows you to decide how the computations should proceed while relieving you of the burden of the tedious computations. In this manner, you get to understand the concepts without being overwhelmed by the computational details. Rest assured that once you have learned how the simplex method works (and it is important that you do understand the concepts), computers will carry out the tedious work and you will *never* again need to solve an LP manually.

Throughout my teaching experience, I have noticed that while students can easily carry out the tedious simplex method computations, in the end, some cannot tell why they are doing them or what the solution is. To assist in overcoming this potential difficulty, the material in the chapter stresses the interpretation of each iteration in terms of the solution to the original problem.

When you complete the material in this chapter, you will be in a position to read and interpret the output reports provided by commercial software. The last section describes how these reports are generated in AMPL, Excel Solver, and TORA.

This chapter includes a summary of 1 real-life application, 11 solved examples, 1 AMPL model, 1 Solver model, 1 TORA model, 107 end-of-section problems, and 3 cases. The cases are in Appendix E on the CD. The AMPL/Excel/Solver/TORA programs are in folder ch3Files.

Real Life Application—Optimization of Heart Valve Production

Biological heart valves in different sizes are bioprostheses manufactured from porcine hearts for human implantation. On the supply side, porcine hearts cannot be "produced" to specific sizes. Moreover, the exact size of a manufactured valve cannot be determined until the biological component of pig heart has been processed. As a result, some sizes may be overstocked and others understocked. A linear programming model was developed to reduce overstocked sizes and increase understocked sizes. The resulting savings exceeded $1,476,000 in 1981, the year the study was made. The details of this study are presented in Case 2, Chapter 24 on the CD.

3.1 LP MODEL IN EQUATION FORM

The development of the simplex method computations is facilitated by imposing two requirements on the constraints of the problem:

1. All the constraints (with the exception of the nonnegativity of the variables) are equations with nonnegative right-hand side.
2. All the variables are nonnegative.

These two requirements are imposed here primarily to standardize and streamline the simplex method calculations. It is important to know that all commercial packages (and TORA) directly accept inequality constraints, nonnegative right-hand side, and unrestricted variables. Any necessary preconditioning of the model is done internally in the software before the simplex method solves the problem.

3.1.1 Converting Inequalities into Equations with Nonnegative Right-Hand Side

In ($\leq$) constraints, the right-hand side can be thought of as representing the limit on the availability of a resource, in which case the left-hand side would represent the usage of this limited resource by the activities (variables) of the model. The difference between the right-hand side and the left-hand side of the ($\leq$) constraint thus yields the *unused* or *slack* amount of the resource.

To convert a ($\leq$)-inequality to an equation, a nonnegative **slack variable** is added to the left-hand side of the constraint. For example, in the Reddy Mikks model (Example 2.1-1), the constraint associated with the use of raw material $M1$ is given as

$$6x_1 + 4x_2 \leq 24$$

Defining s_1 as the slack or unused amount of $M1$, the constraint can be converted to the following equation:

$$6x_1 + 4x_2 + s_1 = 24, s_1 \geq 0$$

Next, a ($\geq$)-constraint sets a lower limit on the activities of the LP model, so that the amount by which the left-hand side exceeds the minimum limit represents a *surplus*. The conversion from ($\geq$) to ($=$) is achieved by subtracting a nonnegative

surplus variable from the left-hand side of the inequality. For example, in the diet model (Example 2.2-2), the constraint representing the minimum feed requirements is

$$x_1 + x_2 \geq 800$$

Defining S_1 as the surplus variable, the constraint can be converted to the following equation

$$x_1 + x_2 - S_1 = 800, S_1 \geq 0$$

The only remaining requirement is for the right-hand side of the resulting equation to be nonnegative. The condition can always be satisfied by multiplying both sides of the resulting equation by -1, where necessary. For example, the constraint

$$-x_1 + x_2 \leq -3$$

is equivalent to the equation

$$-x_1 + x_2 + s_1 = -3, s_1 \geq 0$$

Now, multiplying both sides by -1 will render a nonnegative right-hand side, as desired—that is,

$$x_1 - x_2 - s_1 = 3$$

PROBLEM SET 3.1A

*1. In the Reddy Mikks model (Example 2.2-1), consider the feasible solution $x_1 = 3$ tons and $x_2 = 1$ ton. Determine the value of the associated slacks for raw materials $M1$ and $M2$.

2. In the diet model (Example 2.2-2), determine the surplus amount of feed consisting of 500 lb of corn and 600 lb of soybean meal.

3. Consider the following inequality

$$10x_1 - 3x_2 \geq -5$$

Show that multiplying both sides of the inequality by -1 and then converting the resulting inequality into an equation is the same as converting it first to an equation and then multiplying both sides by -1.

*4. Two different products, $P1$ and $P2$, can be manufactured by one or both of two different machines, $M1$ and $M2$. The unit processing time of either product on either machine is the same. The daily capacity of machine $M1$ is 200 units (of either $P1$ or $P2$, or a mixture of both) and the daily capacity of machine $M2$ is 250 units. The shop supervisor wants to balance the production schedule of the two machines such that the total number of units produced on one machine is within 5 units of the number produced on the other. The profit per unit of $P1$ is \$10 and that of $P2$ is \$15. Set up the problem as an LP in equation form.

5. Show how the following objective function can be presented in equation form:

$$\text{Minimize } z = \max\{|x_1 - x_2 + 3x_3|, |-x_1 + 3x_2 - x_3|\}$$

$$x_1, x_2, x_3 \geq 0$$

(*Hint*: $|a| \leq b$ is equivalent to $a \leq b$ and $a \geq -b$.)

6. Show that the m equations:

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, i = 1, 2, \ldots, m$$

are equivalent to the following $m + 1$ inequalities:

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i, i = 1, 2, \ldots, m$$

$$\sum_{j=1}^{n}\left(\sum_{i=1}^{m} a_{ij}\right)x_j \geq \sum_{i=1}^{m} b_i$$

3.1.2 Dealing with Unrestricted Variables

In Example 2.3-6 we presented a multiperiod production smoothing model in which the workforce at the start of each period is adjusted up or down depending on the demand for that period. Specifically, if x_i (≥ 0) is the workforce size in period i, then x_{i+1} (≥ 0) the workforce size in period $i + 1$ can be expressed as

$$x_{i+1} = x_i + y_{i+1}$$

The variable y_{i+1} must be unrestricted in sign to allow x_{i+1} to increase or decrease relative to x_i depending on whether workers are hired or fired, respectively.

As we will see shortly, the simplex method computations require all the variables be nonnegative. We can always account for this requirement by using the substitution

$$y_{i+1} = y_{i+1}^{-} - y_{i+1}^{+}, \text{ where } y_{i+1}^{-} \geq 0 \text{ and } y_{i+1}^{+} \geq 0$$

To show how this substitution works, suppose that in period 1 the workforce is $x_1 = 20$ workers and that the workforce in period 2 will be increased by 5 to reach 25 workers. In terms of the variables y_2^- and y_2^+, this will be equivalent to $y_2^- = 5$ and $y_2^+ = 0$ or $y_2 = 5 - 0 = 5$. Similarly, if the workforce in period 2 is reduced to 16, then we have $y_2^- = 0$ and $y_2^+ = 4$, or $y_2 = 0 - 4 = -4$. The substitution also allows for the possibility of no change in the workforce by letting both variables assume a zero value.

You probably are wondering about the possibility that both y_2^- and y_2^+ may assume positive values simultaneously. Intuitively, as we explained in Example 2.3-6, this cannot happen, because it means that we can hire and fire a worker at the same time. This intuition is also supported by a mathematical proof that shows that, in any simplex method solution, it is impossible that both variables will assume positive values simultaneously.

PROBLEM SET 3.1B

1. McBurger fast-food restaurant sells quarter-pounders and cheeseburgers. A quarter-pounder uses a quarter of a pound of meat, and a cheeseburger uses only .2 lb. The restaurant starts the day with 200 lb of meat but may order more at an additional cost of 25 cents per pound to cover the delivery cost. Any surplus meat at the end of the day is donated to charity. McBurger's profits are 20 cents for a quarter-pounder and 15 cents for a cheeseburger. McBurger does not expect to sell more than 900 sandwiches in any one

day. How many of each type sandwich should McBurger plan for the day? Solve the problem using TORA, Solver, or AMPL.

2. Two products are manufactured in a machining center. The productions times per unit of products 1 and 2 are 10 and 12 minutes, respectively. The total regular machine time is 2500 minutes per day. In any one day, the manufacturer can produce between 150 and 200 units of product 1, but no more than 45 units of product 2. Overtime may be used to meet the demand at an additional cost of $.50 per minute. Assuming that the unit profits for products 1 and 2 are $6.00 and $7.50, respectively, formulate the problem as an LP model, then solve with TORA, Solver, or AMPL to determine the optimum production level for each product as well as any overtime needed in the center.

*3. JoShop manufactures three products whose unit profits are $2, $5, and $3, respectively. The company has budgeted 80 hours of labor time and 65 hours of machine time for the production of three products. The labor requirements per unit of products 1, 2, and 3 are 2, 1, and 2 hours, respectively. The corresponding machine-time requirements per unit are 1, 1, and 2 hours. JoShop regards the budgeted labor and machine hours as goals that may be exceeded, if necessary, but at the additional cost of $15 per labor hour and $10 per machine hour. Formulate the problem as an LP, and determine its optimum solution using TORA, Solver, or AMPL.

4. In an LP in which there are several unrestricted variables, a transformation of the type $x_j = x_j^- - x_j^+, x_j^-, x_j^+ \geq 0$ will double the corresponding number of nonnegative variables. We can, instead, replace k unrestricted variables with exactly $k + 1$ nonnegative variables by using the substitution $x_j = x_j' - w, x_j', w \geq 0$. Use TORA, Solver, or AMPL to show that the two methods produce the same solution for the following LP:

$$\text{Maximize } z = -2x_1 + 3x_2 - 2x_3$$

subject to

$$4x_1 - x_2 - 5x_3 = 10$$

$$2x_1 + 3x_2 + 2x_3 = 12$$

$$x_1 \geq 0, x_2, x_3 \text{ unrestricted}$$

3.2 TRANSITION FROM GRAPHICAL TO ALGEBRAIC SOLUTION

The ideas conveyed by the graphical LP solution in Section 2.2 lay the foundation for the development of the algebraic simplex method. Figure 3.1 draws a parallel between the two methods. In the graphical method, the solution space is delineated by the half-spaces representing the constraints, and in the simplex method the solution space is represented by m simultaneous linear equations and n nonnegative variables.

We can see visually why the graphical solution space has an infinite number of solution points, but how can we draw a similar conclusion from the algebraic representation of the solution space? The answer is that in the algebraic representation the number of equations m is always *less than or equal to* the number of variables n.[1] If $m = n$, and the equations are consistent, the system has only one solution; but if $m < n$ (which

[1]If the number of equations m is larger than the number of variables n, then at least $m - n$ equations must be redundant.

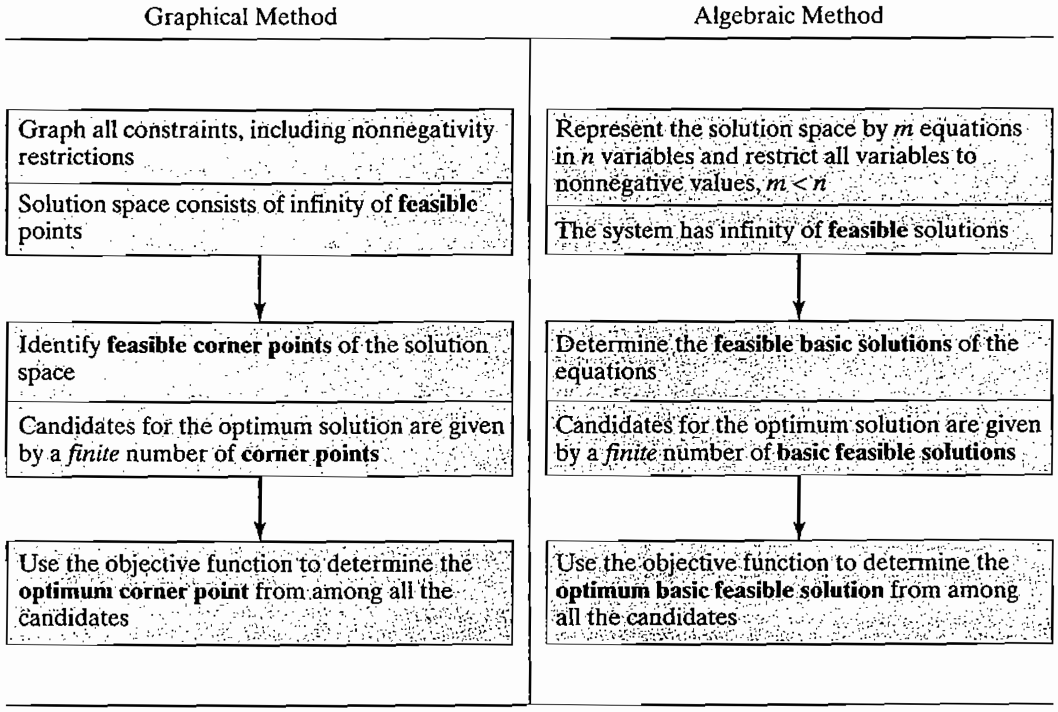


FIGURE 3.1

Transition from graphical to algebraic solution

represents the majority of LPs), then the system of equations, again if consistent, will yield an infinite number of solutions. To provide a simple illustration, the equation $x = 2$ has $m = n = 1$, and the solution is obviously unique. But, the equation $x + y = 1$ has $m = 1$ and $n = 2$, and it yields an infinite number of solutions (any point on the straight line $x + y = 1$ is a solution).

Having shown how the LP solution space is represented algebraically, the candidates for the optimum (i.e., corner points) are determined from the simultaneous linear equations in the following manner:

Algebraic Determination of Corner Points.

In a set of $m \times n$ equations $(m < n)$, if we set $n - m$ variables equal to zero and then solve the m equations for the remaining m variables, the resulting solution, if unique, is called a **basic solution** and must correspond to a (feasible or infeasible) corner point of the solution space. This means that the *maximum* number of corner points is

$$C_m^n = \frac{n!}{m!(n-m)!}$$

The following example demonstrates the procedure.

Example 3.2-1

Consider the following LP with two variables:

$$\text{Maximize } z = 2x_1 + 3x_2$$

subject to

$$2x_1 + x_2 \leq 4$$
$$x_1 + 2x_2 \leq 5$$
$$x_1, x_2 \geq 0$$

Figure 3.2 provides the graphical solution space for the problem.

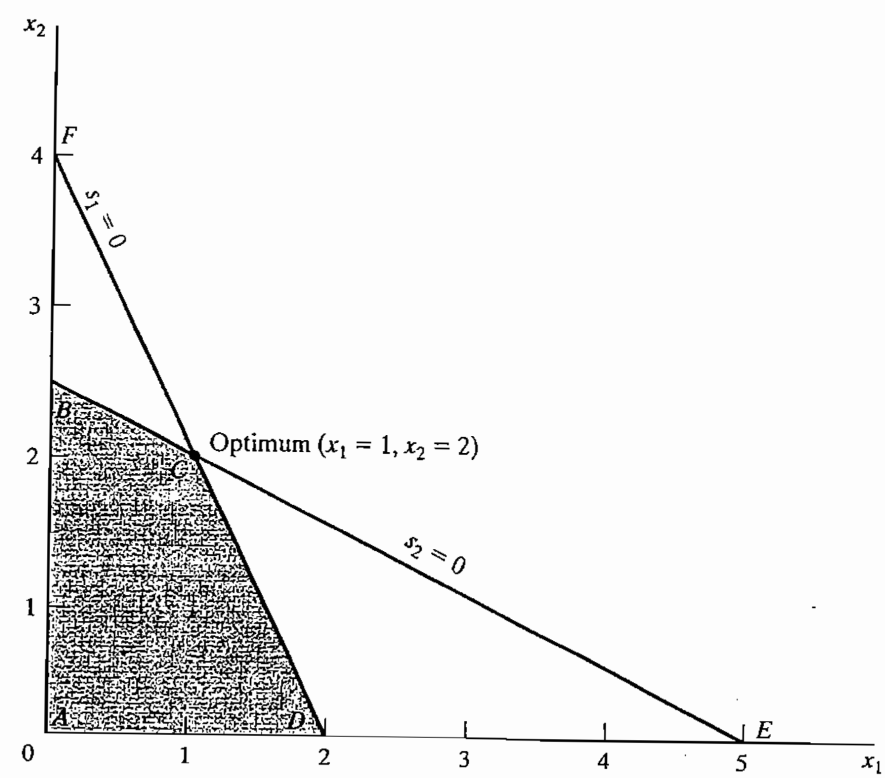
Algebraically, the solution space of the LP is represented as:

$$2x_1 + x_2 + s_1 = 4$$
$$x_1 + 2x_2 + s_2 = 5$$
$$x_1, x_2, s_1, s_2 \geq 0$$

The system has $m = 2$ equations and $n = 4$ variables. Thus, according to the given rule, the corner points can be determined algebraically by setting $n - m = 4 - 2 = 2$ variables equal to

FIGURE 3.2

LP solution space of Example 3.2-1



zero and then solving for the remaining $m = 2$ variables. For example, if we set $x_1 = 0$ and $x_2 = 0$, the equations provide the unique (basic) solution

$$s_1 = 4, s_2 = 5$$

This solution corresponds to point A in Figure 3.2 (convince yourself that $s_1 = 4$ and $s_2 = 5$ at point A). Another point can be determined by setting $s_1 = 0$ and $s_2 = 0$ and then solving the two equations

$$2x_1 + x_2 = 4$$

$$x_1 + 2x_2 = 5$$

This yields the basic solution ($x_1 = 1, x_2 = 2$), which is point C in Figure 3.2.

You probably are wondering how one can decide which $n - m$ variables should be set equal to zero to target a specific corner point. Without the benefit of the graphical solution (which is available only for two or three variables), we cannot say which $(n - m)$ zero variables are associated with which corner point. But that does not prevent us from enumerating *all* the corner points of the solution space. Simply consider *all* combinations in which $n - m$ variables are set to zero and solve the resulting equations. Once done, the optimum solution is the feasible basic solution (corner point) that yields the best objective value.

In the present example we have $C_2^4 = \frac{4!}{2!2!} = 6$ corner points. Looking at Figure 3.2, we can immediately spot the four corner points A, B, C, and D. Where, then, are the remaining two? In fact, points E and F also are corner points for the problem, but they are *infeasible* because they do not satisfy all the constraints. These infeasible corner points are not candidates for the optimum.

To summarize the transition from the graphical to the algebraic solution, the zero $n - m$ variables are known as **nonbasic variables**. The remaining m variables are called **basic variables** and their solution (obtained by solving the m equations) is referred to as **basic solution**. The following table provides all the basic and nonbasic solutions of the current example.

Nonbasic (zero) variables	Basic variables	Basic solution	Associated corner point	Feasible?	Objective value, z
(x_1, x_2)	(s_1, s_2)	(4, 5)	A	Yes	0
(x_1, s_1)	(x_2, s_2)	(4, −3)	F	No	—
(x_1, s_2)	(x_2, s_1)	(2.5, 1.5)	B	Yes	7.5
(x_2, s_1)	(x_1, s_2)	(2, 3)	D	Yes	4
(x_2, s_2)	(x_1, s_1)	(5, −6)	E	No	—
$\mathbf{(s_1, s_2)}$	$\mathbf{(x_1, x_2)}$	**(1, 2)**	C	**Yes**	**8 (optimum)**

Remarks. We can see from the computations above that as the problem size increases (that is, m and n become large), the procedure of enumerating all the corner points involves prohibitive computations. For example, for $m = 10$ and $n = 20$, it is necessary to solve $C_{10}^{20} = 184{,}756$ sets of 10×10 equations, a staggering task indeed, particularly when we realize that a (10×20)-LP is a small size in most real-life situations, where hundreds or even thousands of variables and constraints are not unusual. The simplex method alleviates this computational burden dramatically by investigating only a fraction of all possible basic feasible solutions (corner points) of the solution space. In essence, the simplex method utilizes an intelligent search procedure that locates the optimum corner point in an efficient manner.

PROBLEM SET 3.2A

1. Consider the following LP:

$$\text{Maximize } z = 2x_1 + 3x_2$$

subject to

$$x_1 + 3x_2 \le 6$$
$$3x_1 + 2x_2 \le 6$$
$$x_1, x_2 \ge 0$$

   (a) Express the problem in equation form.
   (b) Determine all the basic solutions of the problem, and classify them as feasible and infeasible.
   ***(c)** Use direct substitution in the objective function to determine the optimum basic feasible solution.
   (d) Verify graphically that the solution obtained in (c) is the optimum LP solution—hence, conclude that the optimum solution can be determined algebraically by considering the basic feasible solutions only.
   ***(e)** Show how the *infeasible* basic solutions are represented on the graphical solution space.

2. Determine the optimum solution for each of the following LPs by enumerating all the basic solutions.

   (a) Maximize $z = 2x_1 - 4x_2 + 5x_3 - 6x_4$
   subject to

$$x_1 + 4x_2 - 2x_3 + 8x_4 \le 2$$
$$-x_1 + 2x_2 + 3x_3 + 4x_4 \le 1$$
$$x_1, x_2, x_3, x_4 \ge 0$$

   (b) Minimize $z = x_1 + 2x_2 - 3x_3 - 2x_4$
   subject to

$$x_1 + 2x_2 - 3x_3 + x_4 = 4$$
$$x_1 + 2x_2 + x_3 + 2x_4 = 4$$
$$x_1, x_2, x_3, x_4 \ge 0$$

*3. Show algebraically that all the basic solutions of the following LP are infeasible.

$$\text{Maximize } z = x_1 + x_2$$

subject to

$$x_1 + 2x_2 \le 6$$
$$2x_1 + x_2 \le 16$$
$$x_1, x_2 \ge 0$$

4. Consider the following LP:

$$\text{Maximize } z = 2x_1 + 3x_2 + 5x_3$$

subject to

$$-6x_1 + 7x_2 - 9x_3 \geq 4$$
$$x_1 + x_2 + 4x_3 = 10$$
$$x_1, x_3 \geq 0$$
$$x_2 \text{ unrestricted}$$

Conversion to the equation form involves using the substitution $x_2 = x_2^- - x_2^+$. Show that a basic solution cannot include both x_2^- and x_2^+ simultaneously.

5. Consider the following LP:

$$\text{Maximize } z = x_1 + 3x_2$$

subject to

$$x_1 + x_2 \leq 2$$
$$-x_1 + x_2 \leq 4$$
$$x_1 \text{ unrestricted}$$
$$x_2 \geq 0$$

(a) Determine all the basic feasible solutions of the problem.
(b) Use direct substitution in the objective function to determine the best basic solution.
(c) Solve the problem graphically, and verify that the solution obtained in (c) is the optimum.

3.3 THE SIMPLEX METHOD

Rather than enumerating *all* the basic solutions (corner points) of the LP problem (as we did in Section 3.2), the simplex method investigates only a "select few" of these solutions. Section 3.3.1 describes the *iterative* nature of the method, and Section 3.3.2 provides the computational details of the simplex algorithm.

3.3.1 Iterative Nature of the Simplex Method

Figure 3.3 provides the solution space of the LP of Example 3.2-1. Normally, the simplex method starts at the origin (point A) where $x_1 = x_2 = 0$. At this starting point, the value of the objective function, z, is zero, and the logical question is whether an increase in nonbasic x_1 and/or x_2 above their current zero values can improve (increase) the value of z. We answer this question by investigating the objective function:

$$\text{Maximize } z = 2x_1 + 3x_2$$

The function shows that an increase in either x_1 or x_2 (or both) above their current zero values will *improve* the value of z. The design of the simplex method calls for increasing *one variable at a time*, with the selected variable being the one with the *largest*

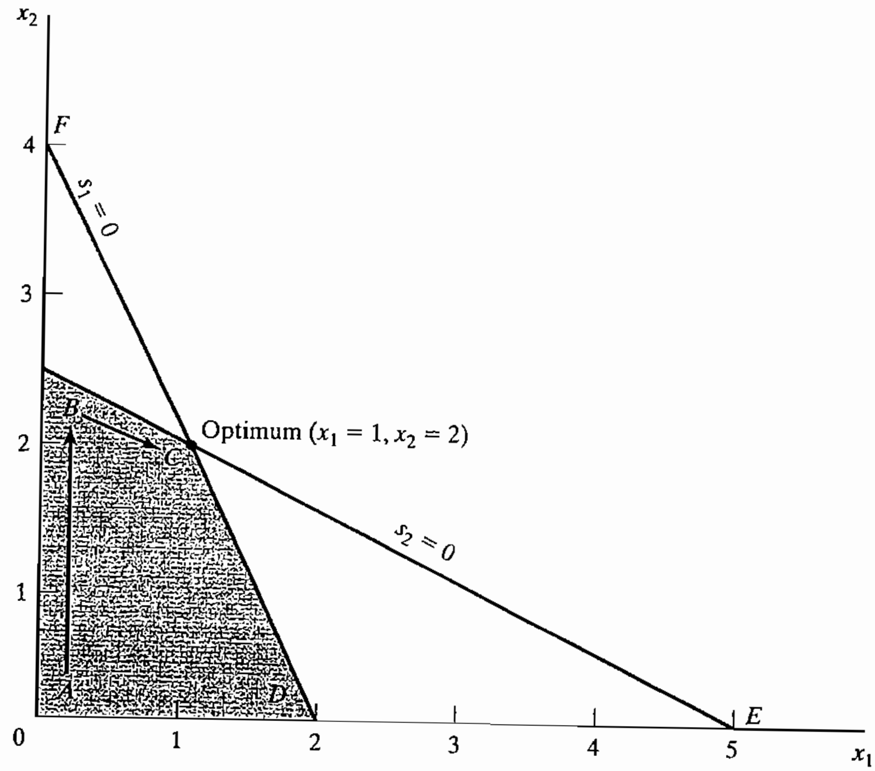


FIGURE 3.3
Iterative process of the simplex method

rate of improvement in z. In the present example, the value of z will increase by 2 for each unit increase in x_1 and by 3 for each unit increase in x_2. This means that the *rate* of improvement in the value of z is 2 for x_1 and 3 for x_2. We thus elect to increase x_2, the variable with the largest rate of improvement. Figure 3.3 shows that the value of x_2 must be increased until corner point B is reached (recall that stopping short of reaching corner point B is not optimal because a candidate for the optimum must be a corner point). At point B, the simplex method will then increase the value of x_1 to reach the improved corner point C, which is the optimum. The path of the simplex algorithm is thus defined as $A \rightarrow B \rightarrow C$. Each corner point along the path is associated with an **iteration**. It is important to note that the simplex method moves alongside the **edges** of the solution space, which means that the method cannot cut across the solution space, going from A to C directly.

We need to make the transition from the graphical solution to the algebraic solution by showing how the points A, B, and C are represented by their basic and nonbasic variables. The following table summarizes these representations:

Corner point	Basic variables	Nonbasic (zero) variables
A	s_1, s_2	x_1, x_2
B	s_1, x_2	x_1, s_2
C	x_1, x_2	s_1, s_2

Notice the change pattern in the basic and nonbasic variables as the solution moves along the path $A \rightarrow B \rightarrow C$. From A to B, nonbasic x_2 at A becomes basic at B and basic s_2 at A becomes nonbasic at B. In the terminology of the simplex method, we say that x_2 is the **entering variable** (because it enters the basic solution) and s_2 is the **leaving variable** (because it leaves the basic solution). In a similar manner, at point B, x_1 *enters* (the basic solution) and s_1 *leaves*, thus leading to point C.

PROBLEM SET 3.3A

1. In Figure 3.3, suppose that the objective function is changed to

$$\text{Maximize } z = 8x_1 + 4x_2$$

   Identify the path of the simplex method and the basic and nonbasic variables that define this path.

2. Consider the graphical solution of the Reddy Mikks model given in Figure 2.2. Identify the path of the simplex method and the basic and nonbasic variables that define this path.

*3. Consider the three-dimensional LP solution space in Figure 3.4, whose feasible extreme points are $A, B, \ldots,$ and J.
   - **(a)** Which of the following pairs of corner points cannot represent *successive* simplex iterations: (A, B), (B, D), (E, H), and (A, I)? Explain the reason.
   - **(b)** Suppose that the simplex iterations start at A and that the optimum occurs at H. Indicate whether any of the following paths are *not* legitimate for the simplex algorithm, and state the reason.
     - **(i)** $A \rightarrow B \rightarrow G \rightarrow H$.
     - **(ii)** $A \rightarrow E \rightarrow I \rightarrow H$.
     - **(iii)** $A \rightarrow C \rightarrow E \rightarrow B \rightarrow A \rightarrow D \rightarrow G \rightarrow H$.

4. For the solution space in Figure 3.4, all the constraints are of the type $\leq$ and all the variables x_1, x_2, and x_3 are nonnegative. Suppose that s_1, s_2, s_3, and s_4 (≥ 0) are the slacks associated with constraints represented by the planes $CEIJF$, $BEIHG$, $DFJHG$, and IJH, respectively. Identify the basic and nonbasic variables associated with each feasible extreme point of the solution space.

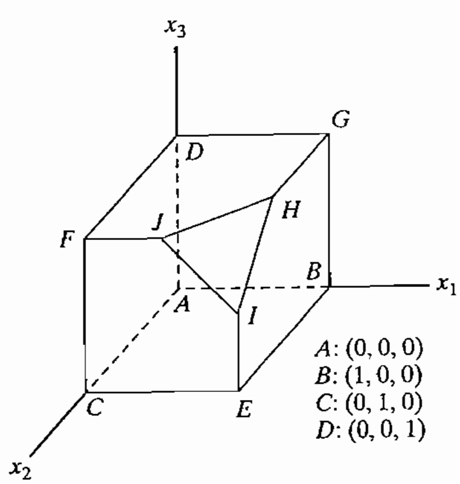


FIGURE 3.4

Solution space of Problem 3, Set 3.2b

5. Consider the solution space in Figure 3.4, where the simplex algorithm starts at point A. Determine the entering variable in the *first* iteration together with its value and the improvement in z for each of the following objective functions:
   *(a) Maximize $z = x_1 - 2x_2 + 3x_3$
   (b) Maximize $z = 5x_1 + 2x_2 + 4x_3$
   (c) Maximize $z = -2x_1 + 7x_2 + 2x_3$
   (d) Maximize $z = x_1 + x_2 + x_3$

3.3.2 Computational Details of the Simplex Algorithm

This section provides the computational details of a simplex iteration, including the rules for determining the entering and leaving variables as well as for stopping the computations when the optimum solution has been reached. The vehicle of explanation is a numerical example.

Example 3.3-1

We use the Reddy Mikks model (Example 2.1-1) to explain the details of the simplex method. The problem is expressed in equation form as

$$\text{Maximize } z = 5x_1 + 4x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$$

subject to

$$\begin{aligned}
6x_1 + 4x_2 + s_1 \qquad\qquad &= 24 \quad \text{(Raw material } M1\text{)}\\
x_1 + 2x_2 \quad + s_2 \qquad &= \ \ 6 \quad \text{(Raw material } M2\text{)}\\
-x_1 + \ x_2 \qquad + s_3 \quad &= \ \ 1 \quad \text{(Market limit)}\\
x_2 \qquad\qquad + s_4 &= \ \ 2 \quad \text{(Demand limit)}\\
x_1, x_2, s_1, s_2, s_3, s_4 &\geq 0
\end{aligned}$$

The variables s_1, s_2, s_3, and s_4 are the slacks associated with the respective constraints.
Next, we write the objective equation as

$$z - 5x_1 - 4x_2 = 0$$

In this manner, the starting simplex tableau can be represented as follows:

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution	
z	1	−5	−4	0	0	0	0	0	z-row
s_1	0	6	4	1	0	0	0	24	s_1-row
s_2	0	1	2	0	1	0	0	6	s_2-row
s_3	0	−1	1	0	0	1	0	1	s_3-row
s_4	0	0	1	0	0	0	1	2	s_4-row

The design of the tableau specifies the set of basic and nonbasic variables as well as provides the solution associated with the starting iteration. As explained in Section 3.3.1, the simplex iterations start at the origin $(x_1, x_2) = (0, 0)$ whose associated set of nonbasic and basic variables are defined as

$$\text{Nonbasic (zero) variables: } (x_1, x_2)$$

$$\text{Basic variables: } (s_1, s_2, s_3, s_4)$$

Substituting the nonbasic variables $(x_1, x_2) = (0, 0)$ and noting the special 0-1 arrangement of the coefficients of z and the basic variables (s_1, s_2, s_3, s_4) in the tableau, the following solution is immediately available (without any calculations):

$$z = 0$$
$$s_1 = 24$$
$$s_2 = 6$$
$$s_3 = 1$$
$$s_4 = 2$$

This information is shown in the tableau by listing the basic variables in the leftmost *Basic* column and their values in the rightmost *Solution* column. In effect, the tableau defines the current corner point by specifying its basic variables and their values, as well as the corresponding value of the objective function, z. Remember that the nonbasic variables (those not listed in the *Basic* column) always equal zero.

Is the starting solution optimal? The objective function $z = 5x_1 + 4x_2$ shows that the solution can be improved by increasing x_1 or x_2. Using the argument in Section 3.3.1, x_1 with the *most positive* coefficient is selected as the *entering variable*. Equivalently, because the simplex tableau expresses the objective function as $z - 5x_1 - 4x_2 = 0$, the entering variable will correspond to the variable with the *most negative* coefficient in the objective equation. This rule is referred to as the **optimality condition.**

The mechanics of determining the leaving variable from the simplex tableau calls for computing the *nonnegative* **ratios** of the right-hand side of the equations (*Solution* column) to the corresponding constraint coefficients under the entering variable, x_1, as the following table shows.

Basic	Entering x_1	Solution	Ratio (or Intercept)
s_1	6	24	$x_1 = \frac{24}{6} = 4 \leftarrow$ minimum
s_2	1	6	$x_1 = \frac{6}{1} = 6$
s_3	−1	1	$x_1 = \frac{1}{-1} = -1$ (ignore)
s_4	0	2	$x_1 = \frac{2}{0} = \infty$ (ignore)
Conclusion: x_1 enters and s_1 leaves			

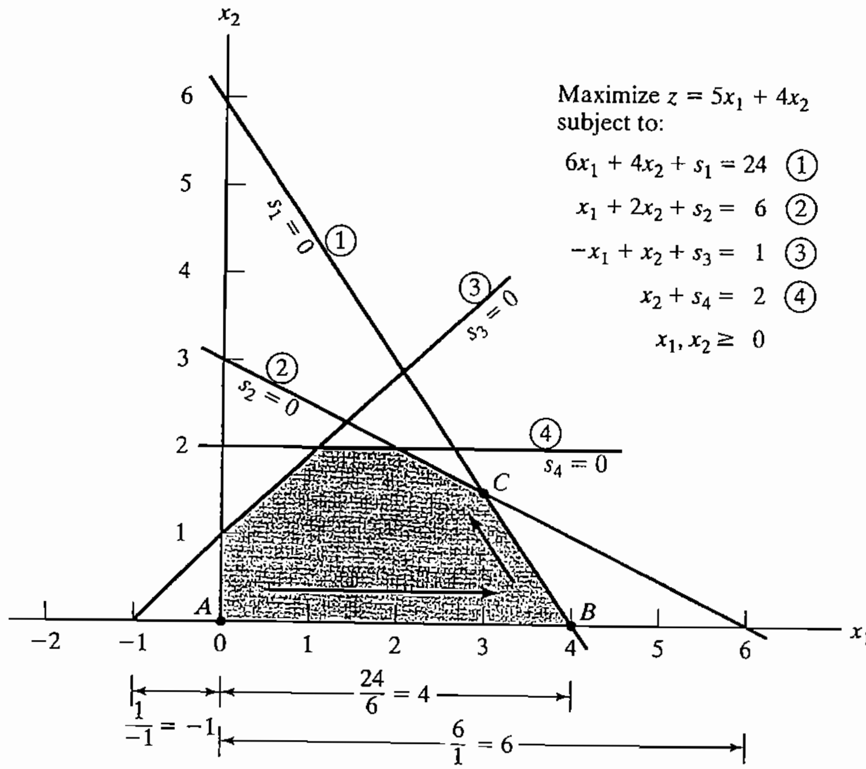


FIGURE 3.5
Graphical interpretation of the simplex method ratios in the Reddy Mikks model

The *minimum nonnegative* ratio automatically identifies the current basic variable s_1 as the leaving variable and assigns the entering variable x_1 the new value of 4.

How do the computed ratios determine the leaving variable and the value of the entering variable? Figure 3.5 shows that the computed ratios are actually the intercepts of the constraints with the entering variable (x_1) axis. We can see that the value of x_1 must be increased to 4 at corner point B, which is the smallest nonnegative intercept with the x_1-axis. An increase beyond B is infeasible. At point B, the current basic variable s_1 associated with constraint 1 assumes a zero value and becomes the *leaving variable*. The rule associated with the ratio computations is referred to as the **feasibility condition** because it guarantees the feasibility of the new solution.

The new solution point B is determined by "swapping" the entering variable x_1 and the leaving variable s_1 in the simplex tableau to produce the following sets of nonbasic and basic variables:

Nonbasic (zero) variables at B: (s_1, x_2)

Basic variables at B: (x_1, s_2, s_3, s_4)

The swapping process is based on the **Gauss-Jordan row operations**. It identifies the entering variable column as the **pivot column** and the leaving variable row as the **pivot row**. The intersection of the pivot column and the pivot row is called the **pivot element**. The following tableau is a restatement of the starting tableau with its pivot row and column highlighted.

	Basic	z	x_1 (Enter ↓)	x_2	s_1	s_2	s_3	s_4	Solution	
	z	1	−5	−4	0	0	0	0	0	
Leave ←	s_1	0	6	4	1	0	0	0	24	Pivot row
	s_2	0	1	2	0	1	0	0	6	
	s_3	0	−1	1	0	0	1	0	1	
	s_4	0	0	1	0	0	0	1	2	
			Pivot column							

The Gauss-Jordan computations needed to produce the new basic solution include two types.

1. *Pivot row*

   a. Replace the leaving variable in the *Basic* column with the entering variable.

   b. New pivot row = Current pivot row ÷ Pivot element

2. *All other rows, including z*

   New Row = (Current row) − (Its pivot column coefficient) ×
   (New pivot row)

   These computations are applied to the preceding tableau in the following manner:

   1. Replace s_1 in the *Basic* column with x_1:

      New x_1-row = Current s_1-row ÷ 6

      $= \frac{1}{6}(0\ 6\ 4\ 1\ 0\ 0\ 0\ 24)$

      $= \left(0\ 1\ \frac{2}{3}\ \frac{1}{6}\ 0\ 0\ 0\ 4\right)$

   2. New z-row = Current z-row − (−5) × New x_1-row

      $= \left(1\ -5\ -4\ 0\ 0\ 0\ 0\ 0\right) - (-5) \times \left(0\ 1\ \frac{2}{3}\ \frac{1}{6}\ 0\ 0\ 0\ 4\right)$

      $= \left(1\ 0\ -\frac{2}{3}\ \frac{5}{6}\ 0\ 0\ 0\ 20\right)$

   3. New s_2-row = Current s_2-row − (1) × New x_1-row

      $= (0\ 1\ 2\ 0\ 1\ 0\ 0\ 6) - (1) \times \left(0\ 1\ \frac{2}{3}\ \frac{1}{6}\ 0\ 0\ 0\ 4\right)$

      $= \left(0\ 0\ \frac{4}{3}\ -\frac{1}{6}\ 1\ 0\ 0\ 2\right)$

   4. New s_3-row = Current s_3-row − (−1) × New x_1-row

      $= (0\ -1\ 1\ 0\ 0\ 1\ 0\ 1) - (-1) \times \left(0\ 1\ \frac{2}{3}\ \frac{1}{6}\ 0\ 0\ 0\ 4\right)$

      $= \left(0\ 0\ \frac{5}{3}\ \frac{1}{6}\ 0\ 1\ 0\ 5\right)$

   5. New s_4-row = Current s_4-row − (0) × New x_1-row

      $= (0\ 0\ 1\ 0\ 0\ 0\ 1\ 2) - (0)\left(0\ 1\ \frac{2}{3}\ \frac{1}{6}\ 0\ 0\ 0\ 4\right)$

      $= (0\ 0\ 1\ 0\ 0\ 0\ 1\ 2)$

The new basic solution is (x_1, s_2, s_3, s_4), and the new tableau becomes

	Basic	z	x_1	↓ x_2	s_1	s_2	s_3	s_4	Solution
	z	1	0	$-\frac{2}{3}$	$\frac{5}{6}$	0	0	0	20
	x_1	0	1	$\frac{2}{3}$	$\frac{1}{6}$	0	0	0	4
←	s_2	0	0	$\frac{4}{3}$	$-\frac{1}{6}$	1	0	0	2
	s_3	0	0	$\frac{5}{3}$	$\frac{1}{6}$	0	1	0	5
	s_4	0	0	1	0	0	0	1	2

Observe that the new tableau has the same properties as the starting tableau. When we set the new nonbasic variables x_2 and s_1 to zero, the *Solution* column automatically yields the new basic solution ($x_1 = 4, s_2 = 2, s_3 = 5, s_4 = 2$). This "conditioning" of the tableau is the result of the application of the Gauss-Jordan row operations. The corresponding new objective value is $z = 20$, which is consistent with

$$\text{New } z = \text{Old } z + \text{New } x_1\text{-value} \times \text{its objective coefficient}$$
$$= 0 + 4 \times 5 = 20$$

In the last tableau, the *optimality condition* shows that x_2 is the entering variable. The feasibility condition produces the following

Basic	Entering x_2	Solution	Ratio
x_1	$\frac{2}{3}$	4	$x_2 = 4 \div \frac{2}{3} = 6$
s_2	$\frac{4}{3}$	2	$x_2 = 2 \div \frac{4}{3} = 1.5$ (minimum)
s_3	$\frac{5}{3}$	5	$x_2 = 5 \div \frac{5}{3} = 3$
s_4	1	2	$x_2 = 2 \div 1 = 2$

Thus, s_2 leaves the basic solution and new value of x_2 is 1.5. The corresponding increase in z is $\frac{2}{3}x_2 = \frac{2}{3} \times 1.5 = 1$, which yields new $z = 20 + 1 = 21$.

Replacing s_2 in the *Basic* column with entering x_2, the following Gauss-Jordan row operations are applied:

1. New pivot x_2-row = Current s_2-row $\div \frac{4}{3}$
2. New z-row = Current z-row $- \left(-\frac{2}{3}\right) \times$ New x_2-row
3. New x_1-row = Current x_1-row $- \left(\frac{2}{3}\right) \times$ New x_2-row
4. New s_3-row = Current s_3-row $- \left(\frac{5}{3}\right) \times$ New x_2-row
5. New s_4-row = Current s_4-row $- (1) \times$ New x_2-row

These computations produce the following tableau:

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution
z	1	0	0	$\frac{3}{4}$	$\frac{1}{2}$	0	0	21
x_1	0	1	0	$\frac{1}{4}$	$-\frac{1}{2}$	0	0	3
x_2	0	0	1	$-\frac{1}{8}$	$\frac{3}{4}$	0	0	$\frac{3}{2}$
s_3	0	0	0	$\frac{3}{8}$	$-\frac{5}{4}$	1	0	$\frac{5}{2}$
s_4	0	0	0	$\frac{1}{8}$	$-\frac{3}{4}$	0	1	$\frac{1}{2}$

Based on the optimality condition, *none* of the z-row coefficients associated with the nonbasic variables, s_1 and s_2, are negative. Hence, the last tableau is optimal.

The optimum solution can be read from the simplex tableau in the following manner. The optimal values of the variables in the *Basic* column are given in the right-hand-side *Solution* column and can be interpreted as

Decision variable	Optimum value	Recommendation
x_1	3	Produce 3 tons of exterior paint daily
x_2	$\frac{3}{2}$	Produce 1.5 tons of interior paint daily
z	21	Daily profit is \$21,000

You can verify that the values $s_1 = s_2 = 0$, $s_3 = \frac{5}{2}$, $s_4 = \frac{1}{2}$ are consistent with the given values of x_1 and x_2 by substituting out the values of x_1 and x_2 in the constraints.

The solution also gives the status of the resources. A resource is designated as **scarce** if the activities (variables) of the model use the resource completely. Otherwise, the resource is **abundant**. This information is secured from the optimum tableau by checking the value of the slack variable associated with the constraint representing the resource. If the slack value is zero, the resource is used completely and, hence, is classified as scarce. Otherwise, a positive slack indicates that the resource is abundant. The following table classifies the constraints of the model:

Resource	Slack value	Status
Raw material, $M1$	$s_1 = 0$	Scarce
Raw material, $M2$	$s_2 = 0$	Scarce
Market limit	$s_3 = \frac{5}{2}$	Abundant
Demand limit	$s_4 = \frac{1}{2}$	Abundant

Remarks. The simplex tableau offers a wealth of additional information that includes:

1. *Sensitivity analysis,* which deals with determining the conditions that will keep the current solution unchanged.
2. *Post-optimal analysis,* which deals with finding a new optimal solution when the data of the model are changed.

Section 3.6 deals with sensitivity analysis. The more involved topic of post-optimal analysis is covered in Chapter 4.

TORA Moment.

The Gauss-Jordan computations are tedious, voluminous, and, above all, boring. Yet, they are the least important, because in practice these computations are carried out by the computer. What is important is that you understand *how* the simplex method works. TORA's interactive *user-guided* option (with instant feedback) can be of help in this regard because it allows you to decide the course of the computations in the simplex method without the burden of carrying out the Gauss-Jordan calculations. To use TORA with the Reddy Mikks problem, enter the model and then, from the SOLVE/MODIFY menu, select Solve ⇒ Algebraic ⇒ Iterations ⇒ All-Slack. (The All-Slack selection indicates that the starting basic solution consists of slack variables only. The remaining options will be presented in Sections 3.4, 4.3, and 7.4.2.) Next, click Go To Output Screen. You can generate one or all iterations by clicking Next Iteration or All Iterations. If you opt to generate the iterations one at a time, you can interactively specify the entering and leaving variables by clicking the headings of their corresponding column and row. If your selections are correct, the column turns green and the row turns red. Else, an error message will be posted.

3.3.3 Summary of the Simplex Method

So far we have dealt with the maximization case. In minimization problems, the *optimality condition* calls for selecting the entering variable as the nonbasic variable with the most *positive* objective coefficient in the objective equation, the exact opposite rule of the maximization case. This follows because max z is equivalent to min $(-z)$. As for the *feasibility condition* for selecting the leaving variable, the rule remains unchanged.

Optimality condition. The entering variable in a maximization (minimization) problem is the *nonbasic* variable having the most negative (positive) coefficient in the z-row. Ties are broken arbitrarily. The optimum is reached at the iteration where all the z-row coefficients of the nonbasic variables are nonnegative (nonpositive).

Feasibility condition. For both the maximization and the minimization problems, the leaving variable is the *basic* variable associated with the smallest nonnegative ratio (with *strictly positive* denominator). Ties are broken arbitrarily.

Gauss-Jordan row operations.

1. Pivot row

a. Replace the leaving variable in the *Basic* column with the entering variable.
b. New pivot row = Current pivot row ÷ Pivot element

2. All other rows, including z
New row = (Current row) − (pivot column coefficient) × (New pivot row)

The steps of the simplex method are

Step 1. Determine a starting basic feasible solution.

Step 2. Select an *entering variable* using the optimality condition. Stop if there is no entering variable; the last solution is optimal. Else, go to step 3.

Step 3. Select a *leaving variable* using the feasibility condition.

Step 4. Determine the new basic solution by using the appropriate Gauss-Jordan computations. Go to step 2.

PROBLEM SET 3.3B

1. This problem is designed to reinforce your understanding of the simplex feasibility condition. In the first tableau in Example 3.3-1, we used the minimum (nonnegative) ratio test to determine the leaving variable. Such a condition guarantees that none of the new values of the basic variables will become negative (as stipulated by the definition of the LP). To demonstrate this point, force s_2, instead of s_1, to leave the basic solution. Now, look at the resulting simplex tableau, and you will note that s_1 assumes a negative value $(= -12)$, meaning that the new solution is infeasible. This situation will never occur if we employ the minimum-ratio feasibility condition.

2. Consider the following set of constraints:

$$\begin{aligned} x_1 + 2x_2 + 2x_3 + 4x_4 &\le 40 \\ 2x_1 - x_2 + x_3 + 2x_4 &\le 8 \\ 4x_1 - 2x_2 + x_3 - x_4 &\le 10 \\ x_1, x_2, x_3, x_4 &\ge 0 \end{aligned}$$

Solve the problem for each of the following objective functions.

(a) Maximize $z = 2x_1 + x_2 - 3x_3 + 5x_4$.

(b) Maximize $z = 8x_1 + 6x_2 + 3x_3 - 2x_4$.

(c) Maximize $z = 3x_1 - x_2 + 3x_3 + 4x_4$.

(d) Minimize $z = 5x_1 - 4x_2 + 6x_3 - 8x_4$.

***3.** Consider the following system of equations:

$$\begin{aligned} x_1 + 2x_2 - 3x_3 + 5x_4 + x_5 &= 4 \\ 5x_1 - 2x_2 + 6x_4 + x_6 &= 8 \\ 2x_1 + 3x_2 - 2x_3 + 3x_4 + x_7 &= 3 \\ -x_1 + x_3 - 2x_4 + x_8 &= 0 \\ x_1, x_2, \ldots, x_8 &\ge 0 \end{aligned}$$

Let $x_5, x_6, \ldots,$ and x_8 be a given initial basic feasible solution. Suppose that x_1 becomes basic. Which of the given basic variables must become nonbasic at zero level to guarantee that all the variables remain nonnegative, and what is the value of x_1 in the new solution? Repeat this procedure for x_2, x_3, and x_4.

4. Consider the following LP:

$$\text{Maximize } z = x_1$$

subject to

$$5x_1 + x_2 = 4$$
$$6x_1 + x_3 = 8$$
$$3x_1 + x_4 = 3$$
$$x_1, x_2, x_3, x_4 \geq 0$$

(a) Solve the problem *by inspection* (do not use the Gauss-Jordan row operations), and justify the answer in terms of the basic solutions of the simplex method.

(b) Repeat (a) assuming that the objective function calls for minimizing $z = x_1$.

5. Solve the following problem by *inspection*, and justify the method of solution in terms of the basic solutions of the simplex method.

$$\text{Maximize } z = 5x_1 - 6x_2 + 3x_3 - 5x_4 + 12x_5$$

subject to

$$x_1 + 3x_2 + 5x_3 + 6x_4 + 3x_5 \leq 90$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

(*Hint:* A basic solution consists of one variable only.)

6. The following tableau represents a specific simplex iteration. All variables are nonnegative. The tableau is not optimal for either a maximization or a minimization problem. Thus, when a nonbasic variable enters the solution it can either increase or decrease z or leave it unchanged, depending on the parameters of the entering nonbasic variable.

Basic	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	Solution
z	0	−5	0	4	−1	−10	0	0	620
x_8	0	3	0	−2	−3	−1	5	1	12
x_3	0	1	1	3	1	0	3	0	6
x_1	1	−1	0	0	6	−4	0	0	0

(a) Categorize the variables as basic and nonbasic and provide the current values of all the variables.

***(b)** Assuming that the problem is of the maximization type, identify the nonbasic variables that have the potential to improve the value of z. If each such variable enters the basic solution, determine the associated leaving variable, if any, and the associated change in z. Do not use the Gauss-Jordan row operations.

(c) Repeat part (b) assuming that the problem is of the minimization type.

(d) Which nonbasic variable(s) will not cause a change in the value of z when selected to enter the solution?

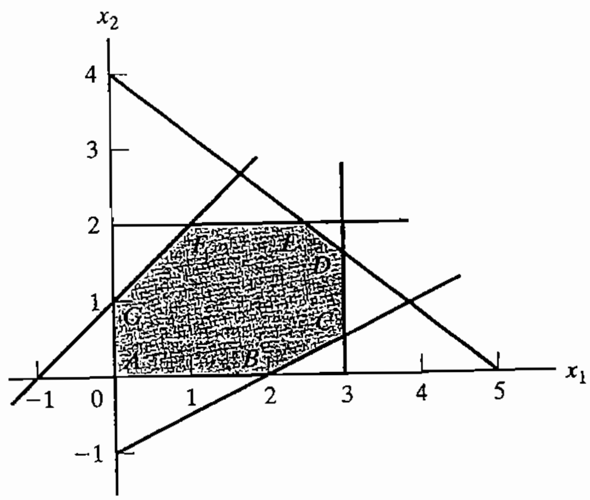


FIGURE 3.6
Solution space for Problem 7, Set 3.3b

7. Consider the two-dimensional solution space in Figure 3.6.
   (a) Suppose that the objective function is given as

$$\text{Maximize } z = 3x_1 + 6x_2$$

   If the simplex iterations start at point A, identify the path to the optimum point E.

   (b) Determine the entering variable, the corresponding ratios of the feasibility condition, and the change in the value of z, assuming that the starting iteration occurs at point A and that the objective function is given as

$$\text{Maximize } z = 4x_1 + x_2$$

   (c) Repeat (b), assuming that the objective function is

$$\text{Maximize } z = x_1 + 4x_2$$

8. Consider the following LP:

$$\text{Maximize } z = 16x_1 + 15x_2$$

subject to

$$\begin{aligned} 40x_1 + 31x_2 &\leq 124 \\ -x_1 + \quad x_2 &\leq 1 \\ x_1 \qquad &\leq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

   (a) Solve the problem by the simplex method, where the entering variable is the nonbasic variable with the *most* negative z-row coefficient.
   (b) Resolve the problem by the simplex algorithm, always selecting the entering variable as the nonbasic variable with the *least* negative z-row coefficient.
   (c) Compare the number of iterations in (a) and (b). Does the selection of the entering variable as the nonbasic variable with the *most* negative z-row coefficient lead to a smaller number of iterations? What conclusion can be made regarding the optimality condition?
   (d) Suppose that the sense of optimization is changed to minimization by multiplying z by -1. How does this change affect the simplex iterations?

*9. In Example 3.3-1, show how the second best optimal value of z can be determined from the optimal tableau.

10. Can you extend the procedure in Problem 9 to determine the third best optimal value of z?

11. The Gutchi Company manufactures purses, shaving bags, and backpacks. The construction includes leather and synthetics, leather being the scarce raw material. The production process requires two types of skilled labor: sewing and finishing. The following table gives the availability of the resources, their usage by the three products, and the profits per unit.

	Resource requirements per unit			
Resource	*Purse*	*Bag*	*Backpack*	Daily availability
Leather (ft^2)	2	1	3	42 ft^2
Sewing (hr)	2	1	2	40 hr
Finishing (hr)	1	.5	1	45 hr
Selling price ($)	24	22	45	

(a) Formulate the problem as a linear program and find the optimum solution (using TORA, Excel Solver, or AMPL).

(b) From the optimum solution determine the status of each resource.

12. *TORA experiment.* Consider the following LP:

$$\text{Maximize } z = x_1 + x_2 + 3x_3 + 2x_4$$

subject to

$$\begin{aligned} x_1 + 2x_2 - 3x_3 + 5x_4 &\leq 4 \\ 5x_1 - 2x_2 \qquad\quad + 6x_4 &\leq 8 \\ 2x_1 + 3x_2 - 2x_3 + 3x_4 &\leq 3 \\ -x_1 \qquad\quad + x_3 + 2x_4 &\leq 0 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

(a) Use TORA's iterations option to determine the optimum tableau.

(b) Select any nonbasic variable to "enter" the basic solution, and click Next Iteration to produce the associated iteration. How does the new objective value compare with the optimum in (a)? The idea is to show that the tableau in (a) is optimum because none of the nonbasic variables can improve the objective value.

13. *TORA experiment.* In Problem 12, use TORA to find the next-best optimal solution.

3.4 ARTIFICIAL STARTING SOLUTION

As demonstrated in Example 3.3-1, LPs in which all the constraints are ($\leq$) with nonnegative right-hand sides offer a convenient all-slack starting basic feasible solution. Models involving ($=$) and/or ($\geq$) constraints do not.

The procedure for starting "ill-behaved" LPs with ($=$) and ($\geq$) constraints is to use **artificial variables** that play the role of slacks at the first iteration, and then dispose of them legitimately at a later iteration. Two closely related methods are introduced here: the M-method and the two-phase method.

3.4.1 *M*-Method

The M-method starts with the LP in equation form (Section 3.1). If equation i does not have a slack (or a variable that can play the role of a slack), an artificial variable, R_i, is added to form a starting solution similar to the convenient all-slack basic solution. However, because the artificial variables are not part of the original LP model, they are assigned a very high **penalty** in the objective function, thus forcing them (eventually) to equal zero in the optimum solution. This will always be the case if the problem has a feasible solution. The following rule shows how the penalty is assigned in the cases of maximization and minimization:

Penalty Rule for Artificial Variables.

Given M, a sufficiently large positive value (mathematically, $M \rightarrow \infty$), the objective coefficient of an artificial variable represents an appropriate **penalty** if:

$$\text{Artificial variable objective coefficient} = \begin{cases} -M, & \text{in maximization problems} \\ M, & \text{in minimization problems} \end{cases}$$

Example 3.4-1

$$\text{Minimize } z = 4x_1 + x_2$$

subject to

$$\begin{aligned} 3x_1 + \ \ x_2 &= 3 \\ 4x_1 + 3x_2 &\geq 6 \\ x_1 + 2x_2 &\leq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Using x_3 as a surplus in the second constraint and x_4 as a slack in the third constraint, the equation form of the problem is given as

$$\text{Minimize } z = 4x_1 + x_2$$

subject to

$$\begin{aligned} 3x_1 + \ \ x_2 \qquad\qquad\quad &= 3 \\ 4x_1 + 3x_2 - x_3 \qquad &= 6 \\ x_1 + 2x_2 \qquad + x_4 &= 4 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

The third equation has its slack variable, x_4, but the first and second equations do not. Thus, we add the artificial variables R_1 and R_2 in the first two equations and penalize them in the objective function with $MR_1 + MR_2$ (because we are minimizing). The resulting LP is given as

$$\text{Minimize } z = 4x_1 + x_2 + MR_1 + MR_2$$

subject to

$$
\begin{aligned}
3x_1 + x_2 \quad\quad + R_1 \quad &= 3\\
4x_1 + 3x_2 - x_3 \quad\quad + R_2 &= 6\\
x_1 + 2x_2 \quad + x_4 \quad\quad &= 4\\
x_1, x_2, x_3, x_4, R_1, R_2 &\geq 0
\end{aligned}
$$

The associated starting basic solution is now given by $(R_1, R_2, x_4) = (3, 6, 4)$.

From the standpoint of solving the problem on the computer, M must assume a numeric value. Yet, in practically all textbooks, including the first seven editions of this book, M is manipulated algebraically in all the simplex tableaus. The result is an added, and unnecessary, layer of difficulty which can be avoided simply by substituting an appropriate numeric value for M (which is what we do anyway when we use the computer). In this edition, we will break away from the long tradition of manipulating M algebraically and use a numerical substitution instead. The intent, of course, is to simplify the presentation without losing substance.

What value of M should we use? The answer depends on the data of the original LP. Recall that M must be sufficiently large *relative to the original objective coefficients* so it will act as a penalty that forces the artificial variables to zero level in the optimal solution. At the same time, since computers are the main tool for solving LPs, we do not want M to be too large (even though mathematically it should tend to infinity) because potential severe round-off error can result when very large values are manipulated with much smaller values. In the present example, the objective coefficients of x_1 and x_2 are 4 and 1, respectively. It thus appears reasonable to set $M = 100$.

Using $M = 100$, the starting simplex tableau is given as follows (for convenience, the z-column is eliminated because it does not change in all the iterations):

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
z	-4	-1	0	-100	-100	0	0
R_1	3	1	0	1	0	0	3
R_2	4	3	-1	0	1	0	6
x_4	1	2	0	0	0	1	4

Before proceeding with the simplex method computations, we need to make the z-row consistent with the rest of the tableau. Specifically, in the tableau, $x_1 = x_2 = x_3 = 0$, which yields the starting basic solution $R_1 = 3$, $R_2 = 6$, and $x_4 = 4$. This solution yields $z = 100 \times 3 + 100 \times 6 = 900$ (instead of 0, as the right-hand side of the z-row currently shows). This inconsistency stems from the fact that R_1 and R_2 have nonzero coefficients $(-100, -100)$ in the z-row (compare with the all-slack starting solution in Example 3.3-1, where the z-row coefficients of the slacks are zero).

We can eliminate this inconsistency by substituting out R_1 and R_2 in the z-row using the appropriate constraint equations. In particular, notice the highlighted elements $(= 1)$ in the R_1-row and the R_2-row. Multiplying *each* of R_1-row and R_2-row by 100 and adding the *sum* to the z-row will substitute out R_1 and R_2 in the objective row—that is,

$$\text{New } z\text{-row} = \text{Old } z\text{-row} + (100 \times R_1\text{-row} + 100 \times R_2\text{-row})$$

The modified tableau thus becomes (verify!)

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
z	696	399	−100	0	0	0	900
R_1	3	1	0	1	0	0	3
R_2	4	3	−1	0	1	0	6
x_4	1	2	0	0	0	1	4

Notice that $z = 900$, which is consistent now with the values of the starting basic feasible solution: $R_1 = 3$, $R_2 = 6$, and $x_4 = 4$.

The last tableau is ready for us to apply the simplex method using the simplex optimality and the feasibility conditions, exactly as we did in Section 3.3.2. Because we are minimizing the objective function, the variable x_1 having the most *positive* coefficient in the z-row ($= 696$) enters the solution. The minimum ratio of the feasibility condition specifies R_1 as the leaving variable (verify!).

Once the entering and the leaving variables have been determined, the new tableau can be computed by using the familiar Gauss-Jordan operations.

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
z	0	167	−100	−232	0	0	204
x_1	1	$\frac{1}{3}$	0	$\frac{1}{3}$	0	0	1
R_2	0	$\frac{5}{3}$	−1	$-\frac{4}{3}$	1	0	2
x_4	0	$\frac{5}{3}$	0	$-\frac{1}{3}$	0	1	3

The last tableau shows that x_2 and R_2 are the entering and leaving variables, respectively. Continuing with the simplex computations, two more iterations are needed to reach the optimum: $x_1 = \frac{2}{5}$, $x_2 = \frac{9}{5}$, $z = \frac{17}{5}$ (verify with TORA!).

Note that the artificial variables R_1 and R_2 leave the basic solution in the first and second iterations, a result that is consistent with the concept of penalizing them in the objective function.

Remarks. The use of the penalty M will not force an artificial variable to zero level in the final simplex iteration if the LP does not have a feasible solution (i.e., the constraints are not consistent). In this case, the final simplex iteration will include at least one artificial variable at a positive level. Section 3.5.4 explains this situation.

PROBLEM SET 3.4A

1. Use hand computations to complete the simplex iteration of Example 3.4-1 and obtain the optimum solution.
2. *TORA experiment.* Generate the simplex iterations of Example 3.4-1 using TORA's Iterations ⇒ M-method module (file toraEx3.4-1.txt). Compare the effect of using $M = 1$, $M = 10$, and $M = 1000$ on the solution. What conclusion can be drawn from this experiment?

3. In Example 3.4-1, identify the starting tableau for each of the following (independent) cases, and develop the associated z-row after substituting out all the artificial variables:
   *(a) The third constraint is $x_1 + 2x_2 \geq 4$.
   *(b) The second constraint is $4x_1 + 3x_2 \leq 6$.
   (c) The second constraint is $4x_1 + 3x_2 = 6$.
   (d) The objective function is to maximize $z = 4x_1 + x_2$.
4. Consider the following set of constraints:

$$\begin{aligned} -2x_1 + 3x_2 &= 3 && (1)\\ 4x_1 + 5x_2 &\geq 10 && (2)\\ x_1 + 2x_2 &\leq 5 && (3)\\ 6x_1 + 7x_2 &\leq 3 && (4)\\ 4x_1 + 8x_2 &\geq 5 && (5)\\ x_1, x_2 &\geq 0 \end{aligned}$$

   For each of the following problems, develop the z-row after substituting out the artificial variables:
   (a) Maximize $z = 5x_1 + 6x_2$ subject to (1), (3), and (4).
   (b) Maximize $z = 2x_1 - 7x_2$ subject to (1), (2), (4), and (5).
   (c) Minimize $z = 3x_1 + 6x_2$ subject to (3), (4), and (5).
   (d) Minimize $z = 4x_1 + 6x_2$ subject to (1), (2), and (5).
   (e) Minimize $z = 3x_1 + 2x_2$ subject to (1) and (5).
5. Consider the following set of constraints:

$$\begin{aligned} x_1 + x_2 + x_3 &= 7\\ 2x_1 - 5x_2 + x_3 &\geq 10\\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

   Solve the problem for each of the following objective functions:
   (a) Maximize $z = 2x_1 + 3x_2 - 5x_3$.
   (b) Minimize $z = 2x_1 + 3x_2 - 5x_3$.
   (c) Maximize $z = x_1 + 2x_2 + x_3$.
   (d) Minimize $z = 4x_1 - 8x_2 + 3x_3$.

*6. Consider the problem

$$\text{Maximize } z = 2x_1 + 4x_2 + 4x_3 - 3x_4$$

   subject to

$$\begin{aligned} x_1 + x_2 + x_3 \qquad &= 4\\ x_1 + 4x_2 \qquad + x_4 &= 8\\ x_1, x_2, x_3, x_4, &\geq 0 \end{aligned}$$

   The problem shows that x_3 and x_4 can play the role of slacks for the two equations. They differ from slacks in that they have nonzero coefficients in the objective function. We can use x_3 and x_4 as starting variable, but, as in the case of artificial variables, they must be substituted out in the objective function before the simplex iterations are carried out. Solve the problem with x_3 and x_4 as the starting basic variables and without using any artificial variables.

7. Solve the following problem using x_3 and x_4 as starting basic feasible variables. As in Problem 6, do not use any artificial variables.

$$\text{Minimize } z = 3x_1 + 2x_2 + 3x_3$$

subject to

$$\begin{aligned} x_1 + 4x_2 + x_3 \quad\quad &\geq 7 \\ 2x_1 + x_2 \quad\quad + x_4 &\geq 10 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

8. Consider the problem

$$\text{Maximize } z = x_1 + 5x_2 + 3x_3$$

subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ 2x_1 - x_2 \quad\quad &= 4 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

The variable x_3 plays the role of a slack. Thus, no artificial variable is needed in the first constraint. However, in the second constraint, an artificial variable is needed. Use this starting solution (i.e., x_3 in the first constraint and R_2 in the second constraint) to solve this problem.

9. Show how the M-method will indicate that the following problem has no feasible solution.

$$\text{Maximize } z = 2x_1 + 5x_2$$

subject to

$$\begin{aligned} 3x_1 + 2x_2 &\geq 6 \\ 2x_1 + x_2 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

3.4.2 Two-Phase Method

In the M-method, the use of the penalty M, which by definition must be large relative to the actual objective coefficients of the model, can result in roundoff error that may impair the accuracy of the simplex calculations. The two-phase method alleviates this difficulty by eliminating the constant M altogether. As the name suggests, the method solves the LP in two phases: Phase I attempts to find a starting basic feasible solution, and, if one is found, Phase II is invoked to solve the original problem.

Summary of the Two-Phase Method

Phase I. Put the problem in equation form, and add the necessary artificial variables to the constraints (exactly as in the M-method) to secure a starting basic solution. Next, find a basic solution of the resulting equations that, regardless of whether the LP is maximization or minimization, *always* minimizes the sum of the artificial variables. If the minimum value of the sum is positive, the LP problem has no feasible solution, which ends the process (recall that a positive artificial variable signifies that an original constraint is not satisfied). Otherwise, proceed to Phase II.

Phase II. Use the feasible solution from Phase I as a starting basic feasible solution for the *original* problem.

Example 3.4-2

We use the same problem in Example 3.4-1.

Phase I

$$\text{Minimize } r = R_1 + R_2$$

subject to

$$\begin{aligned} 3x_1 + x_2 \quad + R_1 \quad &= 3 \\ 4x_1 + 3x_2 - x_3 \quad + R_2 \quad &= 6 \\ x_1 + 2x_2 \quad + x_4 &= 4 \\ x_1, x_2, x_3, x_4, R_1, R_2 &\geq 0 \end{aligned}$$

The associated tableau is given as

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
r	0	0	0	−1	−1	0	0
R_1	3	1	0	1	0	0	3
R_2	4	3	−1	0	1	0	6
x_4	1	2	0	0	0	1	4

As in the M-method, R_1 and R_2 are substituted out in the r-row by using the following computations:

$$\text{New } r\text{-row} = \text{Old } r\text{-row} + (1 \times R_1\text{-row} + 1 \times R_2\text{-row})$$

The new r-row is used to solve Phase I of the problem, which yields the following optimum tableau (verify with TORA's Iterations ⇒ Two-phase Method):

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
r	0	0	0	−1	−1	0	0
x_1	1	0	$\frac{1}{5}$	$\frac{3}{5}$	$-\frac{1}{5}$	0	$\frac{3}{5}$
x_2	0	1	$-\frac{3}{5}$	$-\frac{4}{5}$	$\frac{3}{5}$	0	$\frac{6}{5}$
x_4	0	0	1	1	−1	1	1

Because minimum $r = 0$, Phase I produces the basic feasible solution $x_1 = \frac{3}{5}$, $x_2 = \frac{6}{5}$, and $x_4 = 1$. At this point, the artificial variables have completed their mission, and we can eliminate their columns altogether from the tableau and move on to Phase II.

Phase II

After deleting the artificial columns, we write the *original* problem as

$$\text{Minimize } z = 4x_1 + x_2$$

subject to

$$\begin{aligned} x_1 \quad + \tfrac{1}{5}x_3 \qquad &= \tfrac{3}{5} \\ x_2 - \tfrac{3}{5}x_3 \qquad &= \tfrac{6}{5} \\ x_3 + x_4 &= 1 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

Essentially, Phase I is a procedure that transforms the original constraint equations in a manner that provides a starting basic feasible solution for the problem, if one exists. The tableau associated with Phase II problem is thus given as

Basic	x_1	x_2	x_3	x_4	Solution
z	-4	-1	0	0	0
x_1	1	0	$\frac{1}{5}$	0	$\frac{3}{5}$
x_2	0	1	$-\frac{3}{5}$	0	$\frac{6}{5}$
x_4	0	0	1	1	1

Again, because the basic variables x_1 and x_2 have nonzero coefficients in the z-row, they must be substituted out, using the following computations.

$$\text{New } z\text{-row} = \text{Old } z\text{-row} + (4 \times x_1\text{-row} + 1 \times x_2\text{-row})$$

The initial tableau of Phase II is thus given as

Basic	x_1	x_2	x_3	x_4	Solution
z	0	0	$\frac{1}{5}$	0	$\frac{18}{5}$
x_1	1	0	$\frac{1}{5}$	0	$\frac{3}{5}$
x_2	0	1	$-\frac{3}{5}$	0	$\frac{6}{5}$
x_4	0	0	1	1	1

Because we are minimizing, x_3 must enter the solution. Application of the simplex method will produce the optimum in one iteration (verify with TORA).

Remarks. Practically all commercial packages use the two-phase method to solve LP. The M-method with its potential adverse roundoff error is probably never used in practice. Its inclusion in this text is purely for historical reasons, because its development predates the development of the two-phase method.

The removal of the artificial variables and their columns at the end of Phase I can take place only when they are all *nonbasic* (as Example 3.4-2 illustrates). If one or more artificial variables are *basic* (at *zero* level) at the end of Phase I, then the following additional steps must be undertaken to remove them prior to the start of Phase II.

Step 1. Select a zero artificial variable to leave the basic solution and designate its row as the *pivot row*. The entering variable can be *any* nonbasic (nonartificial) variable with a *nonzero* (positive or negative) coefficient in the pivot row. Perform the associated simplex iteration.

Step 2. Remove the column of the (just-leaving) artificial variable from the tableau. If all the zero artificial variables have been removed, go to Phase II. Otherwise, go back to Step 1.

The logic behind Step 1 is that the feasibility of the remaining basic variables will not be affected when a zero artificial variable is made nonbasic regardless of whether the pivot element is positive or negative. Problems 5 and 6, Set 3.4b illustrate this situation. Problem 7 provides an additional detail about Phase I calculations.

PROBLEM SET 3.4B

***1.** In Phase I, if the LP is of the maximization type, explain why we do not maximize the sum of the artificial variables in Phase I.

2. For each case in Problem 4, Set 3.4a, write the corresponding Phase I objective function.

3. Solve Problem 5, Set 3.4a, by the two-phase method.

4. Write Phase I for the following problem, and then solve (with TORA for convenience) to show that the problem has no feasible solution.

$$\text{Maximize } z = 2x_1 + 5x_2$$

subject to

$$3x_1 + 2x_2 \geq 6$$
$$2x_1 + x_2 \leq 2$$
$$x_1, x_2 \geq 0$$

5. Consider the following problem:

$$\text{Maximize } z = 2x_1 + 2x_2 + 4x_3$$

subject to

$$2x_1 + x_2 + x_3 \leq 2$$
$$3x_1 + 4x_2 + 2x_3 \geq 8$$
$$x_1, x_2, x_3 \geq 0$$

(a) Show that Phase I will terminate with an artificial *basic* variable at zero level (you may use TORA for convenience).

(b) Remove the zero artificial variable prior to the start of Phase II, then carry out Phase II iterations.

6. Consider the following problem:

$$\text{Maximize } z = 3x_1 + 2x_2 + 3x_3$$

subject to

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 2 \\ x_1 + 3x_2 + x_3 &= 6 \\ 3x_1 + 4x_2 + 2x_3 &= 8 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

(a) Show that Phase I terminates with two zero artificial variables in the basic solution (use TORA for convenience).

(b) Show that when the procedure of Problem 5(b) is applied at the end of Phase I, only one of the two zero artificial variables can be made nonbasic.

(c) Show that the original constraint associated with the zero artificial variable that cannot be made nonbasic in (b) must be redundant—hence, its row and its column can be dropped altogether at the start of Phase II.

*7. Consider the following LP:

$$\text{Maximize } z = 3x_1 + 2x_2 + 3x_3$$

subject to

$$\begin{aligned} 2x_1 + x_2 + x_3 &\leq 2 \\ 3x_1 + 4x_2 + 2x_3 &\geq 8 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

The optimal simplex tableau at the end of Phase I is given as

Basic	x_1	x_2	x_3	x_4	x_5	R	Solution
z	−5	0	−2	−1	−4	0	0
x_2	2	1	1	0	1	0	2
R	−5	0	−2	−1	−4	1	0

Explain why the nonbasic variables x_1, x_3, x_4, and x_5 can never assume positive values at the end of Phase II. Hence, conclude that their columns can dropped before we start Phase II. In essence, the removal of these variables reduces the constraint equations of the problem to $x_2 = 2$. This means that it will not be necessary to carry out Phase II at all, because the solution space is reduced to one point only.

8. Consider the LP model

$$\text{Minimize } z = 2x_1 - 4x_2 + 3x_3$$

subject to

$$\begin{aligned} 5x_1 - 6x_2 + 2x_3 &\geq 5 \\ -x_1 + 3x_2 + 5x_3 &\geq 8 \\ 2x_1 + 5x_2 - 4x_3 &\leq 4 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Show how the inequalities can be modified to a set of equations that requires the use of a single artificial variable only (instead of two).