CHAPTER 2

Modeling with Linear Programming

Chapter Guide. This chapter concentrates on model formulation and computations in linear programming (LP). It starts with the modeling and graphical solution of a two-variable problem which, though highly simplified, provides a concrete understanding of the basic concepts of LP and lays the foundation for the development of the general *simplex* algorithm in Chapter 3. To illustrate the use of LP in the real world, applications are formulated and solved in the areas of urban planning, currency arbitrage, investment, production planning and inventory control, gasoline blending, manpower planning, and scheduling. On the computational side, two distinct types of software are used in this chapter. (1) TORA, a totally menu-driven and self-documenting tutorial program, is designed to help you understand the basics of LP through interactive feedback. (2) Spreadsheet-based Excel Solver and the AMPL modeling language are commercial packages designed for practical problems.

The material in Sections 2.1 and 2.2 is crucial for understanding later LP developments in the book. You will find TORA's interactive graphical module especially helpful in conjunction with Section 2.2. Section 2.3 presents diverse LP applications, each followed by targeted problems.

Section 2.4 introduces the commercial packages Excel Solver and AMPL. Models in Section 2.3 are solved with AMPL and Solver, and all the codes are included in folder ch2Files. Additional Solver and AMPL models are included opportunely in the succeeding chapters, and a detailed presentation of AMPL syntax is given in Appendix A. A good way to learn AMPL and Solver is to experiment with the numerous models presented throughout the book and to try to adapt them to the end-of-section problems. The AMPL codes are cross-referenced with the material in Appendix A to facilitate the learning process.

The TORA, Solver, and AMPL materials have been deliberately compartmentalized either in separate sections or under the subheadings *TORA/Solver/AMPL moment* to minimize disruptions in the main text. Nevertheless, you are encouraged to work end-of-section problems on the computer. The reason is that, at times, a model may look "correct" until you try to obtain a solution, and only then will you discover that the formulation needs modifications.

This chapter includes summaries of 2 real-life applications, 12 solved examples, 2 Solver models, 4 AMPL models, 94 end-of-section problems, and 4 cases. The cases are in Appendix E on the CD. The AMPL/Excel/Solver/TORA programs are in folder ch2Files.

Real-Life Application–Frontier Airlines Purchases Fuel Economically

The fueling of an aircraft can take place at any of the stopovers along the flight route. Fuel price varies among the stopovers, and potential savings can be realized by loading extra fuel (called *tankering*) at a cheaper location for use on subsequent flight legs. The disadvantage of tankering is the excess burn of gasoline resulting from the extra weight. LP (and heuristics) is used to determine the optimum amount of tankering that balances the cost of excess burn against the savings in fuel cost. The study, carried out in 1981, resulted in net savings of about $350,000 per year. Case 1 in Chapter 24 on the CD provides the details of the study. Interestingly, with the recent rise in the cost of fuel, many airlines are now using LP-based tankering software to purchase fuel.

2.1 TWO-VARIABLE LP MODEL

This section deals with the graphical solution of a two-variable LP. Though two-variable problems hardly exist in practice, the treatment provides concrete foundations for the development of the general simplex algorithm presented in Chapter 3.

Example 2.1-1 (The Reddy Mikks Company)

Reddy Mikks produces both interior and exterior paints from two raw materials, $M1$ and $M2$. The following table provides the basic data of the problem:

	Tons of raw material per ton of		Maximum daily availability (tons)
	Exterior paint	*Interior paint*	
Raw material, $M1$	6	4	24
Raw material, $M2$	1	2	6
Profit per ton ($1000)	5	4	

A market survey indicates that the daily demand for interior paint cannot exceed that for exterior paint by more than 1 ton. Also, the maximum daily demand for interior paint is 2 tons.

Reddy Mikks wants to determine the optimum (best) product mix of interior and exterior paints that maximizes the total daily profit.

The LP model, as in any OR model, has three basic components.

1. Decision **variables** that we seek to determine.
2. **Objective** (goal) that we need to optimize (maximize or minimize).
3. **Constraints** that the solution must satisfy.

The proper definition of the decision variables is an essential first step in the development of the model. Once done, the task of constructing the objective function and the constraints becomes more straightforward.

For the Reddy Mikks problem, we need to determine the daily amounts to be produced of exterior and interior paints. Thus the variables of the model are defined as

$$x_1 = \text{Tons produced daily of exterior paint}$$

$$x_2 = \text{Tons produced daily of interior paint}$$

To construct the objective function, note that the company wants to *maximize* (i.e., increase as much as possible) the total daily profit of both paints. Given that the profits per ton of exterior and interior paints are 5 and 4 (thousand) dollars, respectively, it follows that

$$\text{Total profit from exterior paint} = 5x_1 \text{ (thousand) dollars}$$

$$\text{Total profit from interior paint} = 4x_2 \text{ (thousand) dollars}$$

Letting z represent the total daily profit (in thousands of dollars), the objective of the company is

$$\text{Maximize } z = 5x_1 + 4x_2$$

Next, we construct the constraints that restrict raw material usage and product demand. The raw material restrictions are expressed verbally as

$$\begin{pmatrix}\text{Usage of a raw material}\\ \text{by both paints}\end{pmatrix} \leq \begin{pmatrix}\text{Maximum raw material}\\ \text{availability}\end{pmatrix}$$

The daily usage of raw material $M1$ is 6 tons per ton of exterior paint and 4 tons per ton of interior paint. Thus

$$\text{Usage of raw material } M1 \text{ by exterior paint} = 6x_1 \text{ tons/day}$$

$$\text{Usage of raw material } M1 \text{ by interior paint} = 4x_2 \text{ tons/day}$$

Hence

$$\text{Usage of raw material } M1 \text{ by both paints} = 6x_1 + 4x_2 \text{ tons/day}$$

In a similar manner,

$$\text{Usage of raw material } M2 \text{ by both paints} = 1x_1 + 2x_2 \text{ tons/day}$$

Because the daily availabilities of raw materials $M1$ and $M2$ are limited to 24 and 6 tons, respectively, the associated restrictions are given as

$$\begin{aligned} 6x_1 + 4x_2 &\leq 24 \quad \text{(Raw material } M1\text{)} \\ x_1 + 2x_2 &\leq 6 \quad \text{(Raw material } M2\text{)} \end{aligned}$$

The first demand restriction stipulates that the excess of the daily production of interior over exterior paint, $x_2 - x_1$, should not exceed 1 ton, which translates to

$$x_2 - x_1 \leq 1 \qquad \text{(Market limit)}$$

The second demand restriction stipulates that the maximum daily demand of interior paint is limited to 2 tons, which translates to

$$x_2 \leq 2 \text{ (Demand limit)}$$

An implicit (or "understood-to-be") restriction is that variables x_1 and x_2 cannot assume negative values. The **nonnegativity restrictions,** $x_1 \geq 0$, $x_2 \geq 0$, account for this requirement.

The complete Reddy Mikks model is

$$\text{Maximize } z = 5x_1 + 4x_2$$

subject to

$$6x_1 + 4x_2 \leq 24 \quad (1)$$

$$x_1 + 2x_2 \leq 6 \quad (2)$$

$$-x_1 + x_2 \leq 1 \quad (3)$$

$$x_2 \leq 2 \quad (4)$$

$$x_1, x_2 \geq 0 \quad (5)$$

Any values of x_1 and x_2 that satisfy *all* five constraints constitute a **feasible solution**. Otherwise, the solution is **infeasible**. For example, the solution, $x_1 = 3$ tons per day and $x_2 = 1$ ton per day, is feasible because it does not violate *any* of the constraints, including the nonnegativity restrictions. To verify this result, substitute $(x_1 = 3, x_2 = 1)$ in the left-hand side of each constraint. In constraint (1) we have $6x_1 + 4x_2 = 6 \times 3 + 4 \times 1 = 22$, which is less than the right-hand side of the constraint (= 24). Constraints 2 through 5 will yield similar conclusions (verify!). On the other hand, the solution $x_1 = 4$ and $x_2 = 1$ is infeasible because it does not satisfy constraint (1)—namely, $6 \times 4 + 4 \times 1 = 28$, which is larger than the right-hand side (= 24).

The goal of the problem is to find the best *feasible* solution, or the **optimum**, that maximizes the total profit. Before we can do that, we need to know how many *feasible* solutions the Reddy Mikks problem has. The answer, as we will see from the graphical solution in Section 2.2, is "an infinite number," which makes it impossible to solve the problem by enumeration. Instead, we need a systematic procedure that will locate the optimum solution in a finite number of steps. The graphical method in Section 2.2 and its algebraic generalization in Chapter 3 will explain how this can be accomplished.

Properties of the LP Model. In Example 2.1-1, the objective and the constraints are all linear functions. **Linearity** implies that the LP must satisfy three basic properties:

1. **Proportionality:** This property requires the contribution of each decision variable in both the objective function and the constraints to be *directly proportional* to the value of the variable. For example, in the Reddy Mikks model, the quantities $5x_1$ and $4x_2$ give the profits for producing x_1 and x_2 tons of exterior and interior paint, respectively, with the unit profits per ton, 5 and 4, providing the constants of proportionality. If, on the other hand, Reddy Mikks grants some sort of quantity discounts when sales exceed certain amounts, then the profit will no longer be proportional to the production amounts, x_1 and x_2, and the profit function becomes nonlinear.

2. **Additivity:** This property requires the total contribution of all the variables in the objective function and in the constraints to be the direct sum of the individual contributions of each variable. In the Reddy Mikks model, the total profit equals the

sum of the two individual profit components. If, however, the two products *compete* for market share in such a way that an increase in sales of one adversely affects the other, then the additivity property is not satisfied and the model is no longer linear.

3. Certainty: All the objective and constraint coefficients of the LP model are deterministic. This means that they are known constants—a rare occurrence in real life, where data are more likely to be represented by probabilistic distributions. In essence, LP coefficients are average-value approximations of the probabilistic distributions. If the standard deviations of these distributions are sufficiently small, then the approximation is acceptable. Large standard deviations can be accounted for directly by using stochastic LP algorithms (Section 19.2.3) or indirectly by applying sensitivity analysis to the optimum solution (Section 3.6).

PROBLEM SET 2.1A

1. For the Reddy Mikks model, construct each of the following constraints and express it with a linear left-hand side and a constant right-hand side:

***(a)** The daily demand for interior paint exceeds that of exterior paint by *at least* 1 ton.

(b) The daily usage of raw material *M2* in tons is *at most* 6 and *at least* 3.

***(c)** The demand for interior paint cannot be less than the demand for exterior paint.

(d) The minimum quantity that should be produced of both the interior and the exterior paint is 3 tons.

***(e)** The proportion of interior paint to the total production of both interior and exterior paints must not exceed .5.

2. Determine the best *feasible* solution among the following (feasible and infeasible) solutions of the Reddy Mikks model:

(a) $x_1 = 1, x_2 = 4$.

(b) $x_1 = 2, x_2 = 2$.

(c) $x_1 = 3, x_2 = 1.5$.

(d) $x_1 = 2, x_2 = 1$.

(e) $x_1 = 2, x_2 = -1$.

***3.** For the feasible solution $x_1 = 2, x_2 = 2$ of the Reddy Mikks model, determine the unused amounts of raw materials *M1* and *M2*.

4. Suppose that Reddy Mikks sells its exterior paint to a single wholesaler at a quantity discount. The profit per ton is $5000 if the contractor buys no more than 2 tons daily and $4500 otherwise. Express the objective function mathematically. Is the resulting function linear?

2.2 GRAPHICAL LP SOLUTION

The graphical procedure includes two steps:

1. Determination of the feasible solution space.
2. Determination of the optimum solution from among all the feasible points in the solution space.

The procedure uses two examples to show how maximization and minimization objective functions are handled.

2.2.1 Solution of a Maximization Model

Example 2.2-1

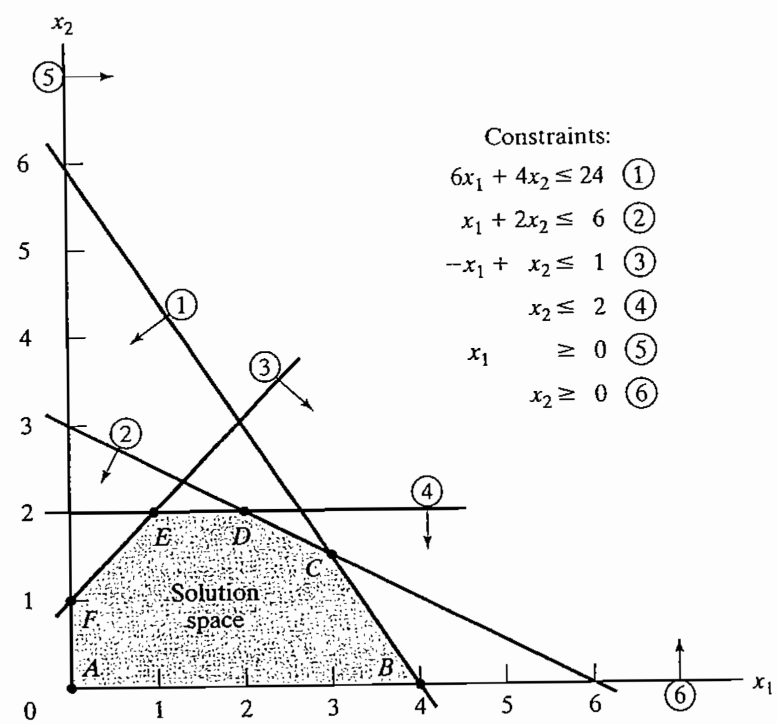
This example solves the Reddy Mikks model of Example 2.1-1.

Step 1. *Determination of the Feasible Solution Space:*
First, we account for the nonnegativity constraints $x_1 \geq 0$ and $x_2 \geq 0$. In Figure 2.1, the horizontal axis x_1 and the vertical axis x_2 represent the exterior- and interior-paint variables, respectively. Thus, the nonnegativity of the variables restricts the solution-space area to the first quadrant that lies above the x_1-axis and to the right of the x_2-axis.

To account for the remaining four constraints, first replace each inequality with an equation and then graph the resulting straight line by locating two distinct points on it. For example, after replacing $6x_1 + 4x_2 \leq 24$ with the straight line $6x_1 + 4x_2 = 24$, we can determine two distinct points by first setting $x_1 = 0$ to obtain $x_2 = \frac{24}{4} = 6$ and then setting $x_2 = 0$ to obtain $x_1 = \frac{24}{6} = 4$. Thus, the line passes through the two points (0, 6) and (4, 0), as shown by line (1) in Figure 2.1.

Next, consider the effect of the inequality. All it does is divide the (x_1, x_2)-plane into two half-spaces, one on each side of the graphed line. Only one of these two halves satisfies the inequality. To determine the correct side, choose (0, 0) as a *reference point*. If it satisfies the inequality, then the side in which it lies is the

FIGURE 2.1
Feasible space of the Reddy Mikks model



feasible half-space, otherwise the other side is. The use of the reference point $(0, 0)$ is illustrated with the constraint $6x_1 + 4x_2 \leq 24$. Because $6 \times 0 + 4 \times 0 = 0$ is less than 24, the half-space representing the inequality includes the origin (as shown by the arrow in Figure 2.1).

It is convenient computationally to select $(0, 0)$ as the reference point, unless the line happens to pass through the origin, in which case any other point can be used. For example, if we use the reference point $(6, 0)$, the left-hand side of the first constraint is $6 \times 6 + 4 \times 0 = 36$, which is larger than its right-hand side $(= 24)$, which means that the side in which $(6, 0)$ lies is not feasible for the inequality $6x_1 + 4x_2 \leq 24$. The conclusion is consistent with the one based on the reference point $(0, 0)$.

Application of the reference-point procedure to all the constraints of the model produces the constraints shown in Figure 2.1 (verify!). The **feasible solution space** of the problem represents the area in the first quadrant in which all the constraints are satisfied simultaneously. In Figure 2.1, any point in or on the boundary of the area *ABCDEF* is part of the feasible solution space. All points outside this area are infeasible.

TORA Moment.

The menu-driven TORA graphical LP module should prove helpful in reinforcing your understanding of how the LP constraints are graphed. Select Linear Programming from the MAIN menu. After inputting the model, select Solve ⇒ Graphical from the SOLVE/MODIFY menu. In the output screen, you will be able to experiment interactively with graphing the constraints one at a time, so you can see how each constraint affects the solution space.

Step 2. *Determination of the Optimum Solution:*

The feasible space in Figure 2.1 is delineated by the line segments joining the points *A*, *B*, *C*, *D*, *E*, and *F*. Any point within or on the boundary of the space *ABCDEF* is feasible. Because the feasible space *ABCDEF* consists of an *infinite* number of points, we need a systematic procedure to identify the optimum solution.

The determination of the optimum solution requires identifying the direction in which the profit function $z = 5x_1 + 4x_2$ increases (recall that we are *maximizing* z). We can do so by assigning *arbitrary* increasing values to z. For example, using $z = 10$ and $z = 15$ would be equivalent to graphing the two lines $5x_1 + 4x_2 = 10$ and $5x_1 + 4x_2 = 15$. Thus, the direction of increase in z is as shown Figure 2.2. The optimum solution occurs at C, which is the point in the solution space beyond which any further increase will put z outside the boundaries of *ABCDEF*.

The values of x_1 and x_2 associated with the optimum point C are determined by solving the equations associated with lines (1) and (2)—that is,

$$6x_1 + 4x_2 = 24$$
$$x_1 + 2x_2 = 6$$

The solution is $x_1 = 3$ and $x_2 = 1.5$ with $z = 5 \times 3 + 4 \times 1.5 = 21$. This calls for a daily product mix of 3 tons of exterior paint and 1.5 tons of interior paint. The associated daily profit is \$21,000.

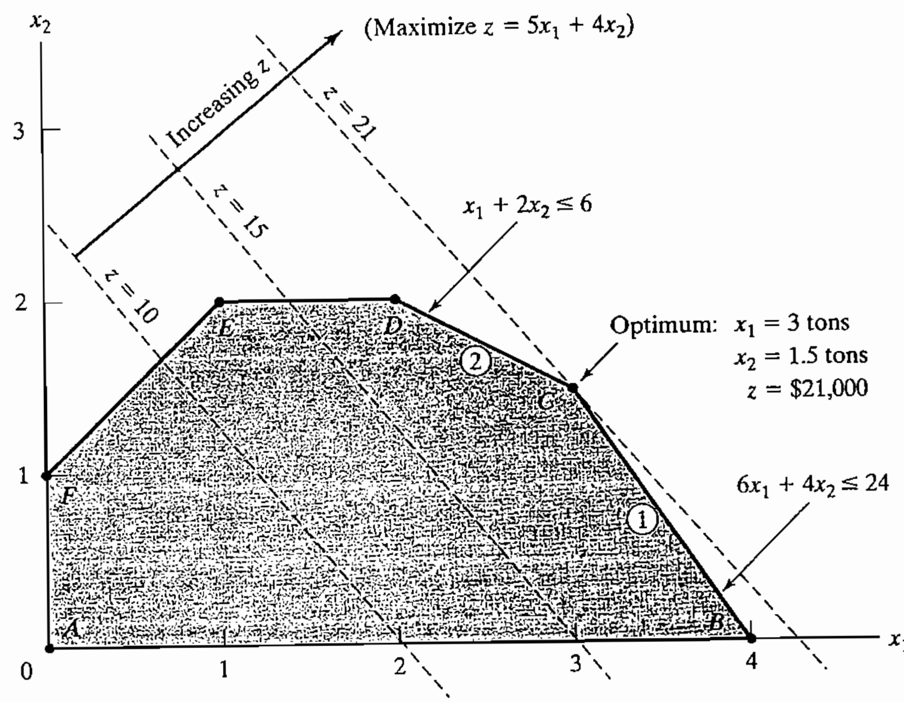


FIGURE 2.2
Optimum solution of the Reddy Mikks model

An important characteristic of the optimum LP solution is that it is *always* associated with a **corner point** of the solution space (where two lines intersect). This is true even if the objective function happens to be parallel to a constraint. For example, if the objective function is $z = 6x_1 + 4x_2$, which is parallel to constraint 1, we can always say that the optimum occurs at either corner point B or corner point C. Actually any point on the line segment BC will be an *alternative* optimum (see also Example 3.5-2), but the important observation here is that the line segment BC is totally defined by the *corner points* B and C.

TORA Moment.

You can use TORA interactively to see that the optimum is always associated with a corner point. From the output screen, you can click View/Modify Input Data to modify the objective coefficients and re-solve the problem graphically. You may use the following objective functions to test the proposed idea:

(a) $z = 5x_1 + x_2$
(b) $z = 5x_1 + 4x_2$
(c) $z = x_1 + 3x_2$
(d) $z = -x_1 + 2x_2$
(e) $z = -2x_1 + x_2$
(f) $z = -x_1 - x_2$

The observation that the LP optimum is always associated with a corner point means that the optimum solution can be found simply by enumerating all the corner points as the following table shows:

Corner point	(x_1, x_2)	z
A	(0, 0)	0
B	(4, 0)	20
C	**(3, 1.5)**	**21 (OPTIMUM)**
D	(2, 2)	18
E	(1, 2)	13
F	(0, 1)	4

As the number of constraints and variables increases, the number of corner points also increases, and the proposed enumeration procedure becomes less tractable computationally. Nevertheless, the idea shows that, from the standpoint of determining the LP optimum, the solution space $ABCDEF$ with its *infinite* number of solutions can, in fact, be replaced with a *finite* number of promising solution points—namely, the corner points, A, B, C, D, E, and F. This result is key for the development of the general algebraic algorithm, called the *simplex method*, which we will study in Chapter 3.

PROBLEM SET 2.2A

1. Determine the feasible space for each of the following independent constraints, given that $x_1, x_2 \geq 0$.
   *(a) $-3x_1 + x_2 \leq 6$.
   (b) $x_1 - 2x_2 \geq 5$.
   (c) $2x_1 - 3x_2 \leq 12$.
   *(d) $x_1 - x_2 \leq 0$.
   (e) $-x_1 + x_2 \geq 0$.
2. Identify the direction of increase in z in each of the following cases:
   *(a) Maximize $z = x_1 - x_2$.
   (b) Maximize $z = -5x_1 - 6x_2$.
   (c) Maximize $z = -x_1 + 2x_2$.
   *(d) Maximize $z = -3x_1 + x_2$.
3. Determine the solution space and the optimum solution of the Reddy Mikks model for each of the following independent changes:
   (a) The maximum daily demand for exterior paint is at most 2.5 tons.
   (b) The daily demand for interior paint is at least 2 tons.
   (c) The daily demand for interior paint is exactly 1 ton higher than that for exterior paint.
   (d) The daily availability of raw material $M1$ is at least 24 tons.
   (e) The daily availability of raw material $M1$ is at least 24 tons, and the daily demand for interior paint exceeds that for exterior paint by at least 1 ton.

4. A company that operates 10 hours a day manufactures two products on three sequential processes. The following table summarizes the data of the problem:

	Minutes per unit			
Product	*Process 1*	*Process 2*	*Process 3*	Unit profit
1	10	6	8	\$2
2	5	20	10	\$3

Determine the optimal mix of the two products.

***5.** A company produces two products, A and B. The sales volume for A is at least 80% of the total sales of both A and B. However, the company cannot sell more than 100 units of A per day. Both products use one raw material, of which the maximum daily availability is 240 lb. The usage rates of the raw material are 2 lb per unit of A and 4 lb per unit of B. The profit units for A and B are \$20 and \$50, respectively. Determine the optimal product mix for the company.

6. Alumco manufactures aluminum sheets and aluminum bars. The maximum production capacity is estimated at either 800 sheets or 600 bars per day. The maximum daily demand is 550 sheets and 580 bars. The profit per ton is \$40 per sheet and \$35 per bar. Determine the optimal daily production mix.

***7.** An individual wishes to invest \$5000 over the next year in two types of investment: Investment A yields 5% and investment B yields 8%. Market research recommends an allocation of at least 25% in A and at most 50% in B. Moreover, investment in A should be at least half the investment in B. How should the fund be allocated to the two investments?

8. The Continuing Education Division at the Ozark Community College offers a total of 30 courses each semester. The courses offered are usually of two types: practical, such as woodworking, word processing, and car maintenance; and humanistic, such as history, music, and fine arts. To satisfy the demands of the community, at least 10 courses of each type must be offered each semester. The division estimates that the revenues of offering practical and humanistic courses are approximately \$1500 and \$1000 per course, respectively.

(a) Devise an optimal course offering for the college.

(b) Show that the worth per additional course is \$1500, which is the same as the revenue per practical course. What does this result mean in terms of offering additional courses?

9. ChemLabs uses raw materials I and II to produce two domestic cleaning solutions, A and B. The daily availabilities of raw materials I and II are 150 and 145 units, respectively. One unit of solution A consumes .5 unit of raw material I and .6 unit of raw material II, and one unit of solution B uses .5 unit of raw material I and .4 unit of raw material II. The profits per unit of solutions A and B are \$8 and \$10, respectively. The daily demand for solution A lies between 30 and 150 units, and that for solution B between 40 and 200 units. Find the optimal production amounts of A and B.

10. In the Ma-and-Pa grocery store, shelf space is limited and must be used effectively to increase profit. Two cereal items, Grano and Wheatie, compete for a total shelf space of 60 ft^2. A box of Grano occupies .2 ft^2 and a box of Wheatie needs .4 ft^2. The maximum daily demands of Grano and Wheatie are 200 and 120 boxes, respectively. A box of Grano nets \$1.00 in profit and a box of Wheatie \$1.35. Ma-and-Pa thinks that because the unit profit of Wheatie is 35% higher than that of Grano, Wheatie should be allocated

35% more space than Grano, which amounts to allocating about 57% to Wheatie and 43% to Grano. What do you think?

11. Jack is an aspiring freshman at Ulern University. He realizes that "all work and no play make Jack a dull boy." As a result, Jack wants to apportion his available time of about 10 hours a day between work and play. He estimates that play is twice as much fun as work. He also wants to study at least as much as he plays. However, Jack realizes that if he is going to get all his homework assignments done, he cannot play more than 4 hours a day. How should Jack allocate his time to maximize his pleasure from both work and play?

12. Wild West produces two types of cowboy hats. A type 1 hat requires twice as much labor time as a type 2. If the all available labor time is dedicated to Type 2 alone, the company can produce a total of 400 Type 2 hats a day. The respective market limits for the two types are 150 and 200 hats per day. The profit is $8 per Type 1 hat and $5 per Type 2 hat. Determine the number of hats of each type that would maximize profit.

13. Show & Sell can advertise its products on local radio and television (TV). The advertising budget is limited to $10,000 a month. Each minute of radio advertising costs $15 and each minute of TV commercials $300. Show & Sell likes to advertise on radio at least twice as much as on TV. In the meantime, it is not practical to use more than 400 minutes of radio advertising a month. From past experience, advertising on TV is estimated to be 25 times as effective as on radio. Determine the optimum allocation of the budget to radio and TV advertising.

***14.** Wyoming Electric Coop owns a steam-turbine power-generating plant. Because Wyoming is rich in coal deposits, the plant generates its steam from coal. This, however, may result in emission that does not meet the Environmental Protection Agency standards. EPA regulations limit sulfur dioxide discharge to 2000 parts per million per ton of coal burned and smoke discharge from the plant stacks to 20 lb per hour. The Coop receives two grades of pulverized coal, *C*1 and *C*2, for use in the steam plant. The two grades are usually mixed together before burning. For simplicity, it can be assumed that the amount of sulfur pollutant discharged (in parts per million) is a weighted average of the proportion of each grade used in the mixture. The following data are based on consumption of 1 ton per hour of each of the two coal grades.

Coal grade	Sulfur discharge in parts per million	Smoke discharge in lb per hour	Steam generated in lb per hour
*C*1	1800	2.1	12,000
*C*2	2100	.9	9,000

(a) Determine the optimal ratio for mixing the two coal grades.

(b) Determine the effect of relaxing the smoke discharge limit by 1 lb on the amount of generated steam per hour.

15. Top Toys is planning a new radio and TV advertising campaign. A radio commercial costs $300 and a TV ad costs $2000. A total budget of $20,000 is allocated to the campaign. However, to ensure that each medium will have at least one radio commercial and one TV ad, the most that can be allocated to either medium cannot exceed 80% of the total budget. It is estimated that the first radio commercial will reach 5000 people, with each additional commercial reaching only 2000 new ones. For TV, the first ad will reach 4500 people and each additional ad an additional 3000. How should the budgeted amount be allocated between radio and TV?

16. The Burroughs Garment Company manufactures men's shirts and women's blouses for Walmark Discount Stores. Walmark will accept all the production supplied by Burroughs. The production process includes cutting, sewing, and packaging. Burroughs employs 25 workers in the cutting department, 35 in the sewing department, and 5 in the packaging department. The factory works one 8-hour shift, 5 days a week. The following table gives the time requirements and profits per unit for the two garments:

	Minutes per unit			
Garment	*Cutting*	*Sewing*	*Packaging*	Unit profit ($)
Shirts	20	70	12	8
Blouses	60	60	4	12

Determine the optimal weekly production schedule for Burroughs.

17. A furniture company manufactures desks and chairs. The sawing department cuts the lumber for both products, which is then sent to separate assembly departments. Assembled items are sent for finishing to the painting department. The daily capacity of the sawing department is 200 chairs or 80 desks. The chair assembly department can produce 120 chairs daily and the desk assembly department 60 desks daily. The paint department has a daily capacity of either 150 chairs or 110 desks. Given that the profit per chair is $50 and that of a desk is $100, determine the optimal production mix for the company.

***18.** An assembly line consisting of three consecutive stations produces two radio models: HiFi-1 and HiFi-2. The following table provides the assembly times for the three workstations.

	Minutes per unit	
Workstation	*HiFi-1*	*HiFi-2*
1	6	4
2	5	5
3	4	6

The daily maintenance for stations 1, 2, and 3 consumes 10%, 14%, and 12%, respectively, of the maximum 480 minutes available for each station each day. Determine the optimal product mix that will minimize the idle (or unused) times in the three workstations.

19. *TORA Experiment.* Enter the following LP into TORA and select the graphic solution mode to reveal the LP graphic screen.

$$\text{Minimize } z = 3x_1 + 8x_2$$

subject to

$$\begin{aligned} x_1 + x_2 &\geq 8 \\ 2x_1 - 3x_2 &\leq 0 \\ x_1 + 2x_2 &\leq 30 \\ 3x_1 - x_2 &\geq 0 \\ x_1 &\leq 10 \\ x_2 &\geq 9 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Next, on a sheet of paper, graph and scale the x_1- and x_2-axes for the problem (you may also click Print Graph on the top of the right window to obtain a ready-to-use scaled sheet). Now, graph a constraint manually on the prepared sheet, then click it on the left window of the screen to check your answer. Repeat the same for each constraint and then terminate the procedure with a graph of the objective function. The suggested process is designed to test and reinforce your understanding of the graphical LP solution through immediate feedback from TORA.

20. *TORA Experiment.* Consider the following LP model:

$$\text{Maximize } z = 5x_1 + 4x_2$$

subject to

$$\begin{aligned} 6x_1 + 4x_2 &\leq 24 \\ 6x_1 + 3x_2 &\leq 22.5 \\ x_1 + x_2 &\leq 5 \\ x_1 + 2x_2 &\leq 6 \\ -x_1 + x_2 &\leq 1 \\ x_2 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

In LP, a constraint is said to be *redundant* if its removal from the model leaves the feasible solution space unchanged. Use the graphical facility of TORA to identify the redundant constraints, then show that their removal (simply by not graphing them) does not affect the solution space or the optimal solution.

21. *TORA Experiment.* In the Reddy Mikks model, use TORA to show that the removal of the raw material constraints (constraints 1 and 2) would result in an *unbounded solution space.* What can be said in this case about the optimal solution of the model?

22. *TORA Experiment.* In the Reddy Mikks model, suppose that the following constraint is added to the problem.

$$x_2 \geq 3$$

Use TORA to show that the resulting model has conflicting constraints that cannot be satisfied simultaneously and hence it has *no feasible solution.*

2.2.2 Solution of a Minimization Model

Example 2.2-2 (Diet Problem)

Ozark Farms uses at least 800 lb of special feed daily. The special feed is a mixture of corn and soybean meal with the following compositions:

	lb per lb of feedstuff		
Feedstuff	*Protein*	*Fiber*	Cost ($/lb)
Corn	.09	.02	.30
Soybean meal	.60	.06	.90

The dietary requirements of the special feed are at least 30% protein and at most 5% fiber. Ozark Farms wishes to determine the daily minimum-cost feed mix.

Because the feed mix consists of corn and soybean meal, the decision variables of the model are defined as

$$x_1 = \text{lb of corn in the daily mix}$$
$$x_2 = \text{lb of soybean meal in the daily mix}$$

The objective function seeks to minimize the total daily cost (in dollars) of the feed mix and is thus expressed as

$$\text{Minimize } z = .3x_1 + .9x_2$$

The constraints of the model reflect the daily amount needed and the dietary requirements. Because Ozark Farms needs at least 800 lb of feed a day, the associated constraint can be expressed as

$$x_1 + x_2 \geq 800$$

As for the protein dietary requirement constraint, the amount of protein included in x_1 lb of corn and x_2 lb of soybean meal is $(.09x_1 + .6x_2)$ lb. This quantity should equal at least 30% of the total feed mix $(x_1 + x_2)$ lb—that is,

$$.09x_1 + .6x_2 \geq .3(x_1 + x_2)$$

In a similar manner, the fiber requirement of at most 5% is constructed as

$$.02x_1 + .06x_2 \leq .05(x_1 + x_2)$$

The constraints are simplified by moving the terms in x_1 and x_2 to the left-hand side of each inequality, leaving only a constant on the right-hand side. The complete model thus becomes

$$\text{minimize } z = .3x_1 + .9x_2$$

subject to

$$\begin{aligned} x_1 + \quad x_2 &\geq 800 \\ .21x_1 - .30x_2 &\leq 0 \\ .03x_1 - .01x_2 &\geq 0 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Figure 2.3 provides the graphical solution of the model. Unlike those of the Reddy Mikks model (Example 2.2-1), the second and third constraints pass through the origin. To plot the associated straight lines, we need one additional point, which can be obtained by assigning a value to one of the variables and then solving for the other variable. For example, in the second constraint, $x_1 = 200$ will yield $.21 \times 200 - .3x_2 = 0$, or $x_2 = 140$. This means that the straight line $.21x_1 - .3x_2 = 0$ passes through (0, 0) and (200, 140). Note also that (0, 0) cannot be used as a reference point for constraints 2 and 3, because both lines pass through the origin. Instead, any other point [e.g., (100, 0) or (0, 100)] can be used for that purpose.

Solution:

Because the present model seeks the minimization of the objective function, we need to reduce the value of z as much as possible in the direction shown in Figure 2.3. The optimum solution is the intersection of the two lines $x_1 + x_2 = 800$ and $.21x_1 - .3x_2 = 0$, which yields $x_1 = 470.59$ lb and $x_2 = 329.41$ lb. The associated minimum cost of the feed mix is $z = .3 \times 470.59 + .9 \times 329.42 =$ \$437.65 per day.

Remarks. We need to take note of the way the constraints of the problem are constructed. Because the model is minimizing the total cost, one may argue that the solution will seek exactly 800 tons of feed. Indeed, this is what the optimum solution given above does. Does this mean then that the first constraint can be deleted altogether simply by including the amount 800 tons

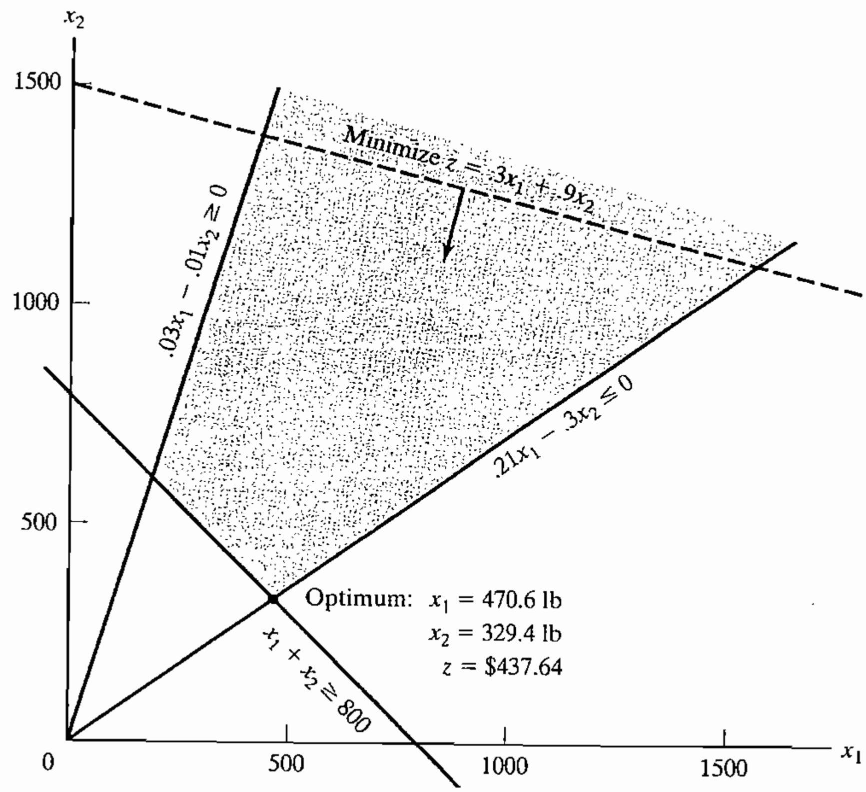


FIGURE 2.3
Graphical solution of the diet model

in the remaining constraints? To find the answer, we state the new protein and fiber constraints as

$$.09x_1 + .6x_2 \geq .3 \times 800$$
$$.02x_1 + .06x_2 \leq .05 \times 800$$

or

$$.09x_1 + .6x_2 \geq 240$$
$$.02x_1 + .06x_2 \leq 40$$

The new formulation yields the solution $x_1 = 0$, and $x_2 = 400$ lb (verify with TORA!), which does not satisfy the *implied* requirement for 800 lb of feed. This means that the constraint $x_1 + x_2 \geq 800$ must be used explicitly and that the protein and fiber constraints must remain exactly as given originally.

Along the same line of reasoning, one may be tempted to replace $x_1 + x_2 \geq 800$ with $x_1 + x_2 = 800$. In the present example, the two constraints yield the same answer. But in general this may not be the case. For example, suppose that the daily mix must include at least 500 lb of corn. In this case, the optimum solution will call for using 500 lb of corn and 350 lb of soybean (verify with TORA!), which is equivalent to a daily feed mix of 500 + 350 = 850 lb. Imposing the equality constraint a priori will lead to the conclusion that the problem has no feasible solution (verify with TORA!). On the other hand, the use of the inequality is inclusive of the equality case, and hence its use does not prevent the model from producing exactly 800 lb of feed mix, should the remaining constraints allow it. The conclusion is that we should not "pre-guess" the solution by imposing the additional equality restriction, and we should always use inequalities unless the situation explicitly stipulates the use of equalities.

PROBLEM SET 2.2B

1. Identify the direction of decrease in z in each of the following cases:
   - *(a) Minimize $z = 4x_1 - 2x_2$.
   - (b) Minimize $z = -3x_1 + x_2$.
   - (c) Minimize $z = -x_1 - 2x_2$.
2. For the diet model, suppose that the daily availability of corn is limited to 450 lb. Identify the new solution space, and determine the new optimum solution.
3. For the diet model, what type of optimum solution would the model yield if the feed mix should not exceed 800 lb a day? Does the solution make sense?
4. John must work at least 20 hours a week to supplement his income while attending school. He has the opportunity to work in two retail stores. In store 1, he can work between 5 and 12 hours a week, and in store 2 he is allowed between 6 and 10 hours. Both stores pay the same hourly wage. In deciding how many hours to work in each store, John wants to base his decision on work stress. Based on interviews with present employees, John estimates that, on an ascending scale of 1 to 10, the stress factors are 8 and 6 at stores 1 and 2, respectively. Because stress mounts by the hour, he assumes that the total stress for each store at the end of the week is proportional to the number of hours he works in the store. How many hours should John work in each store?

*5. OilCo is building a refinery to produce four products: diesel, gasoline, lubricants, and jet fuel. The minimum demand (in bbl/day) for each of these products is 14,000, 30,000, 10,000, and 8,000, respectively. Iran and Dubai are under contract to ship crude to OilCo. Because of the production quotas specified by OPEC (Organization of Petroleum Exporting Countries) the new refinery can receive at least 40% of its crude from Iran and the remaining amount from Dubai. OilCo predicts that the demand and crude oil quotas will remain steady over the next ten years.

   The specifications of the two crude oils lead to different product mixes: One barrel of Iran crude yields .2 bbl of diesel, .25 bbl of gasoline, .1 bbl of lubricant, and .15 bbl of jet fuel. The corresponding yields from Dubai crude are .1, .6, .15, and .1, respectively. OilCo needs to determine the minimum capacity of the refinery (in bbl/ day).

6. Day Trader wants to invest a sum of money that would generate an annual yield of at least $10,000. Two stock groups are available: blue chips and high tech, with average annual yields of 10% and 25%, respectively. Though high-tech stocks provide higher yield, they are more risky, and Trader wants to limit the amount invested in these stocks to no more than 60% of the total investment. What is the minimum amount Trader should invest in each stock group to accomplish the investment goal?

*7. An industrial recycling center uses two scrap aluminum metals, A and B, to produce a special alloy. Scrap A contains 6% aluminum, 3% silicon, and 4% carbon. Scrap B has 3% aluminum, 6% silicon, and 3% carbon. The costs per ton for scraps A and B are $100 and $80, respectively. The specifications of the special alloy require that (1) the aluminum content must be at least 3% and at most 6%, (2) the silicon content must lie between 3%

and 5%, and (3) the carbon content must be between 3% and 7%. Determine the optimum mix of the scraps that should be used in producing 1000 tons of the alloy.

8. *TORA Experiment.* Consider the Diet Model and let the objective function be given as

$$\text{Minimize } z = .8x_1 + .8x_2$$

Use TORA to show that the optimum solution is associated with *two* distinct corner points and that both points yield the same objective value. In this case, the problem is said to have *alternative optima.* Explain the conditions leading to this situation and show that, in effect, the problem has an infinite number of alternative optima, then provide a formula for determining all such solutions.

2.3 SELECTED LP APPLICATIONS

This section presents realistic LP models in which the definition of the variables and the construction of the objective function and constraints are not as straightforward as in the case of the two-variable model. The areas covered by these applications include the following:

1. Urban planning.
2. Currency arbitrage.
3. Investment.
4. Production planning and inventory control.
5. Blending and oil refining.
6. Manpower planning.

Each model is fully developed and its optimum solution is analyzed and interpreted.

2.3.1 Urban Planning[1]

Urban planning deals with three general areas: (1) building new housing developments, (2) upgrading inner-city deteriorating housing and recreational areas, and (3) planning public facilities (such as schools and airports). The constraints associated with these projects are both economic (land, construction, financing) and social (schools, parks, income level). The objectives in urban planning vary. In new housing developments, profit is usually the motive for undertaking the project. In the remaining two categories, the goals involve social, political, economic, and cultural considerations. Indeed, in a publicized case in 2004, the mayor of a city in Ohio wanted to condemn an old area of the city to make way for a luxury housing development. The motive was to increase tax collection to help alleviate budget shortages. The example presented in this section is fashioned after the Ohio case.

[1]This section is based on Laidlaw (1972).

Example 2.3-1 (Urban Renewal Model)

The city of Erstville is faced with a severe budget shortage. Seeking a long-term solution, the city council votes to improve the tax base by condemning an inner-city housing area and replacing it with a modern development.

The project involves two phases: (1) demolishing substandard houses to provide land for the new development, and (2) building the new development. The following is a summary of the situation.

1. As many as 300 substandard houses can be demolished. Each house occupies a .25-acre lot. The cost of demolishing a condemned house is $2000.
2. Lot sizes for new single-, double-, triple-, and quadruple-family homes (units) are .18, .28, .4, and .5 acre, respectively. Streets, open space, and utility easements account for 15% of available acreage.
3. In the new development the triple and quadruple units account for at least 25% of the total. Single units must be at least 20% of all units and double units at least 10%.
4. The tax levied per unit for single, double, triple, and quadruple units is $1,000, $1,900, $2,700, and $3,400, respectively.
5. The construction cost per unit for single-, double-, triple-, and quadruple- family homes is $50,000, $70,000, $130,000, and $160,000, respectively. Financing through a local bank can amount to a maximum of $15 million.

How many units of each type should be constructed to maximize tax collection?

Mathematical Model: Besides determining the number of units to be constructed of each type of housing, we also need to decide how many houses must be demolished to make room for the new development. Thus, the variables of the problem can be defined as follows:

x_1 = Number of units of single-family homes

x_2 = Number of units of double-family homes

x_3 = Number of units of triple-family homes

x_4 = Number of units of quadruple-family homes

x_5 = Number of old homes to be demolished

The objective is to maximize total tax collection from all four types of homes—that is,

$$\text{Maximize } z = 1000x_1 + 1900x_2 + 2700x_3 + 3400x_4$$

The first constraint of the problem deals with land availability.

$$\begin{pmatrix}\text{Acreage used for new}\\ \text{home construction}\end{pmatrix} \leq \begin{pmatrix}\text{Net available}\\ \text{acreage}\end{pmatrix}$$

From the data of the problem we have

$$\text{Acreage needed for new homes} = .18x_1 + .28x_2 + .4x_3 + .5x_4$$

To determine the available acreage, each demolished home occupies a .25-acre lot, thus netting $.25x_5$ acres. Allowing for 15% open space, streets, and easements, the net acreage available is $.85(.25x_5) = .2125x_5$. The resulting constraint is

$$.18x_1 + .28x_2 + .4x_3 + .5x_4 \leq .2125x_5$$

or

$$.18x_1 + .28x_2 + .4x_3 + .5x_4 - .2125x_5 \leq 0$$

The number of demolished homes cannot exceed 300, which translates to

$$x_5 \leq 300$$

Next we add the constraints limiting the number of units of each home type.

$$(\text{Number of single units}) \geq (20\% \text{ of all units})$$

$$(\text{Number of double units}) \geq (10\% \text{ of all units})$$

$$(\text{Number of triple and quadruple units}) \geq (25\% \text{ of all units})$$

These constraints translate mathematically to

$$x_1 \geq .2(x_1 + x_2 + x_3 + x_4)$$

$$x_2 \geq .1(x_1 + x_2 + x_3 + x_4)$$

$$x_3 + x_4 \geq .25(x_1 + x_2 + x_3 + x_4)$$

The only remaining constraint deals with keeping the demolishition/construction cost within the allowable budget—that is,

$$(\text{Construction and demolition cost}) \leq (\text{Available budget})$$

Expressing all the costs in thousands of dollars, we get

$$(50x_1 + 70x_2 + 130x_3 + 160x_4) + 2x_5 \leq 15000$$

The complete model thus becomes

$$\text{Maximize } z = 1000x_1 + 1900x_2 + 2700x_3 + 3400x_4$$

subject to

$$.18x_1 + .28x_2 + .4x_3 + .5x_4 - .2125x_5 \leq 0$$

$$x_5 \leq 300$$

$$-.8x_1 + .2x_2 + .2x_3 + .2x_4 \leq 0$$

$$.1x_1 - .9x_2 + .1x_3 + .1x_4 \leq 0$$

$$.25x_1 + .25x_2 - .75x_3 - .75x_4 \leq 0$$

$$50x_1 + 70x_2 + 130x_3 + 160x_4 + 2x_5 \leq 15000$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Solution:

The optimum solution (using file amplEX2.3-1.txt or solverEx2.3-1.xls) is:

Total tax collection = z = \$343,965
Number of single homes = $x_1 = 35.83 \simeq 36$ units
Number of double homes = $x_2 = 98.53 \simeq 99$ units
Number of triple homes = $x_3 = 44.79 \simeq 45$ units
Number of quadruple homes = $x_4 = 0$ units
Number of homes demolished = $x_5 = 244.49 \simeq 245$ units

Remarks. Linear programming does not guarantee an integer solution automatically, and this is the reason for rounding the continuous values to the closest integer. The rounded solution calls for constructing 180 (= 36 + 99 + 45) units and demolishing 245 old homes, which yields \$345,600 in taxes. Keep in mind, however, that, in general, the rounded solution may not be feasible. In fact, the current rounded solution violates the budget constraint by \$70,000 (verify!). Interestingly, the true optimum integer solution (using the algorithms in Chapter 9) is $x_1 = 36, x_2 = 98, x_3 = 45, x_4 = 0$, and $x_5 = 245$ with z = \$343,700. Carefully note that the rounded solution yields a better objective value, which appears contradictory. The reason is that the rounded solution calls for producing an extra double home, which is feasible only if the budget is increased by \$70,000.

PROBLEM SET 2.3A

1. A realtor is developing a rental housing and retail area. The housing area consists of efficiency apartments, duplexes, and single-family homes. Maximum demand by potential renters is estimated to be 500 efficiency apartments, 300 duplexes, and 250 single-family homes, but the number of duplexes must equal at least 50% of the number of efficiency apartments and single homes. Retail space is proportionate to the number of home units at the rates of at least 10 ft^2, 15 ft^2, and 18 ft^2 for efficiency, duplex, and single family units, respectively. However, land availability limits retail space to no more than 10,000 ft^2. The monthly rental income is estimated at \$600, \$750, and \$1200 for efficiency-, duplex-, and single-family units, respectively. The retail space rents for \$100/ft^2. Determine the optimal retail space area and the number of family residences.
2. The city council of Fayetteville is in the process of approving the construction of a new 200,000-ft^2 convention center. Two sites have been proposed, and both require exercising the "eminent domain" law to acquire the property. The following table provides data about proposed (contiguous) properties in both sites together with the acquisition cost.

	Site 1		Site 2	
Property	*Area* (1000 ft^2)	*Cost* (1000 \$)	*Area* (1000 ft^2)	*Cost* (1000 \$)
1	20	1,000	80	2,800
2	50	2,100	60	1,900
3	50	2,350	50	2,800
4	30	1,850	70	2,500
5	60	2,950		

Partial acquisition of property is allowed. At least 75% of property 4 must be acquired if site 1 is selected, and at least 50% of property 3 must be acquired if site 2 is selected.

Although site 1 property is more expensive (on a per ft^2 basis), the construction cost is less than at site 2, because the infrastructure at site 1 is in a much better shape. Construction cost is $25 million at site 1 and $27 million at site 2. Which site should be selected, and what properties should be acquired?

*3. A city will undertake five urban renewal housing projects over the next five years. Each project has a different starting year and a different duration. The following table provides the basic data of the situation:

	Year 1	Year 2	Year 3	Year 4	Year 5	Cost (million $)	Annual income (million $)
Project 1	Start		End			5.0	.05
Project 2		Start			End	8.0	.07
Project 3	Start				End	15.0	.15
Project 4			Start	End		1.2	.02
Budget (million $)	3.0	6.0	7.0	7.0	7.0		

Projects 1 and 4 must be finished completely within their durations. The remaining two projects can be finished partially within budget limitations, if necessary. However, each project must be at least 25% completed within its duration. At the end of each year, the completed section of a project is immediately occupied by tenants and a proportional amount of income is realized. For example, if 40% of project 1 is completed in year 1 and 60% in year 3, the associated income over the five-year planning horizon is $.4 \times \$50,000$ (for year 2) $+ .4 \times \$50,000$ (for year 3) $+ (.4 + .6) \times \$50,000$ (for year 4) $+ (.4 + .6) \times \$50,000$ (for year 5) $= (4 \times .4 + 2 \times .6) \times \$50,000$. Determine the optimal schedule for the projects that will maximize the total income over the five-year horizon. For simplicity, disregard the time value of money.

4. The city of Fayetteville is embarking on an urban renewal project that will include lower- and middle-income row housing, upper-income luxury apartments, and public housing. The project also includes a public elementary school and retail facilities. The size of the elementary school (number of classrooms) is proportional to the number of pupils, and the retail space is proportional to the number of housing units. The following table provides the pertinent data of the situation:

	Lower income	Middle income	Upper income	Public housing	School room	Retail unit
Minimum number of units	100	125	75	300		0
Maximum number of units	200	190	260	600		25
Lot size per unit (acre)	.05	.07	.03	.025	.045	.1
Average number of pupils per unit	1.3	1.2	.5	1.4		
Retail demand per unit (acre)	.023	.034	.046	.023	.034	
Annual income per unit($)	7000	12,000	20,000	5000	—	15,000

The new school can occupy a maximum space of 2 acres at the rate of at most 25 pupils per room. The operating annual cost per school room is $10,000. The project will be located on a 50-acre vacant property owned by the city. Additionally, the project can make use of an adjacent property occupied by 200 condemned slum homes. Each condemned home occupies .25 acre. The cost of buying and demolishing a slum unit is $7000. Open space, streets, and parking lots consume 15% of total available land.

Develop a linear program to determine the optimum plan for the project.

5. Realco owns 800 acres of undeveloped land on a scenic lake in the heart of the Ozark Mountains. In the past, little or no regulation was imposed upon new developments around the lake. The lake shores are now dotted with vacation homes, and septic tanks, most of them improperly installed, are in extensive use. Over the years, seepage from the septic tanks led to severe water pollution. To curb further degradation of the lake, county officials have approved stringent ordinances applicable to all future developments: (1) Only single-, double-, and triple-family homes can be constructed, with single-family homes accounting for at least 50% of the total. (2) To limit the number of septic tanks, minimum lot sizes of 2, 3, and 4 acres are required for single-, double-, and triple-family homes, respectively. (3) Recreation areas of 1 acre each must be established at the rate of one area per 200 families. (4) To preserve the ecology of the lake, underground water may not be pumped out for house or garden use. The president of Realco is studying the possibility of developing the 800-acre property. The new development will include single-, double-, and triple-family homes. It is estimated that 15% of the acreage will be allocated to streets and utility easements. Realco estimates the returns from the different housing units as follows:

Housing unit	Single	Double	Triple
Net return per unit ($)	10,000	12,000	15,000

The cost of connecting water service to the area is proportionate to the number of units constructed. However, the county charges a minimum of $100,000 for the project. Additionally, the expansion of the water system beyond its present capacity is limited to 200,000 gallons per day during peak periods. The following data summarize the water service connection cost as well as the water consumption, assuming an average size family:

Housing unit	Single	Double	Triple	Recreation
Water service connection cost per unit ($)	1000	1200	1400	800
Water consumption per unit (gal/day)	400	600	840	450

Develop an optimal plan for Realco.

6. Consider the Realco model of Problem 5. Suppose that an additional 100 acres of land can be purchased for $450,000, which will increase the total acreage to 900 acres. Is this a profitable deal for Realco?

2.3.2 Currency Arbitrage[2]

In today's global economy, a multinational company must deal with currencies of the countries in which it operates. Currency arbitrage, or simultaneous purchase and sale of currencies in different markets, offers opportunities for advantageous movement of money from one currency to another. For example, converting £1000 to U.S. dollars in 2001 with an exchange rate of $1.60 to £1 will yield $1600. Another way of making the conversion is to first change the British pound to Japanese yen and then convert the yen to U.S. dollars using the 2001 exchange rates of £1 = ¥175 and $1 = ¥105. The

[2]This section is based on J. Kornbluth and G. Salkin (1987, Chapter 6).

resulting dollar amount is $\frac{(£1{,}000 \times ¥175)}{¥105} = \$1{,}666.67$. This example demonstrates the advantage of converting the British money first to Japanese yen and then to dollars. This section shows how the arbitrage problem involving many currencies can be formulated and solved as a linear program.

Example 2.3-2 (Currency Arbitrage Model)

Suppose that a company has a total of 5 million dollars that can be exchanged for euros (€), British pounds (£), yen (¥), and Kuwaiti dinars (KD). Currency dealers set the following limits on the amount of any single transaction: 5 million dollars, 3 million euros, 3.5 million pounds, 100 million yen, and 2.8 million KDs. The table below provides typical spot exchange rates. The bottom diagonal rates are the reciprocal of the top diagonal rates. For example, rate(€ → \$) = 1/rate(\$ → €) = 1/.769 = 1.30.

	\$	€	£	¥	KD
\$	1	.769	.625	105	.342
€	$\frac{1}{.769}$	1	.813	137	.445
£	$\frac{1}{.625}$	$\frac{1}{.813}$	1	169	.543
¥	$\frac{1}{105}$	$\frac{1}{137}$	$\frac{1}{169}$	1	.0032
KD	$\frac{1}{.342}$	$\frac{1}{.445}$	$\frac{1}{.543}$	$\frac{1}{.0032}$	1

Is it possible to increase the dollar holdings (above the initial \$5 million) by circulating currencies through the currency market?

Mathematical Model: The situation starts with \$5 million. This amount goes through a number of conversions to other currencies before ultimately being reconverted to dollars. The problem thus seeks determining the amount of each conversion that will maximize the total dollar holdings.

For the purpose of developing the model and simplifying the notation, the following numeric code is used to represent the currencies.

Currency	\$	€	£	¥	KD
Code	1	2	3	4	5

Define

$$x_{ij} = \text{Amount in currency } i \text{ converted to currency } j,\ i \text{ and } j = 1, 2, \ldots, 5$$

For example, x_{12} is the dollar amount converted to euros and x_{51} is the KD amount converted to dollars. We further define two additional variables representing the input and the output of the arbitrage problem:

$$I = \text{Initial dollar amount } (= \$5 \text{ million})$$
$$y = \text{Final dollar holdings (to be determined from the solution)}$$

Our goal is to determine the maximum final dollar holdings, y, subject to the currency flow restrictions and the maximum limits allowed for the different transactions.

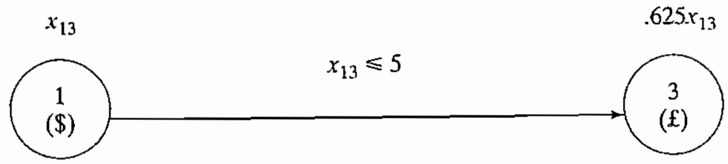


FIGURE 2.4

Definition of the input/output variable, x_{13}, between \$ and £

We start by developing the constraints of the model. Figure 2.4 demonstrates the idea of converting dollars to pounds. The dollar amount x_{13} at originating currency 1 is converted to $.625x_{13}$ pounds at end currency 3. At the same time, the transacted dollar amount cannot exceed the limit set by the dealer, $x_{13} \leq 5$.

To conserve the flow of money from one currency to another, each currency must satisfy the following input-output equation:

$$\begin{pmatrix}\text{Total sum available}\\ \text{of a currency (input)}\end{pmatrix} = \begin{pmatrix}\text{Total sum converted to}\\ \text{other currencies (output)}\end{pmatrix}$$

1. *Dollar* ($i = 1$):

$$\begin{aligned}\text{Total available dollars} &= \text{Initial dollar amount} + \\ &\quad\ \text{dollar amount from other currencies} \\ &= I + (€ \rightarrow \$) + (£ \rightarrow \$) + (¥ \rightarrow \$) + (\text{KD} \rightarrow \$) \\ &= I + \tfrac{1}{.769}x_{21} + \tfrac{1}{.625}x_{31} + \tfrac{1}{105}x_{41} + \tfrac{1}{.342}x_{51}\end{aligned}$$

$$\begin{aligned}\text{Total distributed dollars} &= \text{Final dollar holdings} + \\ &\quad\ \text{dollar amount to other currencies} \\ &= y + (\$ \rightarrow €) + (\$ \rightarrow £) + (\$ \rightarrow ¥) + (\$ \rightarrow \text{KD}) \\ &= y + x_{12} + x_{13} + x_{14} + x_{15}\end{aligned}$$

Given $I = 5$, the dollar constraint thus becomes

$$y + x_{12} + x_{13} + x_{14} + x_{15} - \left(\tfrac{1}{.769}x_{21} + \tfrac{1}{.625}x_{31} + \tfrac{1}{105}x_{41} + \tfrac{1}{.342}x_{51}\right) = 5$$

2. *Euro* ($i = 2$):

$$\begin{aligned}\text{Total available euros} &= (\$ \rightarrow €) + (£ \rightarrow €) + (¥ \rightarrow €) + (\text{KD} \rightarrow €) \\ &= .769x_{12} + \tfrac{1}{.813}x_{32} + \tfrac{1}{137}x_{42} + \tfrac{1}{.445}x_{52}\end{aligned}$$

$$\begin{aligned}\text{Total distributed euros} &= (€ \rightarrow \$) + (€ \rightarrow £) + (€ \rightarrow ¥) + (€ \rightarrow \text{KD}) \\ &= x_{21} + x_{23} + x_{24} + x_{25}\end{aligned}$$

Thus, the constraint is

$$x_{21} + x_{23} + x_{24} + x_{25} - \left(.769x_{12} + \tfrac{1}{.813}x_{32} + \tfrac{1}{137}x_{42} + \tfrac{1}{.445}x_{52}\right) = 0$$

3. *Pound* ($i = 3$):

$$\begin{aligned}\text{Total available pounds} &= (\$ \rightarrow £) + (€ \rightarrow £) + (¥ \rightarrow £) + (\text{KD} \rightarrow £)\\ &= .625x_{13} + .813x_{23} + \tfrac{1}{169}x_{43} + \tfrac{1}{.543}x_{53}\end{aligned}$$

$$\begin{aligned}\text{Total distributed pounds} &= (£ \rightarrow \$) + (£ \rightarrow €) + (£ \rightarrow ¥) + (£ \rightarrow \text{KD})\\ &= x_{31} + x_{32} + x_{34} + x_{35}\end{aligned}$$

Thus, the constraint is

$$x_{31} + x_{32} + x_{34} + x_{35} - .625x_{13} + .813x_{23} + \tfrac{1}{169}x_{43} + \tfrac{1}{.543}x_{53} = 0$$

4. *Yen* ($i = 4$):

$$\begin{aligned}\text{Total available yen} &= (\$ \rightarrow ¥) + (€ \rightarrow ¥) + (£ \rightarrow ¥) + (\text{KD} \rightarrow ¥)\\ &= 105x_{14} + 137x_{24} + 169x_{34} + \tfrac{1}{.0032}x_{54}\end{aligned}$$

$$\begin{aligned}\text{Total distributed yen} &= (¥ \rightarrow \$) + (¥ \rightarrow €) + (¥ \rightarrow £) + (¥ \rightarrow \text{KD})\\ &= x_{41} + x_{42} + x_{43} + x_{45}\end{aligned}$$

Thus, the constraint is

$$x_{41} + x_{42} + x_{43} + x_{45} - \left(105x_{14} + 137x_{24} + 169x_{34} + \tfrac{1}{.0032}x_{54}\right) = 0$$

5. *KD* ($i = 5$):

$$\begin{aligned}\text{Total available KDs} &= (\text{KD} \rightarrow \$) + (\text{KD} \rightarrow €) + (\text{KD} \rightarrow £) + (\text{KD} \rightarrow ¥)\\ &= .342x_{15} + .445x_{25} + .543x_{35} + .0032x_{45}\end{aligned}$$

$$\begin{aligned}\text{Total distributed KDs} &= (\$ \rightarrow \text{KD}) + (€ \rightarrow \text{KD}) + (£ \rightarrow \text{KD}) + (¥ \rightarrow \text{KD})\\ &= x_{51} + x_{52} + x_{53} + x_{54}\end{aligned}$$

Thus, the constraint is

$$x_{51} + x_{52} + x_{53} + x_{54} - (.342x_{15} + .445x_{25} + .543x_{35} + .0032x_{45}) = 0$$

The only remaining constraints are the transaction limits, which are 5 million dollars, 3 million euros, 3.5 million pounds, 100 million yen, and 2.8 million KDs. These can be translated as

$$\begin{aligned} x_{1j} &\le 5, j = 2, 3, 4, 5\\ x_{2j} &\le 3, j = 1, 3, 4, 5\\ x_{3j} &\le 3.5, j = 1, 2, 4, 5\\ x_{4j} &\le 100, j = 1, 2, 3, 5\\ x_{5j} &\le 2.8, j = 1, 2, 3, 4\end{aligned}$$

The complete model is now given as

$$\text{Maximize } z = y$$

subject to

$$y + x_{12} + x_{13} + x_{14} + x_{15} - \left(\tfrac{1}{.769}x_{21} + \tfrac{1}{.625}x_{31} + \tfrac{1}{105}x_{41} + \tfrac{1}{.342}x_{51}\right) = 5$$
$$x_{21} + x_{23} + x_{24} + x_{25} - \left(.769x_{12} + \tfrac{1}{.813}x_{32} + \tfrac{1}{137}x_{42} + \tfrac{1}{.445}x_{52}\right) = 0$$
$$x_{31} + x_{32} + x_{34} + x_{35} - \left(.625x_{13} + .813x_{23} + \tfrac{1}{169}x_{43} + \tfrac{1}{.543}x_{53}\right) = 0$$
$$x_{41} + x_{42} + x_{43} + x_{45} - \left(105x_{14} + 137x_{24} + 169x_{34} + \tfrac{1}{.0032}x_{54}\right) = 0$$
$$x_{51} + x_{52} + x_{53} + x_{54} - (.342x_{15} + .445x_{25} + .543x_{35} + .0032x_{45}) = 0$$

$$x_{1j} \le 5, j = 2, 3, 4, 5$$
$$x_{2j} \le 3, j = 1, 3, 4, 5$$
$$x_{3j} \le 3.5, j = 1, 2, 4, 5$$
$$x_{4j} \le 100, j = 1, 2, 3, 5$$
$$x_{5j} \le 2.8, j = 1, 2, 3, 4$$
$$x_{ij} \ge 0, \text{ for all } i \text{ and } j$$

Solution:

The optimum solution (using file amplEx2.3-2.txt or solverEx2.3-2.xls) is:

Solution	Interpretation
$y = 5.09032$	Final holdings = \$5,090,320. Net dollar gain = \$90,320, which represents a 1.8064% rate of return
$x_{12} = 1.46206$	Buy \$1,462,060 worth of euros
$x_{15} = 5$	Buy \$5,000,000 worth of KD
$x_{25} = 3$	Buy €3,000,000 worth of KD
$x_{31} = 3.5$	Buy £3,500,000 worth of dollars
$x_{32} = 0.931495$	Buy £931,495 worth of euros
$x_{41} = 100$	Buy ¥100,000,000 worth of dollars
$x_{42} = 100$	Buy ¥100,000,000 worth of euros
$x_{43} = 100$	Buy ¥100,000,000 worth of pounds
$x_{53} = 2.085$	Buy KD2,085,000 worth of pounds
$x_{54} = .96$	Buy KD960,000 worth of yen

Remarks. At first it may appear that the solution is nonsensical because it calls for using $x_{12} + x_{15} = 1.46206 + 5 = 6.46206$, or \$6,462,060 to buy euros and KDs when the initial dollar amount is only \$5,000,000. Where do the extra dollars come from? What happens in practice is that the given solution is submitted to the currency dealer as *one* order, meaning we do not wait until we accumulate enough currency of a certain type before making a buy. In the end, the net

2.

result of all these transactions is a net cost of \$5,000,000 to the investor. This can be seen by summing up all the dollar transactions in the solution:

$$I = y + x_{12} + x_{13} + x_{14} + x_{15} - \left(\tfrac{1}{.769}x_{21} + \tfrac{1}{.625}x_{31} + \tfrac{1}{105}x_{41} + \tfrac{1}{.342}x_{51}\right)$$

$$= 5.09032 + 1.46206 + 5 - \left(\tfrac{3.5}{.625} + \tfrac{100}{105}\right) = 5$$

Notice that x_{21}, x_{31}, x_{41} and x_{51} are in euro, pound, yen, and KD, respectively, and hence must be converted to dollars.

PROBLEM SET 2.3B

1. Modify the arbitrage model to account for a commission that amounts to .1% of any currency buy. Assume that the commission does not affect the circulating funds and that it is collected after the entire order is executed. How does the solution compare with that of the original model?
*2. Suppose that the company is willing to convert the initial \$5 million to any other currency that will provide the highest rate of return. Modify the original model to determine which currency is the best.
3. Suppose the initial amount I = \$7 million and that the company wants to convert it optimally to a combination of euros, pounds, and yen. The final mix may not include more than €2 million, £3 million, and ¥200 million. Modify the original model to determine the optimal buying mix of the three currencies.
4. Suppose that the company wishes to buy \$6 million. The transaction limits for different currencies are the same as in the original problem. Devise a buying schedule for this transaction, given that mix may not include more than €3 million, £2 million, and KD2 million.
5. Suppose that the company has \$2 million, €5 million, £4 million. Devise a buy-sell order that will improve the overall holdings converted to yen.

2.3.3 Investment

Today's investors are presented with multitudes of investment opportunities. Examples of investment problems are capital budgeting for projects, bond investment strategy, stock portfolio selection, and establishment of bank loan policy. In many of these situations, linear programming can be used to select the optimal mix of opportunities that will maximize return while meeting the investment conditions set by the investor.

Example 2.3-3 (Loan Policy Model)

Thriftem Bank is in the process of devising a loan policy that involves a maximum of \$12 million. The following table provides the pertinent data about available types of loans.

Type of loan	Interest rate	Bad-debt ratio
Personal	.140	.10
Car	.130	.07
Home	.120	.03
Farm	.125	.05
Commercial	.100	.02

Bad debts are unrecoverable and produce no interest revenue.

Competition with other financial institutions requires that the bank allocate at least 40% of the funds to farm and commercial loans. To assist the housing industry in the region, home loans must equal at least 50% of the personal, car, and home loans. The bank also has a stated policy of not allowing the overall ratio of bad debts on all loans to exceed 4%.

Mathematical Model: The situation seeks to determine the amount of loan in each category, thus leading to the following definitions of the variables:

x_1 = personal loans (in millions of dollars)

x_2 = car loans

x_3 = home loans

x_4 = farm loans

x_5 = commercial loans

The objective of the Thriftem Bank is to maximize its net return, the difference between interest revenue and lost bad debts. The interest revenue is accrued only on loans in good standing. Thus, because 10% of personal loans are lost to bad debt, the bank will receive interest on only 90% of the loan—that is, it will receive 14% interest on $.9x_1$ of the original loan x_1. The same reasoning applies to the remaining four types of loans. Thus,

$$\text{Total interest} = .14(.9x_1) + .13(.93x_2) + .12(.97x_3) + .125(.95x_4) + .1(.98x_5)$$
$$= .126x_1 + .1209x_2 + .1164x_3 + .11875x_4 + .098x_5$$

We also have

$$\text{Bad debt} = .1x_1 + .07x_2 + .03x_3 + .05x_4 + .02x_5$$

The objective function is thus expressed as

$$\begin{aligned}\text{Maximize } z &= \text{Total interest—Bad debt}\\ &= (.126x_1 + .1209x_2 + .1164x_3 + .11875x_4 + .098x_5)\\ &\quad - (.1x_1 + .07x_2 + .03x_3 + .05x_4 + .02x_5)\\ &= .026x_1 + .0509x_2 + .0864x_3 + .06875x_4 + .078x_5\end{aligned}$$

The problem has five constraints:

1. *Total funds should not exceed \$12 (million):*

$$x_1 + x_2 + x_3 + x_4 + x_5 \leq 12$$

2. *Farm and commercial loans equal at least 40% of all loans:*

$$x_4 + x_5 \geq .4(x_1 + x_2 + x_3 + x_4 + x_5)$$

or

$$.4x_1 + .4x_2 + .4x_3 - .6x_4 - .6x_5 \leq 0$$

3. *Home loans should equal at least 50% of personal, car, and home loans:*

$$x_3 \geq .5(x_1 + x_2 + x_3)$$

or

$$.5x_1 + .5x_2 - .5x_3 \leq 0$$

4. *Bad debts should not exceed 4% of all loans:*

$$.1x_1 + .07x_2 + .03x_3 + .05x_4 + .02x_5 \leq .04(x_1 + x_2 + x_3 + x_4 + x_5)$$

or

$$.06x_1 + .03x_2 - .01x_3 + .01x_4 - .02x_5 \leq 0$$

5. *Nonnegativity:*

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0$$

A subtle assumption in the preceding formulation is that all loans are issued at approximately the same time. This assumption allows us to ignore differences in the time value of the funds allocated to the different loans.

Solution:

The optimal solution is

$$z = .99648, x_1 = 0, x_2 = 0, x_3 = 7.2, x_4 = 0, x_5 = 4.8$$

Remarks.

1. You may be wondering why we did not define the right-hand side of the second constraint as $.4 \times 12$ instead of $.4(x_1 + x_2 + x_3 + x_4 + x_5)$. After all, it seems logical that the bank would want to loan out all \$12 (million). The answer is that the second usage does not "rob" the model of this possibility. If the optimum solution needs all \$12 (million), the given constraint will allow it. But there are two important reasons why you should not use $.4 \times 12$: (1) If other constraints in the model are such that all \$12 (million) *cannot* be used (for example, the bank may set caps on the different loans), then the choice $.4 \times 12$ could lead to an infeasible or incorrect solution. (2) If you want to experiment with the effect of changing available funds (say from \$12 to \$13 million) on the optimum solution, there is a real chance that you may forget to change $.4 \times 12$ to $.4 \times 13$, in which case the solution you get will not be correct. A similar reasoning applies to the left-hand side of the fourth constraint.
2. The optimal solution calls for allocating all \$12 million: \$7.2 million to home loans and \$4.8 million to commercial loans. The remaining categories receive none. The return on the investment is computed as

$$\text{Rate of return} = \frac{z}{12} = \frac{.99648}{12} = .08034$$

This shows that the combined annual rate of return is 8.034%, which is less than the best *net* interest rate (=.0864 for home loans), and one wonders why the optimum does not take advantage of this opportunity. The answer is that the restriction stipulating that farm and commercial loans account for at least 40% of all loans (constraint 2) forces the solution to allocate \$4.8 million to commercial loans at the lower *net* rate of .078, hence lowering the overall interest rate to $\frac{.0864 \times 7.2 + .078 \times 4.8}{12} = .08034$. In fact, if we remove constraint 2, the optimum will allocate all the funds to home loans at the higher 8.64% rate.

PROBLEM SET 2.3C

1. Fox Enterprises is considering six projects for possible construction over the next four years. The expected (present value) returns and cash outlays for the projects are given below. Fox can undertake any of the projects partially or completely. A partial undertaking of a project will prorate both the return and cash outlays proportionately.

	Cash outlay ($1000)				
Project	*Year 1*	*Year 2*	*Year 3*	*Year 4*	Return ($1000)
1	10.5	14.4	2.2	2.4	32.40
2	8.3	12.6	9.5	3.1	35.80
3	10.2	14.2	5.6	4.2	17.75
4	7.2	10.5	7.5	5.0	14.80
5	12.3	10.1	8.3	6.3	18.20
6	9.2	7.8	6.9	5.1	12.35
Available funds ($1000)	60.0	70.0	35.0	20.0	

(a) Formulate the problem as a linear program, and determine the optimal project mix that maximizes the total return. Ignore the time value of money.

(b) Suppose that if a portion of project 2 is undertaken then at least an equal portion of project 6 must undertaken. Modify the formulation of the model and find the new optimal solution.

(c) In the original model, suppose that any funds left at the end of a year are used in the next year. Find the new optimal solution, and determine how much each year "borrows" from the preceding year. For simplicity, ignore the time value of money.

(d) Suppose in the original model that the yearly funds available for any year can be exceeded, if necessary, by borrowing from other financial activities within the company. Ignoring the time value of money, reformulate the LP model, and find the optimum solution. Would the new solution require borrowing in any year? If so, what is the rate of return on borrowed money?

*2. Investor Doe has $10,000 to invest in four projects. The following table gives the cash flow for the four investments.

	Cash flow ($1000) at the start of				
Project	*Year 1*	*Year 2*	*Year 3*	*Year 4*	*Year 5*
1	−1.00	0.50	0.30	1.80	1.20
2	−1.00	0.60	0.20	1.50	1.30
3	0.00	−1.00	0.80	1.90	0.80
4	−1.00	0.40	0.60	1.80	0.95

The information in the table can be interpreted as follows: For project 1, $1.00 invested at the start of year 1 will yield $.50 at the start of year 2, $.30 at the start of year 3, $1.80 at the start of year 4, and $1.20 at the start of year 5. The remaining entries can be interpreted similarly. The entry 0.00 indicates that no transaction is taking place. Doe has the additional option of investing in a bank account that earns 6.5% annually. All funds accumulated at the end of one year can be reinvested in the following year. Formulate the problem as a linear program to determine the optimal allocation of funds to investment opportunities.

3. HiRise Construction can bid on two 1-year projects. The following table provides the quarterly cash flow (in millions of dollars) for the two projects.

Project	Cash flow (in millions of $) at				
	1/1/08	*4/1/08*	*7/1/08*	*10/1/08*	*12/31/08*
I	−1.0	−3.1	−1.5	1.8	5.0
II	−3.0	−2.5	1.5	1.8	2.8

HiRise has cash funds of $1 million at the beginning of each quarter and may borrow at most $1 million at a 10% nominal annual interest rate. Any borrowed money must be returned at the end of the quarter. Surplus cash can earn quarterly interest at an 8% nominal annual rate. Net accumulation at the end of one quarter is invested in the next quarter.

(a) Assume that HiRise is allowed partial or full participation in the two projects. Determine the level of participation that will maximize the net cash accumulated on 12/31/2008.

(b) Is it possible in any quarter to borrow money and simultaneously end up with surplus funds? Explain.

4. In anticipation of the immense college expenses, a couple have started an annual investment program on their child's eighth birthday that will last until the eighteenth birthday. The couple estimate that they will be able to invest the following amounts at the beginning of each year:

Year	1	2	3	4	5	6	7	8	9	10
Amount ($)	2000	2000	2500	2500	3000	3500	3500	4000	4000	5000

To avoid unpleasant surprises, they want to invest the money safely in the following options: Insured savings with 7.5% annual yield, six-year government bonds that yield 7.9% and have a current market price equal to 98% of face value, and nine-year municipal bonds yielding 8.5% and having a current market price of 1.02 of face value. How should the couple invest the money?

*5. A business executive has the option to invest money in two plans: Plan A guarantees that each dollar invested will earn $.70 a year later, and plan B guarantees that each dollar invested will earn $2 after 2 years. In plan A, investments can be made annually, and in plan B, investments are allowed for periods that are multiples of two years only. How should the executive invest $100,000 to maximize the earnings at the end of 3 years?

6. A gambler plays a game that requires dividing bet money among four choices. The game has three outcomes. The following table gives the corresponding gain or loss per dollar for the different options of the game.

Outcome	Return per dollar deposited in choice			
	1	*2*	*3*	*4*
1	−3	4	−7	15
2	5	−3	9	4
3	3	−9	10	−8

The gambler has a total of $500, which may be played only once. The exact outcome of the game is not known a priori. Because of this uncertainty, the gambler's strategy is to maximize the *minimum* return produced by the three outcomes. How should the gambler allocate the $500 among the four choices? (*Hint:* The gambler's net return may be positive, zero, or negative.)

7. (Lewis, 1996) Monthly bills in a household are received monthly (e.g., utilities and home mortgage), quarterly (e.g., estimated tax payment), semiannually (e.g., insurance), or annually (e.g., subscription renewals and dues). The following table provides the monthly bills for next year.

Month	Jan.	Feb.	Mar.	Apr.	May	Jun.	Jul.	Aug.	Sep.	Oct.	Nov.	Dec.	Total
$	800	1200	400	700	600	900	1500	1000	900	1100	1300	1600	12000

To account for these expenses, the family sets aside $1000 per month, which is the average of the total divided by 12 months. If the money is deposited in a regular savings account, it can earn 4% annual interest, provided it stays in the account at least one month. The bank also offers 3-month and 6-month certificates of deposit that can earn 5.5% and 7% annual interest, respectively. Develop a 12-month investment schedule that will maximize the family's total return for the year. State any assumptions or requirements needed to reach a feasible solution.

2.3.4 Production Planning and Inventory Control

There is a wealth of LP applications to production and inventory control, ranging from simple allocation of machining capacity to meet demand to the more complex case of using inventory to "dampen" the effect of erratic change in demand over a given planning horizon and of using hiring and firing to respond to changes in workforce needs. This section presents three examples. The first deals with the scheduling of products using common production facilities to meet demand during a single period, the second deals with the use of inventory in a multiperiod production system to fill future demand, and the third deals with the use of a combined inventory and worker hiring/firing to "smooth" production over a multiperiod planning horizon with fluctuating demand.

Example 2.3-4 (Single-Period Production Model)

In preparation for the winter season, a clothing company is manufacturing parka and goose overcoats, insulated pants, and gloves. All products are manufactured in four different departments: cutting, insulating, sewing, and packaging. The company has received firm orders for its products. The contract stipulates a penalty for undelivered items. The following table provides the pertinent data of the situation.

	Time per units (hr)				
Department	*Parka*	*Goose*	*Pants*	*Gloves*	Capacity (hr)
Cutting	.30	.30	.25	.15	1000
Insulating	.25	.35	.30	.10	1000
Sewing	.45	.50	.40	.22	1000
Packaging	.15	.15	.1	.05	1000
Demand	800	750	600	500	
Unit profit	$30	$40	$20	$10	
Unit penalty	$15	$20	$10	$8	

Devise an optimal production plan for the company.

Mathematical Model: The definition of the variables is straightforward. Let

$$x_1 = \text{number of parka jackets}$$
$$x_2 = \text{number of goose jackets}$$
$$x_3 = \text{number of pairs of pants}$$
$$x_4 = \text{number of pairs of gloves}$$

The company is penalized for not meeting demand. This means that the objective of the problem is to maximize the net receipts, defined as

$$\text{Net receipts} = \text{Total profit} - \text{Total penalty}$$

The total profit is readily expressed as $30x_1 + 40x_2 + 20x_3 + 10x_4$. The total penalty is a function of the shortage quantities (= demand − units supplied of each product). These quantities can be determined from the following demand limits:

$$x_1 \le 800,\ x_2 \le 750,\ x_3 \le 600,\ x_4 \le 500$$

A demand is not fulfilled if its constraint is satisfied as a strict inequality. For example, if 650 parka jackets are produced, then $x_1 = 650$, which leads to a shortage of $800 - 650 = 150$ parka jackets. We can express the shortage of any product algebraically by defining a new nonnegative variable—namely,

$$s_j = \text{Number of shortage units of product } j,\ j = 1, 2, 3, 4$$

In this case, the demand constraints can be written as

$$x_1 + s_1 = 800,\ x_2 + s_2 = 750,\ x_3 + s_3 = 600,\ x_4 + s_4 = 500$$
$$x_j \ge 0,\ s_j \ge 0,\ j = 1, 2, 3, 4$$

We can now compute the shortage penalty as $15s_1 + 20s_2 + 10s_3 + 8s_4$. Thus, the objective function can be written as

$$\text{Maximize } z = 30x_1 + 40x_2 + 20x_3 + 10x_4 - (15s_1 + 20s_2 + 10s_3 + 8s_4)$$

To complete the model, the remaining constraints deal with the production capacity restrictions; namely

$$.30x_1 + .30x_2 + .25x_3 + .15x_4 \le 1000 \quad \text{(Cutting)}$$
$$.25x_1 + .35x_2 + .30x_3 + .10x_4 \le 1000 \quad \text{(Insulating)}$$
$$.45x_1 + .50x_2 + .40x_3 + .22x_4 \le 1000 \quad \text{(Sewing)}$$
$$.15x_1 + .15x_2 + .10x_3 + .05x_4 \le 1000 \quad \text{(Packaging)}$$

The complete model thus becomes

$$\text{Maximize } z = 30x_1 + 40x_2 + 20x_3 + 10x_4 - (15s_1 + 20s_2 + 10s_3 + 8s_4)$$

subject to

$$.30x_1 + .30x_2 + .25x_3 + .15x_4 \leq 1000$$
$$.25x_1 + .35x_2 + .30x_3 + .10x_4 \leq 1000$$
$$.45x_1 + .50x_2 + .40x_3 + .22x_4 \leq 1000$$
$$.15x_1 + .15x_2 + .10x_3 + .05x_4 \leq 1000$$
$$x_1 + s_1 = 800, x_2 + s_2 = 750, x_3 + s_3 = 600, x_4 + s_4 = 500$$
$$x_j \geq 0, s_j \geq 0, j = 1, 2, 3, 4$$

Solution:

The optimum solution is $z = \$64{,}625, x_1 = 850, x_2 = 750, x_3 = 387.5, x_4 = 500, s_1 = s_2 = s_4 = 0, s_3 = 212.5$. The solution satisfies all the demand for both types of jackets and the gloves. A shortage of 213 (rounded up from 212.5) pairs of pants will result in a penalty cost of $213 \times \$10 = \$2130$.

Example 2.3-5 (Multiple Period Production-Inventory Model)

Acme Manufacturing Company has contracted to deliver home windows over the next 6 months. The demands for each month are 100, 250, 190, 140, 220, and 110 units, respectively. Production cost per window varies from month to month depending on the cost of labor, material, and utilities. Acme estimates the production cost per window over the next 6 months to be \$50, \$45, \$55, \$48, \$52, and \$50, respectively. To take advantage of the fluctuations in manufacturing cost, Acme may elect to produce more than is needed in a given month and hold the excess units for delivery in later months. This, however, will incur storage costs at the rate of \$8 per window per month assessed on end-of-month inventory. Develop a linear program to determine the optimum production schedule.

Mathematical Model: The variables of the problem include the monthly production amount and the end-of-month inventory. For $i = 1, 2, \ldots, 6$, let

$$x_i = \text{Number of units produced in month } i$$
$$I_i = \text{Inventory units left at the end of month } i$$

The relationship between these variables and the monthly demand over the six-month horizon is represented by the schematic diagram in Figure 2.5. The system starts empty, which means that $I_0 = 0$.

The objective function seeks to minimize the sum of the production and end-of-month inventory costs. Here we have,

$$\text{Total production cost} = 50x_1 + 45x_2 + 55x_3 + 48x_4 + 52x_5 + 50x_6$$
$$\text{Total inventory cost} = 8(I_1 + I_2 + I_3 + I_4 + I_5 + I_6)$$

Thus the objective function is

$$\text{Minimize } z = 50x_1 + 45x_2 + 55x_3 + 48x_4 + 52x_5 + 50x_6 + 8(I_1 + I_2 + I_3 + I_4 + I_5 + I_6)$$

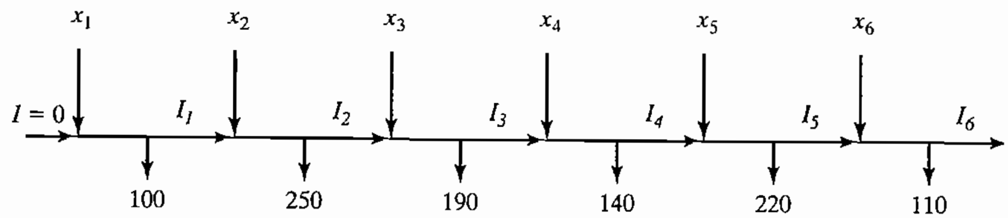


FIGURE 2.5

Schematic representation of the production-inventory system

The constraints of the problem can be determined directly from the representation in Figure 2.5. For each period we have the following balance equation:

$$\text{Beginning inventory} + \text{Production amount} - \text{Ending inventory} = \text{Demand}$$

This is translated mathematically for the individual months as

$$
\begin{aligned}
I_0 + x_1 - I_1 &= 100 \quad \text{(Month 1)}\\
I_1 + x_2 - I_2 &= 250 \quad \text{(Month 2)}\\
I_2 + x_3 - I_3 &= 190 \quad \text{(Month 3)}\\
I_3 + x_4 - I_4 &= 140 \quad \text{(Month 4)}\\
I_4 + x_5 - I_5 &= 220 \quad \text{(Month 5)}\\
I_5 + x_6 - I_6 &= 110 \quad \text{(Month 6)}\\
x_i, I_i \geq 0, &\text{ for all } i = 1, 2, \ldots, 6\\
I_0 &= 0
\end{aligned}
$$

For the problem, $I_0 = 0$ because the situation starts with no initial inventory. Also, in any optimal solution, the ending inventory I_6 will be zero, because it is not logical to end the horizon with positive inventory, which can only incur additional inventory cost without serving any purpose.

The complete model is now given as

$$
\begin{aligned}
\text{Minimize } z = {} & 50x_1 + 45x_2 + 55x_3 + 48x_4 + 52x_5 + 50x_6\\
& + 8(I_1 + I_2 + I_3 + Ix_4 + I_5 + I_6)
\end{aligned}
$$

subject to

$$
\begin{aligned}
x_1 - I_1 &= 100 \quad \text{(Month 1)}\\
I_1 + x_2 - I_2 &= 250 \quad \text{(Month 2)}\\
I_2 + x_3 - I_3 &= 190 \quad \text{(Month 3)}\\
I_3 + x_4 - I_4 &= 140 \quad \text{(Month 4)}\\
I_4 + x_5 - I_5 &= 220 \quad \text{(Month 5)}\\
I_5 + x_6 - I_6 &= 110 \quad \text{(Month 6)}\\
x_i, I_i \geq 0, &\text{ for all } i = 1, 2, \ldots, 6
\end{aligned}
$$

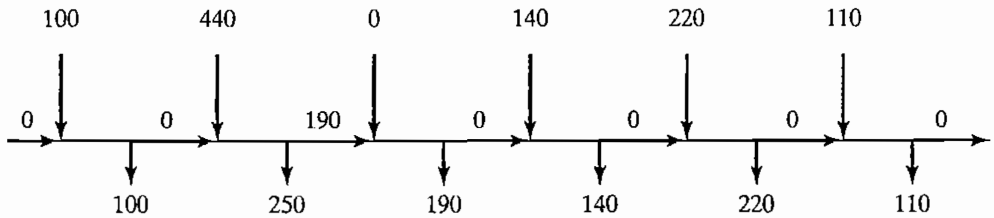


FIGURE 2.6
Optimum solution of the production-inventory problem

Solution:

The optimum solution is summarized in Figure 2.6. It shows that each month's demand is satisfied directly from the month's production, except for month 2 whose production quantity of 440 units covers the demand for both months 2 and 3. The total associated cost is $z = \$49{,}980$.

Example 2.3-6 (Multiperiod Production Smoothing Model)

A company will manufacture a product for the next four months: March, April, May, and June. The demands for each month are 520, 720, 520, and 620 units, respectively. The company has a steady workforce of 10 employees but can meet fluctuating production needs by hiring and firing temporary workers, if necessary. The extra costs of hiring and firing in any month are \$200 and \$400 per worker, respectively. A permanent worker can produce 12 units per month, and a temporary worker, lacking comparable experience, only produce 10 units per month. The company can produce more than needed in any month and carry the surplus over to a succeeding month at a holding cost of \$50 per unit per month. Develop an optimal hiring/firing policy for the company over the four-month planning horizon.

Mathematical Model: This model is similar to that of Example 2.3-5 in the general sense that each month has its production, demand, and ending inventory. There are two exceptions: (1) accounting for the permanent versus the temporary workforce, and (2) accounting for the cost of hiring and firing in each month.

Because the permanent 10 workers cannot be fired, their impact can be accounted for by subtracting the units they produce from the respective monthly demand. The remaining demand, if any, is satisfied through hiring and firing of temps. From the standpoint of the model, the net demand for each month is

$$\text{Demand for March} = 520 - 12 \times 10 = 400 \text{ units}$$

$$\text{Demand for April} = 720 - 12 \times 10 = 600 \text{ units}$$

$$\text{Demand for May} = 520 - 12 \times 10 = 400 \text{ units}$$

$$\text{Demand for June} = 620 - 12 \times 10 = 500 \text{ units}$$

For $i = 1, 2, 3, 4$, the variables of the model can be defined as

x_i = Net number of temps at the start of month i *after* any hiring or firing

S_i = Number of temps hired or fired at the start of month i

I_i = Units of ending inventory for month i

The variables x_i and I_i, by definition, must assume nonnegative values. On the other hand, the variable S_i can be positive when new temps are hired, negative when workers are fired, and zero if no hiring or firing occurs. As a result, the variable must be *unrestricted in sign*. This is the first instance in this chapter of using an unrestricted variable. As we will see shortly, special substitution is needed to allow the implementation of hiring and firing in the model.

The objective is to minimize the sum of the cost of hiring and firing plus the cost of holding inventory from one month to the next. The treatment of the inventory cost is similar to the one given in Example 2.3-5—namely,

$$\text{Inventory holding cost} = 50(I_1 + I_2 + I_3)$$

(Note that $I_4 = 0$ in the optimum solution.) The cost of hiring and firing is a bit more involved. We know that in any optimum solution, at least 40 temps $\left(= \frac{400}{10}\right)$ must be hired at the start of March to meet the month's demand. However, rather than treating this situation as a special case, we can let the optimization process take care of it automatically. Thus, given that the costs of hiring and firing a temp are \$200 and \$400, respectively, we have

$$\begin{pmatrix}\text{Cost of hiring}\\ \text{and firing}\end{pmatrix} = 200\begin{pmatrix}\text{Number of hired temps}\\ \text{at the start of}\\ \text{March, April, May, and June}\end{pmatrix}$$

$$+ 400\begin{pmatrix}\text{Number of fired temps}\\ \text{at the start of}\\ \text{March, April, May, and June}\end{pmatrix}$$

To translate this equation mathematically, we will need to develop the constraints first.

The constraints of the model deal with inventory and hiring and firing. First we develop the inventory constraints. Defining x_i as the number of temps available in month i and given that the productivity of a temp is 10 units per month, the number of units produced in the same month is $10x_i$. Thus the inventory constraints are

$$\begin{aligned}
10x_1 &= 400 + I_1 \quad \text{(March)}\\
I_1 + 10x_2 &= 600 + I_2 \quad \text{(April)}\\
I_2 + 10x_3 &= 400 + I_3 \quad \text{(May)}\\
I_3 + 10x_4 &= 500 \qquad\quad \text{(June)}\\
x_1, x_2, x_3, x_4 &\geq 0, I_1, I_2, I_3 \geq 0
\end{aligned}$$

Next, we develop the constraints dealing with hiring and firing. First, note that the temp workforce starts with x_1 workers at the beginning of March. At the start of April, x_1 will be adjusted (up or down) by S_2 to generate x_2. The same idea applies to x_3 and x_4. These observations lead to the following equations

$$\begin{aligned}
x_1 &= S_1\\
x_2 &= x_1 + S_2\\
x_3 &= x_2 + S_3
\end{aligned}$$

$$x_4 = x_3 + S_4$$
$$S_1, S_2, S_3, S_4 \text{ unrestricted in sign}$$
$$x_1, x_2, x_3, x_4 \geq 0$$

The variables S_1, S_2, S_3, and S_4 represent hiring when they are strictly positive and firing when they are strictly negative. However, this "qualitative" information cannot be used in a mathematical expression. Instead, we use the following substitution:

$$S_i = S_i^- - S_i^+, \text{ where } S_i^-, S_i^+ \geq 0$$

The unrestricted variable S_i is now the difference between two nonnegative variables S_i^- and S_i^+. We can think of S_i^- as the number of temps hired and S_i^+ as the number of temps fired. For example, if $S_i^- = 5$ and $S_i^+ = 0$ then $S_i = 5 - 0 = +5$, which represents hiring. If $S_i^- = 0$ and $S_i^+ = 7$ then $S_i = 0 - 7 = -7$, which represents firing. In the first case, the corresponding cost of hiring is $200S_i^- = 200 \times 5 = \$1000$ and in the second case the corresponding cost of firing is $400S_i^+ = 400 \times 7 = \$2800$. This idea is the basis for the development of the objective function.

First we need to address an important point: What if both S_i^- and S_i^+ are positive? The answer is that this cannot happen because it implies that the solution calls for both hiring and firing in the same month. Interestingly, the theory of linear programming (see Chapter 7) tells us that S_i^- and S_i^+ can never be positive simultaneously, a result that confirms intuition.

We can now write the cost of hiring and firing as follows:

$$\text{Cost of hiring} = 200(S_1^- + S_2^- + S_3^- + S_4^-)$$
$$\text{Cost of firing} = 400(S_1^+ + S_2^+ + S_3^+ + S_4^+)$$

The complete model is

$$\text{Minimize } z = 50(I_1 + I_2 + I_3 + I_4) + 200(S_1^- + S_2^- + S_3^- + S_4^-) + 400(S_1^+ + S_2^+ + S_3^+ + S_4^+)$$

subject to

$$10x_1 = 400 + I_1$$
$$I_1 + 10x_2 = 600 + I_2$$
$$I_2 + 10x_3 = 400 + I_3$$
$$I_3 + 10x_4 = 500$$
$$x_1 = S_1^- - S_1^+$$
$$x_2 = x_1 + S_2^- - S_2^+$$
$$x_3 = x_2 + S_3^- - S_3^+$$
$$x_4 = x_3 + S_4^- - S_4^+$$
$$S_1^-, S_1^+, S_2^-, S_2^+, S_3^-, S_3^+, S_4^-, S_4^+ \geq 0$$
$$x_1, x_2, x_3, x_4 \geq 0$$
$$I_1, I_2, I_3 \geq 0$$

Solution:

The optimum solution is $z = \$19{,}500$, $x_1 = 50, x_2 = 50$, $x_3 = 45, x_4 = 45$, $S_1^- = 50$, $S_3^+ = 5$, $I_1 = 100$, $I_3 = 50$. All the remaining variables are zero. The solution calls for hiring 50 temps in March ($S_1^- = 50$) and holding the workforce steady till May, when 5 temps are fired ($S_3^+ = 5$). No further hiring or firing is recommended until the end of June, when, presumably, all temps are terminated. This solution requires 100 units of inventory to be carried into May and 50 units to be carried into June.

PROBLEM SET 2.3D

1. Toolco has contracted with AutoMate to supply their automotive discount stores with wrenches and chisels. AutoMate's weekly demand consists of at least 1500 wrenches and 1200 chisels. Toolco cannot produce all the requested units with its present one-shift capacity and must use overtime and possibly subcontract with other tool shops. The result is an increase in the production cost per unit, as shown in the following table. Market demand restricts the ratio of chisels to wrenches to at least 2:1.

Tool	Production type	Weekly production range (units)	Unit cost ($)
Wrenches	Regular	0–550	2.00
	Overtime	551–800	2.80
	Subcontracting	$801-\infty$	3.00
Chisel	Regular	0–620	2.10
	Overtime	621–900	3.20
	Subcontracting	$901-\infty$	4.20

   (a) Formulate the problem as a linear program, and determine the optimum production schedule for each tool.

   (b) Relate the fact that the production cost function has increasing unit costs to the validity of the model.

2. Four products are processed sequentially on three machines. The following table gives the pertinent data of the problem.

Machine	Cost per hr ($)	Manufacturing time (hr) per unit: *Product 1*	*Product 2*	*Product 3*	*Product 4*	Capacity (hr)
1	10	2	3	4	2	500
2	5	3	2	1	2	380
3	4	7	3	2	1	450
Unit selling price ($)		75	70	55	45	

Formulate the problem as an LP model, and find the optimum solution.

*3. A manufacturer produces three models, I, II, and III, of a certain product using raw materials A and B. The following table gives the data for the problem:

	Requirements per unit			
Raw material	*I*	*II*	*III*	Availability
A	2	3	5	4000
B	4	2	7	6000
Minimum demand	200	200	150	
Profit per unit($)	30	20	50	

The labor time per unit of model I is twice that of II and three times that of III. The entire labor force of the factory can produce the equivalent of 1500 units of model I. Market requirements specify the ratios 3:2:5 for the production of the three respective models. Formulate the problem as a linear program, and find the optimum solution.

4. The demand for ice cream during the three summer months (June, July, and August) at All-Flavors Parlor is estimated at 500, 600, and 400 20-gallon cartons, respectively. Two wholesalers, 1 and 2, supply All-Flavors with its ice cream. Although the flavors from the two suppliers are different, they are interchangeable. The maximum number of cartons either supplier can provide is 400 per month. Also, the prices the two suppliers charge change from one month to the next according to the following schedule:

	Price per carton in month		
	June	*July*	*August*
Supplier 1	\$100	\$110	\$120
Supplier 2	\$115	\$108	\$125

To take advantage of price fluctuation, All-Flavors can purchase more than is needed for a month and store the surplus to satisfy the demand in a later month. The cost of refrigerating an ice cream carton is \$5 per month. It is realistic in the present situation to assume that the refrigeration cost is a function of the average number of cartons on hand during the month. Develop an optimum schedule for buying ice cream from the two suppliers.

5. The demand for an item over the next four quarters is 300, 400, 450, and 250 units, respectively. The price per unit starts at \$20 in the first quarter and increases by \$2 each quarter thereafter. The supplier can provide no more than 400 units in any one quarter. Although we can take advantage of lower prices in early quarters, a storage cost of \$3.50 is incurred per unit per quarter. In addition, the maximum number of units that can be held over from one quarter to the next cannot exceed 100. Develop an optimum schedule for purchasing the item to meet the demand.

6. A company has contracted to produce two products, A and B, over the months of June, July, and August. The total production capacity (expressed in hours) varies monthly. The following table provides the basic data of the situation:

	June	July	August
Demand for *A* (units)	500	5000	750
Demand for *B* (units)	1000	1200	1200
Capacity (hours)	3000	3500	3000

The production rates in units per hour are 1.25 and 1 for products A and B, respectively. All demand must be met. However, demand for a later month may be filled from the production in an earlier one. For any carryover from one month to the next, holding costs of \$.90 and \$.75 per unit per month are charged for products A and B, respectively. The unit production costs for the two products are \$30 and \$28 for A and B, respectively. Determine the optimum production schedule for the two products.

*7. The manufacturing process of a product consists of two successive operations, I and II. The following table provides the pertinent data over the months of June, July, and August:

	June	July	August
Finished product demand (units)	500	450	600
Capacity of operation I (hr)	800	700	550
Capacity of operation II (hr)	1000	850	700

Producing a unit of the product takes .6 hour on operation I plus .8 hour on operation II. Overproduction of either the semifinished product (operation I) or the finished product (operation II) in any month is allowed for use in a later month. The corresponding holding costs are \$.20 and \$.40 per unit per month. The production cost varies by operation and by month. For operation 1, the unit production cost is \$10, \$12, and \$11 for June, July, and August. For operation 2, the corresponding unit production cost is \$15, \$18, and \$16. Determine the optimal production schedule for the two operations over the 3-month horizon.

8. Two products are manufactured sequentially on two machines. The time available on each machine is 8 hours per day and may be increased by up to 4 hours of overtime, if necessary, at an additional cost of \$100 per hour. The table below gives the production rate on the two machines as well as the price per unit of the two products. Determine the optimum production schedule and the recommended use of overtime, if any.

	Production rate (units/hr)	
	Product 1	*Product 2*
Machine 1	5	5
Machine 2	8	4
Price per unit (\$)	110	118

2.3.5 Blending and Refining

A number of LP applications deal with blending different input materials to produce products that meet certain specifications while minimizing cost or maximizing profit. The input materials could be ores, metal scraps, chemicals, or crude oils and the output products could be metal ingots, paints, or gasoline of various grades. This section presents a (simplified) model for oil refining. The process starts with distilling crude oil to produce intermediate gasoline stocks and then blending these stocks to produce final gasolines. The final products must satisfy certain quality specifications (such as octane rating). In addition, distillation capacities and demand limits can directly affect the level of production of the different grades of gasoline. One goal of the model is determine the optimal mix of final products that will maximize an appropriate profit function. In some cases, the goal may be to minimize a cost function.

Example 2.3-7 (Crude Oil Refining and Gasoline Blending)

Shale Oil, located on the island of Aruba, has a capacity of 1,500,000 bbl of crude oil per day. The final products from the refinery include three types of unleaded gasoline with different octane numbers (ON): regular with ON = 87, premium with ON = 89, and super with ON = 92. The refining process encompasses three stages: (1) a distillation tower that produces feedstock (ON = 82) at the rate of .2 bbl per bbl of crude oil, (2) a cracker unit that produces gasoline stock (ON = 98) by using a portion of the feedstock produced from the distillation tower at the rate of .5 bbl per bbl of feedstock, and (3) a blender unit that blends the gasoline stock from the cracker unit and the feedstock from the distillation tower. The company estimates the net profit per barrel of the three types of gasoline to be \$6.70, \$7.20, and \$8.10, respectively. The input capacity of the cracker unit is 200,000 barrels of feedstock a day. The demand limits for regular, premium, and super gasoline are 50,000, 30,000, and 40,000 barrels per day. Develop a model for determining the optimum production schedule for the refinery.

Mathematical Model: Figure 2.7 summarizes the elements of the model. The variables can be defined in terms of two input streams to the blender (feedstock and cracker gasoline) and the three final products. Let

$$x_{ij} = \text{bbl/day of input stream } i \text{ used to blend final product } j,\ i = 1, 2;\ j = 1, 2, 3$$

Using this definition, we have

$$\text{Daily production of regular gasoline} = x_{11} + x_{21} \text{ bbl/day}$$
$$\text{Daily production of premium gasoline} = x_{12} + x_{22} \text{ bbl/day}$$
$$\text{Daily production of super gasoline} = x_{13} + x_{23} \text{ bbl/day}$$

$$\begin{pmatrix}\text{Daily output}\\ \text{of blender unit}\end{pmatrix} = \begin{pmatrix}\text{Daily production}\\ \text{of regular gas}\end{pmatrix} + \begin{pmatrix}\text{Daily production}\\ \text{of premium gas}\end{pmatrix}$$
$$+ \begin{pmatrix}\text{Daily production}\\ \text{of super gas}\end{pmatrix}$$
$$= (x_{11} + x_{21}) + (x_{12} + x_{22}) + (x_{13} + x_{23}) \text{bbl/day}$$

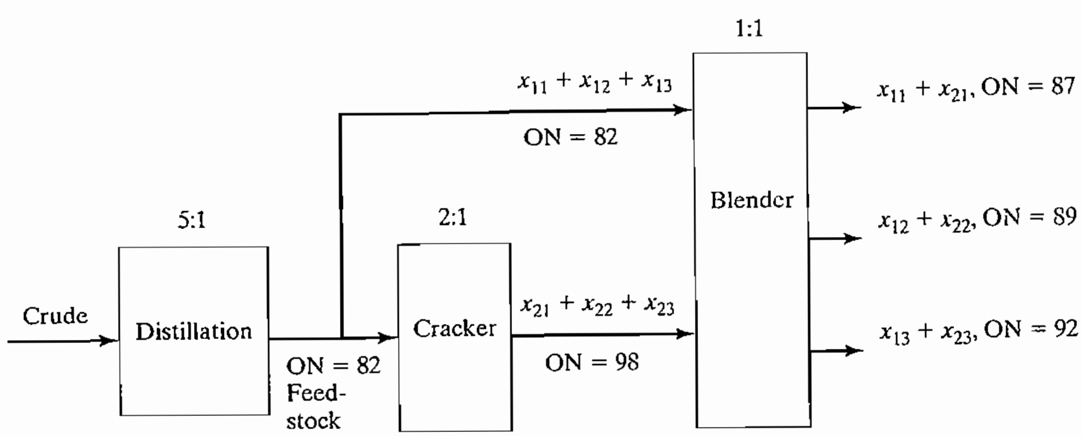


FIGURE 2.7

Product flow in the refinery problem

$$\begin{pmatrix}\text{Daily feedstock}\\ \text{to blender}\end{pmatrix} = x_{11} + x_{12} + x_{13} \text{ bbl/day}$$

$$\begin{pmatrix}\text{Daily cracker unit}\\ \text{feed to blender}\end{pmatrix} = x_{21} + x_{22} + x_{23} \text{ bbl/day}$$

$$\begin{pmatrix}\text{Daily feedstock}\\ \text{to cracker}\end{pmatrix} = 2(x_{21} + x_{22} + x_{23})\text{bbl/day}$$

$$\begin{pmatrix}\text{Daily crude oil used}\\ \text{in the refinery}\end{pmatrix} = 5(x_{11} + x_{12} + x_{13}) + 10(x_{21} + x_{22} + x_{23})\text{bbl/day}$$

The objective of the model is to maximize the total profit resulting from the sale of all three grades of gasoline. From the definitions given above, we get

$$\text{Maximize } z = 6.70(x_{11} + x_{21}) + 7.20(x_{12} + x_{22}) + 8.10(x_{13} + x_{23})$$

The constraints of the problem are developed as follows:

1. *Daily crude oil supply does not exceed 1,500,000 bbl/day:*

$$5(x_{11} + x_{12} + x_{13}) + 10(x_{21} + x_{22} + x_{23}) \leq 1{,}500{,}000$$

2. *Cracker unit input capacity does not exceed 200,000 bbl/day:*

$$2(x_{21} + x_{22} + x_{23}) \leq 200{,}000$$

3. *Daily demand for regular does not exceed 50,000 bbl:*

$$x_{11} + x_{21} \leq 50{,}000$$

4. *Daily demand for premium does not exceed 30,000:*

$$x_{12} + x_{22} \leq 30{,}000$$

5. *Daily demand for super does not exceed 40,000 bbl:*

$$x_{13} + x_{23} \leq 40{,}000$$

6. *Octane number (ON) for regular is at least 87:*

   The octane number of a gasoline product is the weighted average of the octane numbers of the input streams used in the blending process and can be computed as

$$\begin{pmatrix}\text{Average ON of}\\ \text{regular gasoline}\end{pmatrix} = \frac{\text{Feedstock ON} \times \text{feedstock bbl/day} + \text{Cracker unit ON} \times \text{Cracker unit bbl/day}}{\text{Total bbl/day of regular gasoline}}$$

$$= \frac{82x_{11} + 98x_{21}}{x_{11} + x_{21}}$$

Thus, octane number constraint for regular gasoline becomes

$$\frac{82x_{11} + 98x_{21}}{x_{11} + x_{21}} \geq 87$$

The constraint is linearized as

$$82x_{11} + 98x_{21} \geq 87(x_{11} + x_{21})$$

7. *Octane number (ON) for premium is at least 89:*

$$\frac{82x_{12} + 98x_{22}}{x_{12} + x_{22}} \geq 89$$

which is linearized as

$$82x_{12} + 98x_{22} \geq 89(x_{12} + x_{22})$$

8. *Octane number (ON) for super is at least 92:*

$$\frac{82x_{13} + 98x_{23}}{x_{13} + x_{23}} \geq 92$$

or

$$82x_{13} + 98x_{23} \geq 92(x_{13} + x_{23})$$

The complete model is thus summarized as

$$\text{Maximize } z = 6.70(x_{11} + x_{21}) + 7.20(x_{12} + x_{22}) + 8.10(x_{13} + x_{23})$$

subject to

$$5(x_{11} + x_{12} + x_{13}) + 10(x_{21} + x_{22} + x_{23}) \leq 1{,}500{,}000$$
$$2(x_{21} + x_{22} + x_{23}) \leq 200{,}000$$
$$x_{11} + x_{21} \leq 50{,}000$$
$$x_{12} + x_{22} \leq 30{,}000$$
$$x_{13} + x_{23} \leq 40{,}000$$
$$82x_{11} + 98x_{21} \geq 87(x_{11} + x_{21})$$
$$82x_{12} + 98x_{22} \geq 89(x_{12} + x_{22})$$
$$82x_{13} + 98x_{23} \geq 92(x_{13} + x_{23})$$
$$x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23} \geq 0$$

The last three constraints can be simplified to produce a constant right-hand side.

Solution:

The optimum solution (using file amplEx2.3-7.txt) is $z = 1{,}482{,}000$, $x_{11} = 20{,}625$, $x_{21} = 9375$, $x_{12} = 16{,}875$, $x_{22} = 13{,}125$, $x_{13} = 15{,}000$, $x_{23} = 25{,}000$. This translates to

Daily profit = \$1,482,000
Daily amount of regular gasoline = $x_{11} + x_{21} = 20{,}625 + 9375 = 30{,}000$ bbl/day
Daily amount of premium gasoline = $x_{12} + x_{22} = 16{,}875 + 13{,}125 = 30{,}000$ bbl/day
Daily amount of regular gasoline = $x_{13} + x_{23} = 15{,}000 + 25{,}000 = 40{,}000$ bbl/day

The solution shows that regular gasoline production is 20,000 bbl/day short of satisfying the maximum demand. The demand for the remaining two grades is satisfied.

PROBLEM SET 2.3E

1. Hi-V produces three types of canned juice drinks, *A*, *B*, and *C*, using fresh strawberries, grapes, and apples. The daily supply is limited to 200 tons of strawberries, 100 tons of grapes, and 150 tons of apples. The cost per ton of strawberries, grapes, and apples is $200, $100, and $90, respectively. Each ton makes 1500 lb of strawberry juice, 1200 lb of grape juice, and 1000 lb of apple juice. Drink *A* is a 1:1 mix of strawberry and apple juice. Drink *B* is 1:1:2 mix of strawberry, grape, and apple juice. Drink *C* is a 2:3 mix of grape and apple juice. All drinks are canned in 16-oz (1 lb) cans. The price per can is $1.15, $1.25, and $1.20 for drinks *A*, *B*, and *C*. Determine the optimal production mix of the three drinks.

*2. A hardware store packages handyman bags of screws, bolts, nuts, and washers. Screws come in 100-lb boxes and cost $110 each, bolts come in 100-lb boxes and cost $150 each, nuts come in 80-lb boxes and cost $70 each, and washers come in 30-lb boxes and cost $20 each. The handyman package weighs at least 1 lb and must include, by weight, at least 10% screws and 25% bolts, and at most 15% nuts and 10% washers. To balance the package, the number of bolts cannot exceed the number of nuts or the number of washers. A bolt weighs 10 times as much as a nut and 50 times as much as a washer. Determine the optimal mix of the package.

3. All-Natural Coop makes three breakfast cereals, *A*, *B*, and *C*, from four ingredients: rolled oats, raisins, shredded coconuts, and slivered almonds. The daily availabilities of the ingredients are 5 tons, 2 tons, 1 ton, and 1 ton, respectively. The corresponding costs per ton are $100, $120, $110, and $200. Cereal *A* is a 50:5:2 mix of oats, raisins, and almond. Cereal *B* is a 60:2:3 mix of oats, coconut, and almond. Cereal *C* is a 60:3:4:2 mix of oats, raisins, coconut, and almond. The cereals are produced in jumbo 5-lb sizes. All-Natural sells *A*, *B*, and *C* at $2, $2.50, and $3.00 per box, respectively. The minimum daily demand for cereals *A*, *B*, and *C* is 500, 600, and 500 boxes. Determine the optimal production mix of the cereals and the associated amounts of ingredients.

4. A refinery manufactures two grades of jet fuel, *F1* and *F2*, by blending four types of gasoline, *A*, *B*, *C*, and *D*. Fuel *F1* uses gasolines *A*, *B*, *C*, and *D* in the ratio 1:1:2:4, and fuel *F2* uses the ratio 2:2:1:3. The supply limits for *A*, *B*, *C*, and *D* are 1000, 1200, 900, and 1500 bbl/day, respectively. The costs per bbl for gasolines *A*, *B*, *C*, and *D* are $120, $90, $100, and $150, respectively. Fuels *F1* and *F2* sell for $200 and $250 per bbl. The minimum demand for *F1* and *F2* is 200 and 400 bbl/day. Determine the optimal production mix for *F1* and *F2*.

5. An oil company distills two types of crude oil, *A* and *B*, to produce regular and premium gasoline and jet fuel. There are limits on the daily availability of crude oil and the minimum demand for the final products. If the production is not sufficient to cover demand, the shortage must be made up from outside sources at a penalty. Surplus production will not be sold immediately and will incur storage cost. The following table provides the data of the situation:

	Fraction yield per bbl				
Crude	*Regular*	*Premium*	*Jet*	Price/bbl ($)	bbl/day
Crude A	.20	.1	.25	30	2500
Crude B	.25	.3	.10	40	3000
Demand (bbl/day)	500	700	400		
Revenue ($/bbl)	50	70	120		
Storage cost for surplus production ($/bbl)	2	3	4		
Penalty for unfilled demand ($/bbl)	10	15	20		

Determine the optimal product mix for the refinery.

6. In the refinery situation of Problem 5, suppose that the distillation unit actually produces the intermediate products naphtha and light oil. One bbl of crude A produces .35 bbl of naphtha and .6 bbl of light oil, and one bbl of crude B produces .45 bbl of naphtha and .5 bbl of light oil. Naphtha and light oil are blended to produce the three final gasoline products: One bbl of regular gasoline has a blend ratio of 2:1 (naphtha to light oil), one bbl of premium gasoline has a blend ratio of ratio of 1:1, and one bbl of jet fuel has a blend ratio of 1:2. Determine the optimal production mix.

7. Hawaii Sugar Company produces brown sugar, processed (white) sugar, powdered sugar, and molasses from sugar cane syrup. The company purchases 4000 tons of syrup weekly and is contracted to deliver at least 25 tons weekly of each type of sugar. The production process starts by manufacturing brown sugar and molasses from the syrup. A ton of syrup produces .3 ton of brown sugar and .1 ton of molasses. White sugar is produced by processing brown sugar. It takes 1 ton of brown sugar to produce .8 ton of white sugar. Powdered sugar is produced from white sugar through a special grinding process that has a 95% conversion efficiency (1 ton of white sugar produces .95 ton of powdered sugar). The profits per ton for brown sugar, white sugar, powdered sugar, and molasses are $150, $200, $230, and $35, respectively. Formulate the problem as a linear program, and determine the weekly production schedule.

8. Shale Oil refinery blends two petroleum stocks, A and B, to produce two high-octane gasoline products, I and II. Stocks A and B are produced at the maximum rates of 450 and 700 bbl/hour, respectively. The corresponding octane numbers are 98 and 89, and the vapor pressures are 10 and 8 lb/in^2. Gasoline I and gasoline II must have octane numbers of at least 91 and 93, respectively. The vapor pressure associated with both products should not exceed 12 lb/in^2. The profits per bbl of I and II are $7 and $10, respectively. Determine the optimum production rate for I and II and their blend ratios from stocks A and B. (*Hint:* Vapor pressure, like the octane number, is the weighted average of the vapor pressures of the blended stocks.)

9. A foundry smelts steel, aluminum, and cast iron scraps to produce two types of metal ingots, I and II, with specific limits on the aluminum, graphite and silicon contents. Aluminum and silicon briquettes may be used in the smelting process to meet the desired specifications. The following tables set the specifications of the problem: 2.3

	Contents (%)				
Input item	*Aluminum*	*Graphite*	*Silicon*	Cost/ton ($)	Available tons/day
Steel scrap	10	5	4	100	1000
Aluminum scrap	95	1	2	150	500
Cast iron scrap	0	15	8	75	2500
Aluminum briquette	100	0	0	900	Any amount
Silicon briquette	0	0	100	380	Any amount

	Ingot I		Ingot II	
Ingredient	*Minimum*	*Maximum*	*Minimum*	*Maximum*
Aluminum	8.1%	10.8%	6.2%	8.9%
Graphite	1.5%	3.0%	4.1%	∞
Silicon	2.5%	∞	2.8%	4.1%
Demand (tons/day)	130		250	

Determine the optimal input mix the foundry should smelt.

10. Two alloys, A and B, are made from four metals, I, II, III, and IV, according to the following specifications:

Alloy	Specifications	Selling price ($)
A	At most 80% of I	200
	At most 30% of II	
	At least 50% of IV	
B	Between 40% and 60% of II	300
	At least 30% of III	
	At most 70% of IV	

The four metals, in turn, are extracted from three ores according to the following data:

		Constituents (%)					
Ore	Maximum quantity (tons)	*I*	*II*	*III*	*IV*	*Others*	Price/ton ($)
1	1000	20	10	30	30	10	30
2	2000	10	20	30	30	10	40
3	3000	5	5	70	20	0	50

How much of each type of alloy should be produced? (*Hint:* Let x_{kj} be tons of ore i allocated to alloy k, and define w_k as tons of alloy k produced.)

2.3.6 Manpower Planning

Fluctuations in a labor force to meet variable demand over time can be achieved through the process of hiring and firing, as demonstrated in Example 2.3-6. There are situations in which the effect of fluctuations in demand can be "absorbed" by adjusting the start and end times of a work shift. For example, instead of following the traditional three 8-hour-shift start times at 8:00 A.M., 3:00 P.M., and 11:00 P.M., we can use overlapping 8-hour shifts in which the start time of each is made in response to increase or decrease in demand.

The idea of redefining the start of a shift to accommodate fluctuation in demand can be extended to other operating environments as well. Example 2.3-8 deals with the determination of the minimum number of buses needed to meet rush-hour and off-hour transportation needs.

Real-Life Application—Telephone Sales Manpower Planning at Qantas Airways

Australian airline Qantas operates its main reservation offices from 7:00 till 22:00 using 6 shifts that start at different times of the day. Qantas used linear programming (with imbedded queuing analysis) to staff its main telephone sales reservation office efficiently while providing convenient service to its customers. The study, carried out in the late 1970s, resulted in annual savings of over 200,000 Australian dollars per year. The study is detailed in Case 15, Chapter 24 on the CD.

Example 2.3-8 (Bus Scheduling)

Progress City is studying the feasibility of introducing a mass-transit bus system that will alleviate the smog problem by reducing in-city driving. The study seeks the minimum number of buses that can handle the transportation needs. After gathering necessary information, the city engineer noticed that the minimum number of buses needed fluctuated with the time of the day and that the required number of buses could be approximated by constant values over successive 4-hour intervals. Figure 2.8 summarizes the engineer's findings. To carry out the required daily maintenance, each bus can operate 8 successive hours a day only.

Mathematical Model: Determine the number of operating buses in each shift (variables) that will meet the minimum demand (constraints) while minimizing the total number of buses in operation (objective).

You may already have noticed that the definition of the variables is ambiguous. We know that each bus will run for 8 consecutive hours, but we do not know when a shift should start. If we follow a normal three-shift schedule (8:01 A.M.-4:00 P.M., 4:01 P.M.-12:00 midnight, and 12:01 A.M.-8:00 A.M.) and assume that x_1, x_2, and x_3 are the number of buses starting in the first, second, and third shifts, we can see from Figure 2.8 that $x_1 \geq 10$, $x_2 \geq 12$, and $x_3 \geq 8$. The corresponding minimum number of daily buses is $x_1 + x_2 + x_3 = 10 + 12 + 8 = 30$.

The given solution is acceptable only if the shifts *must* coincide with the normal three-shift schedule. It may be advantageous, however, to allow the optimization process to choose the "best" starting time for a shift. A reasonable way to accomplish this is to allow a shift to start every 4 hours. The bottom of Figure 2.8 illustrates this idea where overlapping 8-hour shifts

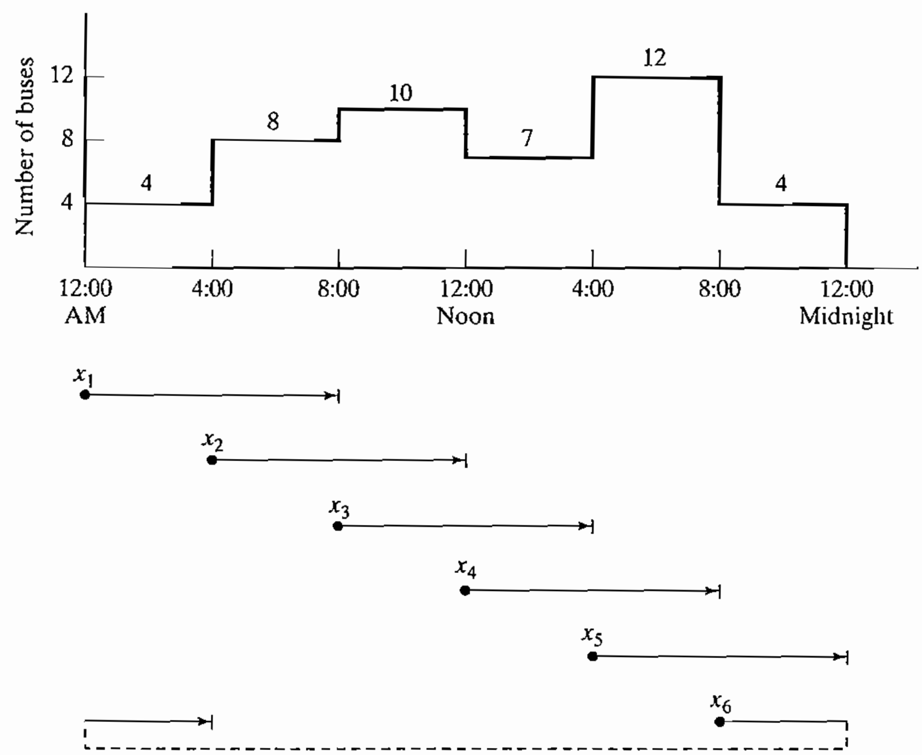


FIGURE 2.8

Number of buses as a function of the time of the day may start at 12:01 A.M., 4:01 A.M., 8:01 A.M., 12:01 P.M., 4:01 P.M., and 8:01 P.M. Thus, the variables may be defined as

x_1 = number of buses starting at 12: 01 A.M.

x_2 = number of buses starting at 4:01 A.M.

x_3 = number of buses starting at 8:01 A.M.

x_4 = number of buses starting at 12:01 P.M.

x_5 = number of buses starting at 4:01 P.M.

x_6 = number of buses starting at 8:01 P.M.

We can see from Figure 2.8 that because of the overlapping of the shifts, the number of buses for the successive 4-hour periods is given as

Time period	Number of buses in operation
12:01 A.M. – 4:00 A.M.	$x_1 + x_6$
4:01 A.M. – 8:00 A.M.	$x_1 + x_2$
8:01 A.M. – 12:00 noon	$x_2 + x_3$
12:01 P.M. – 4:00 P.M.	$x_3 + x_4$
4:01 P.M. – 8:00 P.M.	$x_4 + x_5$
8:01 A.M. – 12:00 A.M.	$x_5 + x_6$

The complete model is thus written as

$$\text{Minimize } z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$$

subject to

$$\begin{aligned}
x_1 \qquad\qquad\qquad\qquad + x_6 &\geq 4 \text{ (12:01 A.M.-4:00 A.M.)}\\
x_1 + x_2 &\geq 8 \text{ (4:01 A.M.-8:00 A.M.)}\\
x_2 + x_3 &\geq 10 \text{ (8:01 A.M.-12:00 noon)}\\
x_3 + x_4 &\geq 7 \text{ (12:01 P.M.-4:00 P.M.)}\\
x_4 + x_5 &\geq 12 \text{ (4:01 P.M.-8:00 P.M.)}\\
x_5 + x_6 &\geq 4 \text{ (8:01 P.M.-12:00 P.M.)}\\
x_j \geq 0, j &= 1, 2, \ldots, 6
\end{aligned}$$

Solution:

The optimal solution calls for using 26 buses to satisfy the demand with $x_1 = 4$ buses to start at 12:01 A.M., $x_2 = 10$ at 4:01 A.M., $x_4 = 8$ at 12:01 P.M., and $x_5 = 4$ at 4:01 P.M.

PROBLEM SET 2.3F

*1. In the bus scheduling example suppose that buses can run either 8- or 12-hour shifts. If a bus runs for 12 hours, the driver must be paid for the extra hours at 150% of the regular hourly pay. Do you recommend the use of 12-hour shifts?

2. A hospital employs volunteers to staff the reception desk between 8:00 A.M. and 10:00 P.M. Each volunteer works three consecutive hours except for those starting at 8:00 P.M. who work for two hours only. The minimum need for volunteers is approximated by a step function over 2-hour intervals starting at 8:00 A.M. as 4, 6, 8, 6, 4, 6, 8. Because most volunteers are retired individuals, they are willing to offer their services at any hour of the day (8:00 A.M. to 10:00 P.M.). However, because of the large number of charities competing for their service, the number needed must be kept as low as possible. Determine an optimal schedule for the start time of the volunteers
3. In Problem 2, suppose that no volunteers will start at noon or 6:00 P.M. to allow for lunch and dinner. Determine the optimal schedule.
4. In an LTL (less-than-truckload) trucking company, terminal docks include *casual* workers who are hired temporarily to account for peak loads. At the Omaha, Nebraska, dock, the minimum demand for casual workers during the seven days of the week (starting on Monday) is 20, 14, 10, 15, 18, 10, 12 workers. Each worker is contracted to work five consecutive days. Determine an optimal weekly hiring practice of casual workers for the company.

*5. On most university campuses students are contracted by academic departments to do errands, such as answering the phone and typing. The need for such service fluctuates during work hours (8:00 A.M. to 5:00 P.M.). In the IE department, the minimum number of students needed is 2 between 8:00 A.M. and 10:00 A.M., 3 between 10:01 A.M. and 11:00 A.M., 4 between 11:01 A.M. and 1:00 P.M., and 3 between 1:01 P.M. and 5:00 P.M. Each student is allotted 3 consecutive hours (except for those starting at 3:01, who work for 2 hours and those who start at 4:01, who work for one hour). Because of their flexible schedule, students can usually report to work at any hour during the work day, except that no student wants to start working at lunch time (12:00 noon). Determine the minimum number of students the IE department should employ and specify the time of the day at which they should report to work.

6. A large department store operates 7 days a week. The manager estimates that the minimum number of salespersons required to provide prompt service is 12 for Monday, 18 for Tuesday, 20 for Wednesday, 28 for Thursday, 32 for Friday, and 40 for each of Saturday and Sunday. Each salesperson works 5 days a week, with the two consecutive off-days staggered throughout the week. For example, if 10 salespersons start on Monday, two can take their off-days on Tuesday and Wednesday, five on Wednesday and Thursday, and three on Saturday and Sunday. How many salespersons should be contracted and how should their off-days be allocated?

2.3.7 Additional Applications

The preceding sections have demonstrated the application of LP to six representative areas. The fact is that LP enjoys diverse applications in an enormous number of areas. The problems at the end of this section demonstrate some of these areas, ranging from agriculture to military applications. This section also presents an interesting application that deals with cutting standard stocks of paper rolls to sizes specified by customers.

Example 2.3-9 (Trim Loss or Stock Slitting)

The Pacific Paper Company produces paper rolls with a standard width of 20 feet each. Special customer orders with different widths are produced by slitting the standard rolls. Typical orders (which may vary daily) are summarized in the following table: