$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}), \qquad \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}). \tag{1}$$

It is quite remarkable that here in complex, functions come together that are unrelated in real. This is not an isolated incident but is typical of the general situation and shows the advantage of working in complex.

Furthermore, as in calculus we define

$$\tan z = \frac{\sin z}{\cos z}, \qquad \cot z = \frac{\cos z}{\sin z} \tag{2}$$

and

$$\sec z = \frac{1}{\cos z}, \qquad \csc z = \frac{1}{\sin z}. \tag{3}$$

Since e^z is entire, $\cos z$ and $\sin z$ are entire functions. $\tan z$ and $\sec z$ are not entire; they are analytic except at the points where $\cos z$ is zero; and $\cot z$ and $\csc z$ are analytic except where $\sin z$ is zero. Formulas for the derivatives follow readily from $(e^z)' = e^z$ and (1)–(3); as in calculus,

$$(\cos z)' = -\sin z, \qquad (\sin z)' = \cos z, \qquad (\tan z)' = \sec^2 z, \tag{4}$$

etc. Equation (1) also shows that **Euler's formula** ***is valid in complex:***

$$e^{iz} = \cos z + i \sin z \qquad \text{for all } z. \tag{5}$$

The real and imaginary parts of $\cos z$ and $\sin z$ are needed in computing values, and they also help in displaying properties of our functions. We illustrate this with a typical example.

EXAMPLE 1 Real and Imaginary Parts. Absolute Value. Periodicity

Show that

$$\text{(6)} \quad \begin{aligned} &\text{(a)} \quad \cos z = \cos x \cosh y - i \sin x \sinh y \\ &\text{(b)} \quad \sin z = \sin x \cosh y + i \cos x \sinh y \end{aligned}$$

and

$$\text{(7)} \quad \begin{aligned} &\text{(a)} \quad |\cos z|^2 = \cos^2 x + \sinh^2 y \\ &\text{(b)} \quad |\sin z|^2 = \sin^2 x + \sinh^2 y \end{aligned}$$

and give some applications of these formulas.

Solution. From (1),

$$\begin{aligned} \cos z &= \tfrac{1}{2}(e^{i(x+iy)} + e^{-i(x+iy)}) \\ &= \tfrac{1}{2}e^{-y}(\cos x + i \sin x) + \tfrac{1}{2}e^{y}(\cos x - i \sin x) \\ &= \tfrac{1}{2}(e^{y} + e^{-y}) \cos x - \tfrac{1}{2}i(e^{y} - e^{-y}) \sin x. \end{aligned}$$

This yields (6a) since, as is known form calculus,

$$\cosh y = \tfrac{1}{2}(e^{y} + e^{-y}), \qquad \sinh y = \tfrac{1}{2}(e^{y} - e^{-y}); \tag{8}$$

(6b) is obtained similarly. From (6a) and $\cosh^2 y = 1 + \sinh^2 y$ we obtain

$$|\cos z|^2 = (\cos^2 x)(1 + \sinh^2 y) + \sin^2 x \sinh^2 y.$$

Since $\sin^2 x + \cos^2 x = 1$, this gives (7a), and (7b) is obtained similarly.

For instance, $\cos(2 + 3i) = \cos 2 \cosh 3 - i \sin 2 \sinh 3 = -4.190 - 9.109i$.

From (6) we see that $\cos z$ and $\sin z$ are ***periodic with period 2π***, just as in real. Periodicity of $\tan z$ and $\cot z$ with period π now follows.

Formula (7) points to an essential difference between the real and the complex cosine and sine; whereas $|\cos x| \leqq 1$ and $|\sin x| \leqq 1$, the complex cosine and sine functions are ***no longer bounded*** but approach infinity in absolute value as $y \to \infty$, since then $\sinh y \to \infty$ in (7). ■

EXAMPLE 2 Solutions of Equations. Zeros of cos z and sin z

Solve (a) $\cos z = 5$ (which has no real solution!), (b) $\cos z = 0$, (c) $\sin z = 0$.

Solution. (a) $e^{2iz} - 10e^{iz} + 1 = 0$ from (1) by multiplication by e^{iz}. This is a quadratic equation in e^{iz}, with solutions (rounded off to 3 decimals)

$$e^{iz} = e^{-y+ix} = 5 \pm \sqrt{25 - 1} = 9.899 \quad \text{and} \quad 0.101.$$

Thus $e^{-y} = 9.899$ or 0.101, $e^{ix} = 1$, $y = \pm 2.292$, $x = 2n\pi$. *Ans.* $z = \pm 2n\pi \pm 2.292i$ $(n = 0, 1, 2, \cdots)$.

Can you obtain this from (6a)?

(b) $\cos x = 0$, $\sinh y = 0$ by (7a), $y = 0$. *Ans.* $z = \pm\frac{1}{2}(2n + 1)\pi$ $(n = 0, 1, 2, \cdots)$.

(c) $\sin x = 0$, $\sinh y = 0$ by (7b). *Ans.* $z = \pm n\pi$ $(n = 0, 1, 2, \cdots)$. Hence the only zeros of $\cos z$ and $\sin z$ are those of the real cosine and sine functions. ■

General formulas *for the real trigonometric functions continue to hold for complex values.* This follows immediately from the definitions. We mention in particular the addition rules

$$\begin{aligned} \cos(z_1 \pm z_2) &= \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2 \\ \sin(z_1 \pm z_2) &= \sin z_1 \cos z_2 \pm \sin z_2 \cos z_1 \end{aligned} \tag{9}$$

and the formula

$$\cos^2 z + \sin^2 z = 1. \tag{10}$$

Some further useful formulas are included in the problem set.

Hyperbolic Functions

The complex **hyperbolic cosine** and **sine** are defined by the formulas

$$\cosh z = \tfrac{1}{2}(e^z + e^{-z}), \qquad \sinh z = \tfrac{1}{2}(e^z - e^{-z}). \tag{11}$$

This is suggested by the familiar definitions for a real variable [see (8)]. These functions are entire, with derivatives

$$(\cosh z)' = \sinh z, \qquad (\sinh z)' = \cosh z, \tag{12}$$

as in calculus. The other hyperbolic functions are defined by

$$\tanh z = \frac{\sinh z}{\cosh z}, \qquad \coth z = \frac{\cosh z}{\sinh z}, \tag{13}$$
$$\operatorname{sech} z = \frac{1}{\cosh z}, \qquad \operatorname{csch} z = \frac{1}{\sinh z}.$$

Complex Trigonometric and Hyperbolic Functions Are Related. If in (11), we replace z by iz and then use (1), we obtain

$$\cosh iz = \cos z, \qquad \sinh iz = i \sin z. \tag{14}$$

Similarly, if in (1) we replace z by iz and then use (11), we obtain conversely

$$\cos iz = \cosh z, \qquad \sin iz = i \sinh z. \tag{15}$$

Here we have another case of *unrelated* real functions that have *related* complex analogs, pointing again to the advantage of working in complex in order to get both a more unified formalism and a deeper understanding of special functions. This is one of the main reasons for the importance of complex analysis to the engineer and physicist.

[illegible]

1. Prove that $\cos z$, $\sin z$, $\cosh z$, $\sinh z$ are entire functions.
2. Verify by differentiation that $\operatorname{Re} \cos z$ and $\operatorname{Im} \sin z$ are harmonic.

3–6 FORMULAS FOR HYPERBOLIC FUNCTIONS

Show that

3. $$\cosh z = \cosh x \cos y + i \sinh x \sin y$$
$$\sinh z = \sinh x \cos y + i \cosh x \sin y.$$

4. $$\cosh (z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$$
$$\sinh (z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2.$$

5. $\cosh^2 z - \sinh^2 z = 1$
6. $\cosh^2 z + \sinh^2 z = \cosh 2z$

7–15 Function Values. Compute (in the form $u + iv$)

7. $\cos (1 + i)$
8. $\sin (1 + i)$
9. $\sin 5i$, $\cos 5i$
10. $\cos 3\pi i$
11. $\cosh (-2 + 3i)$, $\cos (-3 - 2i)$
12. $-i \sinh (-\pi + 2i)$, $\sin (2 + \pi i)$
13. $\cosh (2n + 1)\pi i$, $n = 1, 2, \cdots$
14. $\sinh (4 - 3i)$
15. $\cosh (4 - 6\pi i)$
16. **(Real and imaginary parts)** Show that
$$\operatorname{Re} \tan z = \frac{\sin x \cos x}{\cos^2 x + \sinh^2 y},$$
$$\operatorname{Im} \tan z = \frac{\sinh y \cosh y}{\cos^2 x + \sinh^2 y}.$$

17–21 Equations. Find all solutions of the following equations.

17. $\cosh z = 0$
18. $\sin z = 100$
19. $\cos z = 2i$
20. $\cosh z = -1$
21. $\sinh z = 0$
22. Find all z for which (a) $\cos z$, (b) $\sin z$ has real values.

23–25 Equations and Inequalities. Using the definitions, prove:

23. $\cos z$ is even, $\cos(-z) = \cos z$, and $\sin z$ is odd, $\sin(-z) = -\sin z$.
24. $|\sinh y| \leqq |\cos z| \leqq \cosh y$, $|\sinh y| \leqq |\sin z| \leqq \cosh y$. Conclude that the complex cosine and sine are not bounded in the whole complex plane.
25. $\sin z_1 \cos z_2 = \frac{1}{2}[\sin (z_1 + z_2) + \sin (z_1 - z_2)]$

13.7 Logarithm. General Power

We finally introduce the *complex logarithm*, which is more complicated than the real logarithm (which it includes as a special case) and historically puzzled mathematicians for some time (so if you first get puzzled—which need not happen!—be patient and work through this section with extra care).

The **natural logarithm** of $z = x + iy$ is denoted by $\ln z$ (sometimes also by $\log z$) and is defined as the inverse of the exponential function; that is, $w = \ln z$ is defined for $z \neq 0$ by the relation

$$e^w = z.$$

(Note that $z = 0$ is impossible, since $e^w \neq 0$ for all w; see Sec. 13.5.) If we set $w = u + iv$ and $z = re^{i\theta}$, this becomes

$$e^w = e^{u+iv} = re^{i\theta}.$$

Now from Sec. 13.5 we know that e^{u+iv} has the absolute value e^u and the argument v. These must be equal to the absolute value and argument on the right:

$$e^u = r, \qquad v = \theta.$$

$e^u = r$ gives $u = \ln r$, where $\ln r$ is the familiar *real* natural logarithm of the positive number $r = |z|$. Hence $w = u + iv = \ln z$ is given by

$$\text{(1)} \qquad \ln z = \ln r + i\theta \qquad (r = |z| > 0, \quad \theta = \arg z).$$

Now comes an important point (without analog in real calculus). Since the argument of z is determined only up to integer multiples of 2π, ***the complex natural logarithm*** $\ln z$ ***(***$z \neq 0$***) is infinitely many-valued.***

The value of $\ln z$ corresponding to the principal value $\operatorname{Arg} z$ (see Sec. 13.2) is denoted by $\operatorname{Ln} z$ (Ln with capital L) and is called the **principal value** of $\ln z$. Thus

$$\text{(2)} \qquad \operatorname{Ln} z = \ln |z| + i \operatorname{Arg} z \qquad (z \neq 0).$$

The uniqueness of $\operatorname{Arg} z$ for given z ($\neq 0$) implies that $\operatorname{Ln} z$ is single-valued, that is, a function in the usual sense. Since the other values of $\arg z$ differ by integer multiples of 2π, the other values of $\ln z$ are given by

$$\text{(3)} \qquad \ln z = \operatorname{Ln} z \pm 2n\pi i \qquad (n = 1, 2, \cdots).$$

They all have the same real part, and their imaginary parts differ by integer multiples of 2π.

If z is positive real, then $\operatorname{Arg} z = 0$, and $\operatorname{Ln} z$ becomes identical with the real natural logarithm known from calculus. If z is negative real (so that the natural logarithm of calculus is not defined!), then $\operatorname{Arg} z = \pi$ and

$$\operatorname{Ln} z = \ln |z| + \pi i \qquad (z \text{ negative real}).$$

From (1) and $e^{\ln r} = r$ for positive real r we obtain

$$e^{\ln z} = z \tag{4a}$$

as expected, but since $\arg(e^z) = y \pm 2n\pi$ is multivalued, so is

$$\ln(e^z) = z \pm 2n\pi i, \qquad n = 0, 1, \cdots. \tag{4b}$$

EXAMPLE 1 Natural Logarithm. Principal Value

$$\begin{aligned}
\ln 1 &= 0, \pm 2\pi i, \pm 4\pi i, \cdots & \quad \operatorname{Ln} 1 &= 0 \\
\ln 4 &= 1.386\,294 \pm 2n\pi i & \operatorname{Ln} 4 &= 1.386\,294 \\
\ln(-1) &= \pm\pi i, \pm 3\pi i, \pm 5\pi i, \cdots & \operatorname{Ln}(-1) &= \pi i \\
\ln(-4) &= 1.386\,294 \pm (2n+1)\pi i & \operatorname{Ln}(-4) &= 1.386\,294 + \pi i \\
\ln i &= \pi i/2, -3\pi/2, 5\pi i/2, \cdots & \operatorname{Ln} i &= \pi i/2 \\
\ln 4i &= 1.386\,294 + \pi i/2 \pm 2n\pi i & \operatorname{Ln} 4i &= 1.386\,294 + \pi i/2 \\
\ln(-4i) &= 1.386\,294 - \pi i/2 \pm 2n\pi i & \operatorname{Ln}(-4i) &= 1.386\,294 - \pi i/2 \\
\ln(3-4i) &= \ln 5 + i \arg(3-4i) & \operatorname{Ln}(3-4i) &= 1.609\,438 - 0.927\,295i \\
&= 1.609\,438 - 0.927\,295i \pm 2n\pi i & &\text{(Fig. 334)}
\end{aligned}$$

■

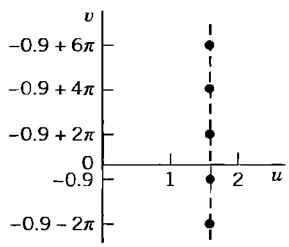

Fig. 334. Some values of $\ln(3 - 4i)$ in Example 1

The familiar relations for the natural logarithm continue to hold for complex values, that is,

$$\text{(a)}\quad \ln(z_1 z_2) = \ln z_1 + \ln z_2, \qquad \text{(b)}\quad \ln(z_1/z_2) = \ln z_1 - \ln z_2 \tag{5}$$

but these relations are to be understood in the sense that each value of one side is also contained among the values of the other side: see the next example.

EXAMPLE 2 Illustration of the Functional Relation (5) in Complex

Let

$$z_1 = z_2 = e^{\pi i} = -1.$$

If we take the principal values

$$\operatorname{Ln} z_1 = \operatorname{Ln} z_2 = \pi i,$$

then (5a) holds provided we write $\ln(z_1 z_2) = \ln 1 = 2\pi i$; however, it is not true for the principal value, $\operatorname{Ln}(z_1 z_2) = \operatorname{Ln} 1 = 0$. ■

THEOREM 1

Analyticity of the Logarithm

For every $n = 0, \pm 1, \pm 2, \cdots$ formula (3) defines a function, which is analytic, except at 0 and on the negative real axis, and has the derivative

$$(6) \qquad (\ln z)' = \frac{1}{z} \qquad (z \text{ not 0 or negative real}).$$

PROOF We show that the Cauchy–Riemann equations are satisfied. From (1)–(3) we have

$$\ln z = \ln r + i(\theta + c) = \frac{1}{2} \ln (x^2 + y^2) + i\left(\arctan \frac{y}{x} + c\right)$$

where the constant c is a multiple of 2π. By differentiation,

$$u_x = \frac{x}{x^2 + y^2} = v_y = \frac{1}{1 + (y/x)^2} \cdot \frac{1}{x}$$

$$u_y = \frac{y}{x^2 + y^2} = -v_x = -\frac{1}{1 + (y/x)^2}\left(-\frac{y}{x^2}\right).$$

Hence the Cauchy–Riemann equations hold. [Confirm this by using these equations in polar form, which we did not use since we proved them only in the problems (to Sec. 13.4).] Formula (4) in Sec. 13.4 now gives (6),

$$(\ln z)' = u_x + iv_x = \frac{x}{x^2 + y^2} + i\,\frac{1}{1 + (y/x)^2}\left(-\frac{y}{x^2}\right) = \frac{x - iy}{x^2 + y^2} = \frac{1}{z}. \qquad \blacksquare$$

Each of the infinitely many functions in (3) is called a **branch** of the logarithm. The negative real axis is known as a **branch cut** and is usually graphed as shown in Fig. 335. The branch for $n = 0$ is called the **principal branch** of $\ln z$.

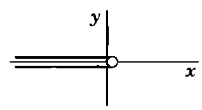

Fig. 335. Branch cut for ln z

General Powers

General powers of a complex number $z = x + iy$ are defined by the formula

$$(7) \qquad z^c = e^{c \ln z} \qquad (c \text{ complex}, z \neq 0).$$

Since $\ln z$ is infinitely many-valued, z^c will, in general, be multivalued. The particular value

$$z^c = e^{c \operatorname{Ln} z}$$

is called the **principal value** *of* z^c.