EXAMPLE 2 Cauchy–Riemann Equations. Exponential Function

Is $f(z) = u(x, y) + iv(x, y) = e^x(\cos y + i \sin y)$ analytic?

Solution. We have $u = e^x \cos y$, $v = e^x \sin y$ and by differentiation

$$u_x = e^x \cos y, \qquad v_y = e^x \cos y$$
$$u_y = -e^x \sin y, \qquad v_x = e^x \sin y.$$

We see that the Cauchy–Riemann equations are satisfied and conclude that $f(z)$ is analytic for all z. ($f(z)$ will be the complex analog of e^x known from calculus.) ■

EXAMPLE 3 An Analytic Function of Constant Absolute Value Is Constant

The Cauchy–Riemann equations also help in deriving general properties of analytic functions.

For instance, show that if $f(z)$ is analytic in a domain D and $|f(z)| = k = const$ in D, then $f(z) = const$ in D. (We shall make crucial use of this in Sec. 18.6 in the proof of Theorem 3.)

Solution. By assumption, $|f|^2 = |u + iv|^2 = u^2 + v^2 = k^2$. By differentiation,

$$uu_x + vv_x = 0,$$
$$uu_y + vv_y = 0.$$

Now use $v_x = -u_y$ in the first equation and $v_y = u_x$ in the second, to get

$$\text{(6)} \qquad \text{(a)} \quad uu_x - vu_y = 0, \qquad \text{(b)} \quad uu_y + vu_x = 0.$$

To get rid of u_y, multiply (6a) by u and (6b) by v and add. Similarly, to eliminate u_x, multiply (6a) by $-v$ and (6b) by u and add. This yields

$$(u^2 + v^2)u_x = 0,$$
$$(u^2 + v^2)u_y = 0.$$

If $k^2 = u^2 + v^2 = 0$, then $u = v = 0$; hence $f = 0$. If $k^2 = u^2 + v^2 \neq 0$, then $u_x = u_y = 0$. Hence, by the Cauchy–Riemann equations, also $v_x = v_y = 0$. Together this implies $u = const$ and $v = const$; hence $f = const$. ■

We mention that if we use the polar form $z = r(\cos\theta + i \sin\theta)$ and set $f(z) = u(r, \theta) + iv(r, \theta)$, then the **Cauchy–Riemann equations** are (Prob. 11)

$$\text{(7)} \qquad u_r = \frac{1}{r} v_\theta, \qquad v_r = -\frac{1}{r} u_\theta \qquad (r > 0).$$

Laplace's Equation. Harmonic Functions

The great importance of complex analysis in engineering mathematics results mainly from the fact that both the real part and the imaginary part of an analytic function satisfy Laplace's equation, the most important PDE of physics, which occurs in gravitation, electrostatics, fluid flow, heat conduction, and so on (see Chaps. 12 and 18).

THEOREM 3

Laplace's Equation

If $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then both u and v satisfy **Laplace's equation**

$$\nabla^2 u = u_{xx} + u_{yy} = 0 \tag{8}$$

(∇^2 read "nabla squared") *and*

$$\nabla^2 v = v_{xx} + v_{yy} = 0, \tag{9}$$

in D and have continuous second partial derivatives in D.

PROOF Differentiating $u_x = v_y$ with respect to x and $u_y = -v_x$ with respect to y, we have

$$u_{xx} = v_{yx}, \qquad u_{yy} = -v_{xy}. \tag{10}$$

Now the derivative of an analytic function is itself analytic, as we shall prove later (in Sec. 14.4). This implies that u and v have continuous partial derivatives of all orders; in particular, the mixed second derivatives are equal: $v_{yx} = v_{xy}$. By adding (10) we thus obtain (8). Similarly, (9) is obtained by differentiating $u_x = v_y$ with respect to y and $u_y = -v_x$ with respect to x and subtracting, using $u_{xy} = u_{yx}$. ■

Solutions of Laplace's equation having ***continuous*** second-order partial derivatives are called **harmonic functions** and their theory is called **potential theory** (see also Sec. 12.10). Hence the real and imaginary parts of an analytic function are harmonic functions.

If two harmonic functions u and v satisfy the Cauchy–Riemann equations in a domain D, they are the real and imaginary parts of an analytic function f in D. Then v is said to be a **harmonic conjugate function** of u in D. (Of course, this has absolutely nothing to do with the use of "conjugate" for $\bar{z}$.)

EXAMPLE 4 **How to Find a Harmonic Conjugate Function by the Cauchy–Riemann Equations**

Verify that $u = x^2 - y^2 - y$ is harmonic in the whole complex plane and find a harmonic conjugate function v of u.

Solution. $\nabla^2 u = 0$ by direct calculation. Now $u_x = 2x$ and $u_y = -2y - 1$. Hence because of the Cauchy–Riemann equations a conjugate v of u must satisfy

$$v_y = u_x = 2x, \qquad v_x = -u_y = 2y + 1.$$

Integrating the first equation with respect to y and differentiating the result with respect to x, we obtain

$$v = 2xy + h(x), \qquad v_x = 2y + \frac{dh}{dx}.$$

A comparison with the second equation shows that $dh/dx = 1$. This gives $h(x) = x + c$. Hence $v = 2xy + x + c$ (c any real constant) is the most general harmonic conjugate of the given u. The corresponding analytic function is

$$f(z) = u + iv = x^2 - y^2 - y + i(2xy + x + c) = z^2 + iz + ic. \qquad ■$$

Example 4 illustrates that *a conjugate of a given harmonic function is uniquely determined up to an arbitrary real additive constant.*

The Cauchy–Riemann equations are the most important equations in this chapter. Their relation to Laplace's equation opens wide ranges of engineering and physical applications, as we shall show in Chap. 18.

13.4

1–10 CAUCHY–RIEMANN EQUATIONS

Are the following functions analytic? [Use (1) or (7).]

1. $f(z) = z^4$
2. $f(z) = \text{Im}(z^2)$
3. $e^{2x}(\cos y + i \sin y)$
4. $f(z) = 1/(1 - z^4)$
5. $e^{-x}(\cos y - i \sin y)$
6. $f(z) = \text{Arg}\, \pi z$
7. $f(z) = \text{Re}\, z + \text{Im}\, z$
8. $f(z) = \ln |z| + i\, \text{Arg}\, z$
9. $f(z) = i/z^8$
10. $f(z) = z^2 + 1/z^2$
11. **(Cauchy–Riemann equations in polar form)** Derive (7) from (1).

12–21 HARMONIC FUNCTIONS

Are the following functions harmonic? If your answer is yes, find a corresponding analytic function $f(z) = u(x, y) + iv(x, y)$.

12. $u = xy$
13. $v = xy$
14. $v = -y/(x^2 + y^2)$
15. $u = \ln |z|$
16. $v = \ln |z|$
17. $u = x^3 - 3xy^2$
18. $u = 1/(x^2 + y^2)$
19. $v = (x^2 - y^2)^2$
20. $u = \cos x \cosh y$
21. $u = e^{-x} \sin 2y$

22–24 Determine a, b, c such that the given functions are harmonic and find a harmonic conjugate.

22. $u = e^{3x} \cos ay$
23. $u = \sin x \cosh cy$
24. $u = ax^3 + by^3$
25. **(Harmonic conjugate)** Show that if u is harmonic and v is a harmonic conjugate of u, then u is a harmonic conjugate of $-v$.
26. **TEAM PROJECT. Conditions for $f(z)$ = *const.*** Let $f(z)$ be analytic. Prove that each of the following conditions is sufficient for $f(z) = const.$
    (a) $\text{Re}\, f(z) = const$
    (b) $\text{Im}\, f(z) = const$
    (c) $f'(z) = 0$
    (d) $|f(z)| = const$ (see Example 3)
27. **(Two further formulas for the derivative).** Formulas (4), (5), and (11) (below) are needed from time to time. Derive

$$(11) \qquad f'(z) = u_x - iu_y, \qquad f'(z) = v_y + iv_x.$$

28. **CAS PROJECT. Equipotential Lines.** Write a program for graphing equipotential lines $u = const$ of a harmonic function u and of its conjugate v on the same axes. Apply the program to (a) $u = x^2 - y^2$, $v = 2xy$, (b) $u = x^3 - 3xy^2$, $v = 3x^2y - y^3$, (c) $u = e^x \cos y$, $v = e^x \sin y$.

13.5 Exponential Function

In the remaining sections of this chapter we discuss the basic elementary complex functions, the exponential function, trigonometric functions, logarithm, and so on. They will be counterparts to the familiar functions of calculus, to which they reduce when $z = x$ is real. They are indispensable throughout applications, and some of them have interesting properties not shared by their real counterparts.

We begin with one of the most important analytic functions, the complex **exponential function**

$$e^z, \qquad \text{also written} \qquad \exp z.$$

The definition of e^z in terms of the real functions e^x, $\cos y$, and $\sin y$ is

$$(1) \qquad e^z = e^x(\cos y + i \sin y).$$