The complex conjugate is important because it permits us to switch from complex to real. Indeed, by multiplication, $z\bar{z} = x^2 + y^2$ (verify!). By addition and subtraction, $z + \bar{z} = 2x$, $z - \bar{z} = 2iy$. We thus obtain for the real part x and the imaginary part y (not iy!) of $z = x + iy$ the important formulas

(8) $$\mathrm{Re}\, z = x = \frac{1}{2}(z + \bar{z}), \qquad \mathrm{Im}\, z = y = \frac{1}{2i}(z - \bar{z}).$$

If z is real, $z = x$, then $\bar{z} = z$ by the definition of $\bar{z}$, and conversely. Working with conjugates is easy, since we have

(9) $$\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2, \qquad \overline{(z_1 - z_2)} = \bar{z}_1 - \bar{z}_2,$$
$$\overline{(z_1 z_2)} = \bar{z}_1 \bar{z}_2, \qquad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}.$$

EXAMPLE 3 **Illustration of (8) and (9)**

Let $z_1 = 4 + 3i$ and $z_2 = 2 + 5i$. Then by (8),

$$\mathrm{Im}\, z_1 = \frac{1}{2i}[(4 + 3i) - (4 - 3i)] = \frac{3i + 3i}{2i} = 3.$$

Also, the multiplication formula in (9) is verified by

$$\overline{(z_1 z_2)} = \overline{(4 + 3i)(2 + 5i)} = \overline{(-7 + 26i)} = -7 - 26i,$$
$$\bar{z}_1 \bar{z}_2 = (4 - 3i)(2 - 5i) = -7 - 26i. \qquad \blacksquare$$

1. (Powers of i) Show that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, $\cdots$ and $1/i = -i$, $1/i^2 = -1$, $1/i^3 = i$, $\cdots$.

2. (Rotation) Multiplication by i is geometrically a counterclockwise rotation through $\pi/2$ (90°). Verify this by graphing z and iz and the angle of rotation for $z = 2 + 2i$, $z = -1 - 5i$, $z = 4 - 3i$.

3. (Division) Verify the calculation in (7).

4. (Multiplication) If the product of two complex numbers is zero, show that at least one factor must be zero.

5. Show that $z = x + iy$ is pure imaginary if and only if $\bar{z} = -z$.

6. (Laws for conjugates) Verify (9) for $z_1 = 24 + 10i$, $z_2 = 4 + 6i$.

7–15   **COMPLEX ARITHMETIC**

Let $z_1 = 2 + 3i$ and $z_2 = 4 - 5i$. Showing the details of your work, find (in the form $x + iy$):

7. $(5z_1 + 3z_2)^2$

8. $\bar{z}_1 \bar{z}_2$

9. $\mathrm{Re}\,(1/z_1^2)$

10. $\mathrm{Re}\,(z_2^2)$, $(\mathrm{Re}\, z_2)^2$

11. z_2/z_1

12. $\bar{z}_1/\bar{z}_2$, $\overline{(z_1/z_2)}$

13. $(4z_1 - z_2)^2$

14. $\bar{z}_1/z_1$, $z_1/\bar{z}_1$

15. $(z_1 + z_2)/(z_1 - z_2)$

16–19   Let $z = x + iy$. Find:

16. $\mathrm{Im}\, z^3$, $(\mathrm{Im}\, z)^3$

17. $\mathrm{Re}\,(1/\bar{z})$

18. $\mathrm{Im}\,[(1 + i)^8 z^2]$

19. $\mathrm{Re}\,(1/\bar{z}^2)$

20. (Laws of addition and multiplication) Derive the following laws for complex numbers from the corresponding laws for real numbers.

$z_1 + z_2 = z_2 + z_1$, $z_1 z_2 = z_2 z_1$ (*Commutative laws*)

$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$,

(*Associative laws*)

$(z_1 z_2)z_3 = z_1(z_2 z_3)$

$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ (*Distributive law*)

$0 + z = z + 0 = z$,

$z + (-z) = (-z) + z = 0$, $\qquad z \cdot 1 = z$.

13.2 Polar Form of Complex Numbers. Powers and Roots

The complex plane becomes even more useful and gives further insight into the arithmetic operations for complex numbers if besides the xy-coordinates we also employ the usual polar coordinates r, θ defined by

$$(1) \qquad\qquad x = r \cos \theta, \qquad y = r \sin \theta.$$

We see that then $z = x + iy$ takes the so-called **polar form**

$$(2) \qquad\qquad z = r(\cos \theta + i \sin \theta).$$

r is called the **absolute value** or **modulus** of z and is denoted by $|z|$. Hence

$$(3) \qquad\qquad |z| = r = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}.$$

Geometrically, $|z|$ is the distance of the point z from the origin (Fig. 320). Similarly, $|z_1 - z_2|$ is the distance between z_1 and z_2 (Fig. 321).

θ is called the **argument** of z and is denoted by $\arg z$. Thus (Fig. 320)

$$(4) \qquad\qquad \theta = \arg z = \arctan \frac{y}{x} \qquad\qquad (z \neq 0).$$

Geometrically, θ is the directed angle from the positive x-axis to OP in Fig. 320. Here. as in calculus, all **angles are measured in radians and positive in the counterclockwise sense.**

For $z = 0$ this angle θ is undefined. (Why?) For a given $z \neq 0$ it is determined only up to integer multiples of 2π since cosine and sine are periodic with period 2π. But one often wants to specify a unique value of $\arg z$ of a given $z \neq 0$. For this reason one defines the **principal value** Arg z (with capital A!) of $\arg z$ by the double inequality

$$(5) \qquad\qquad -\pi < \text{Arg } z \leq \pi.$$

Then we have Arg $z = 0$ for positive real $z = x$, which is practical, and Arg $z = \pi$ (not $-\pi$!) for negative real z, e.g., for $z = -4$. The principal value (5) will be important in connection with roots, the complex logarithm (Sec. 13.7), and certain integrals. Obviously, for a given $z \neq 0$ the other values of $\arg z$ are $\arg z = \text{Arg } z \pm 2n\pi$ ($n = \pm 1, \pm 2, \cdots$).
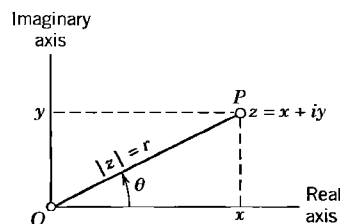

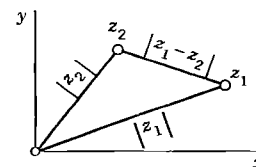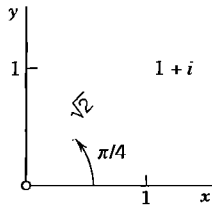**Fig. 320.** Complex plane, polar form of a complex number

Fig. 321. Distance between two points in the complex plane

EXAMPLE 1   **Polar Form of Complex Numbers. Principal Value Arg z**

$z = 1 + i$ (Fig. 322) has the polar form $z = \sqrt{2}\,(\cos\frac{1}{4}\pi + i\sin\frac{1}{4}\pi)$. Hence we obtain

$|z| = \sqrt{2}$, $\arg z = \frac{1}{4}\pi \pm 2n\pi$ $(n = 0, 1, \cdots)$,   and $\operatorname{Arg} z = \frac{1}{4}\pi$ (the principal value).

Similarly, $z = 3 + 3\sqrt{3}i = 6\,(\cos\frac{1}{3}\pi + i\sin\frac{1}{3}\pi)$, $|z| = 6$, and $\operatorname{Arg} z = \frac{1}{3}\pi$.   ∎

g. 322.   Example 1

CAUTION!   In using (4), we must pay attention to the quadrant in which z lies, since $\tan\theta$ has period π, so that the arguments of z and $-z$ have the same tangent. *Example:* for $\theta_1 = \arg(1 + i)$ and $\theta_2 = \arg(-1 - i)$ we have $\tan\theta_1 = \tan\theta_2 = 1$.

Triangle Inequality

Inequalities such as $x_1 < x_2$ make sense for *real* numbers, but not in complex because *there is no natural way of ordering complex numbers*. However, inequalities between absolute values (which are real!), such as $|z_1| < |z_2|$ (meaning that z_1 is closer to the origin than z_2) are of great importance. The daily bread of the complex analyst is the **triangle inequality**

(6) $$|z_1 + z_2| \le |z_1| + |z_2|$$ (Fig. 323)

which we shall use quite frequently. This inequality follows by noting that the three points 0, z_1, and $z_1 + z_2$ are the vertices of a triangle (Fig. 323) with sides $|z_1|$, $|z_2|$, and $|z_1 + z_2|$, and one side cannot exceed the sum of the other two sides. A formal proof is left to the reader (Prob. 35). (The triangle degenerates if z_1 and z_2 lie on the same straight line through the origin.)
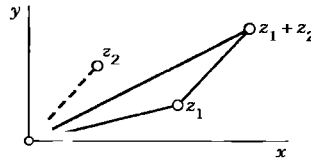

Fig. 323.   Triangle inequality

By induction we obtain from (6) the **generalized triangle inequality**

(6*) $$|z_1 + z_2 + \cdots + z_n| \le |z_1| + |z_2| + \cdots + |z_n|;$$

that is, *the absolute value of a sum cannot exceed the sum of the absolute values of the terms.*

EXAMPLE 2   **Triangle Inequality**

If $z_1 = 1 + i$ and $z_2 = -2 + 3i$, then (sketch a figure!)

$$|z_1 + z_2| = |-1 + 4i| = \sqrt{17} = 4.123 < \sqrt{2} + \sqrt{13} = 5.020.$$ ∎

Multiplication and Division in Polar Form

This will give us a "geometrical" understanding of multiplication and division. Let

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \qquad \text{and} \qquad z_2 = r_2(\cos\theta_2 + i\sin\theta_2).$$

Multiplication. By (3) in Sec. 13.1 the product is at first

$$z_1 z_2 = r_1 r_2 [(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)].$$

The addition rules for the sine and cosine [(6) in App. A3.1] now yield

$$\text{(7)} \qquad\qquad z_1 z_2 = r_1 r_2 [\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)].$$

Taking absolute values on both sides of (7), we see that *the absolute value of a product equals the product of the absolute values of the factors,*

$$\text{(8)} \qquad\qquad\qquad |z_1 z_2| = |z_1||z_2|.$$

Taking arguments in (7) shows that *the argument of a product equals the sum of the arguments of the factors,*

$$\text{(9)} \qquad\qquad \arg (z_1 z_2) = \arg z_1 + \arg z_2 \qquad \text{(up to multiples of } 2\pi\text{)}.$$

Division. We have $z_1 = (z_1/z_2)z_2$. Hence $|z_1| = |(z_1/z_2)z_2| = |z_1/z_2||z_2|$ and by division by $|z_2|$

$$\text{(10)} \qquad\qquad\qquad \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|} \qquad\qquad (z_2 \neq 0).$$

Similarly, $\arg z_1 = \arg [(z_1/z_2)z_2] = \arg (z_1/z_2) + \arg z_2$ and by subtraction of $\arg z_2$

$$\text{(11)} \qquad\qquad \arg \frac{z_1}{z_2} = \arg z_1 - \arg z_2 \qquad \text{(up to multiples of } 2\pi\text{)}.$$

Combining (10) and (11) we also have the analog of (7),

$$\text{(12)} \qquad\qquad \frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos (\theta_1 - \theta_2) + i \sin (\theta_1 - \theta_2)].$$

To comprehend this formula. note that it is the polar form of a complex number of absolute value r_1/r_2 and argument $\theta_1 - \theta_2$. But these are the absolute value and argument of z_1/z_2, as we can see from (10), (11), and the polar forms of z_1 and z_2.

EXAMPLE 3 **Illustration of Formulas (8)–(11)**

Let $z_1 = -2 + 2i$ and $z_2 = 3i$. Then $z_1 z_2 = -6 - 6i$, $z_1/z_2 = 2/3 + (2/3)i$. Hence (make a sketch)

$$|z_1 z_2| = 6\sqrt{2} = 3\sqrt{8} = |z_1||z_2|. \qquad |z_1/z_2| = 2\sqrt{2}/3 = |z_1|/|z_2|,$$

and for the arguments we obtain $\text{Arg } z_1 = 3\pi/4$, $\text{Arg } z_2 = \pi/2$,

$$\text{Arg } (z_1 z_2) = -\frac{3\pi}{4} = \text{Arg } z_1 + \text{Arg } z_2 - 2\pi, \qquad \text{Arg } (z_1/z_2) = \frac{\pi}{4} = \text{Arg } z_1 - \text{Arg } z_2. \qquad \blacksquare$$

EXAMPLE 4   **Integer Powers of z. De Moivre's Formula**

From (8) and (9) with $z_1 = z_2 = z$ we obtain by induction for $n = 0, 1, 2, \cdots$

$$(13) \qquad\qquad z^n = r^n (\cos n\theta + i \sin n\theta).$$

Similarly, (12) with $z_1 = 1$ and $z_2 = z^n$ gives (13) for $n = -1, -2, \cdots$. For $|z| = r = 1$, formula (13) becomes **De Moivre's formula**[3]

$$(13^*) \qquad\qquad (\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$

We can use this to express $\cos n\theta$ and $\sin n\theta$ in terms of powers of $\cos \theta$ and $\sin \theta$. For instance, for $n = 2$ we have on the left $\cos^2 \theta + 2i \cos \theta \sin \theta - \sin^2 \theta$. Taking the real and imaginary parts on both sides of (13^*) with $n = 2$ gives the familiar formulas

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta, \qquad \sin 2\theta = 2 \cos \theta \sin \theta.$$

This shows that *complex* methods often simplify the derivation of *real* formulas. Try $n = 3$.   ■

Roots

If $z = w^n$ ($n = 1, 2, \cdots$), then to each value of w there corresponds *one* value of z. We shall immediately see that, conversely, to a given $z \neq 0$ there correspond precisely n distinct values of w. Each of these values is called an *nth root* of z, and we write

$$(14) \qquad\qquad w = \sqrt[n]{z} .$$

Hence this symbol is *multivalued*, namely, *n-valued*. The n values of $\sqrt[n]{z}$ can be obtained as follows. We write z and w in polar form

$$z = r(\cos \theta + i \sin \theta) \qquad \text{and} \qquad w = R(\cos \phi + i \sin \phi).$$

Then the equation $w^n = z$ becomes, by De Moivre's formula (with ϕ instead of θ)

$$w^n = R^n(\cos n\phi + i \sin n\phi) = z = r(\cos \theta + i \sin \theta).$$

The absolute values on both sides must be equal; thus, $R^n = r$, so that $R = \sqrt[n]{r}$, where $\sqrt[n]{r}$ is positive real (an absolute value must be nonnegative!) and thus uniquely determined. Equating the arguments $n\phi$ and θ and recalling that θ is determined only up to integer multiples of 2π, we obtain

$$n\phi = \theta + 2k\pi, \qquad \text{thus} \qquad \phi = \frac{\theta}{n} + \frac{2k\pi}{n}$$

where k is an integer. For $k = 0, 1, \cdots, n - 1$ we get n *distinct* values of w. Further integers of k would give values already obtained. For instance, $k = n$ gives $2k\pi/n = 2\pi$,

---

[3]ABRAHAM DE MOIVRE (1667–1754), French mathematician, who pioneered the use of complex numbers in trigonometry and also contributed to probability theory (see Sec. 24.8).

hence the w corresponding to $k = 0$, etc. Consequently, $\sqrt[n]{z}$, for $z \neq 0$, has the n distinct values

(15)
$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right)$$

where $k = 0, 1, \cdots, n - 1$. These n values lie on a circle of radius $\sqrt[n]{r}$ with center at the origin and constitute the vertices of a regular polygon of n sides. The value of $\sqrt[n]{z}$ obtained by taking the principal value of arg z and $k = 0$ in (15) is called the **principal value of** $w = \sqrt[n]{z}$.

Taking $z = 1$ in (15), we have $|z| = r = 1$ and Arg $z = 0$. Then (15) gives

(16)
$$\sqrt[n]{1} = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, \qquad k = 0, 1, \cdots, n - 1.$$

These n values are called the **nth roots of unity.** They lie on the circle of radius 1 and center 0, briefly called the **unit circle** (and used quite frequently!). Figures 324–326 show $\sqrt[3]{1} = 1$, $-\frac{1}{2} \pm \frac{1}{2}\sqrt{3}\, i$, $\sqrt[4]{1} = \pm 1$, $\pm i$, and $\sqrt[5]{1}$.

If ω denotes the value corresponding to $k = 1$ in (16), then the n values of $\sqrt[n]{1}$ can be written as

$$1, \omega, \omega^2, \cdots, \omega^{n-1}.$$

More generally, if w_1 is any nth root of an arbitrary complex number z ($\neq 0$), then the n values of $\sqrt[n]{z}$ in (15) are

(17) $w_1,$ $w_1\omega,$ $w_1\omega^2,$ $\cdots,$ $w_1\omega^{n-1}$

because multiplying w_1 by ω^k corresponds to increasing the argument of w_1 by $2k\pi/n$. Formula (17) motivates the introduction of roots of unity and shows their usefulness.
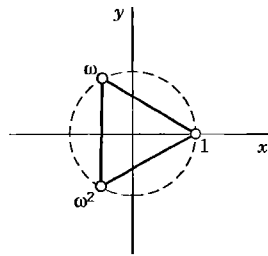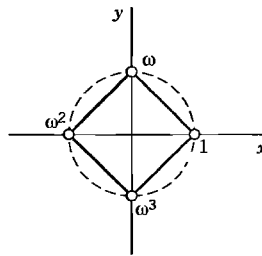
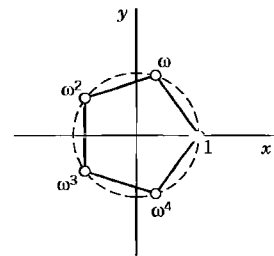**Fig. 324.** $\sqrt[3]{1}$ **Fig. 325.** $\sqrt[4]{1}$ **Fig. 326.** $\sqrt[5]{1}$

PROBLEM SET 13.2

|1–8| **POLAR FORM**

Do these problems very carefully since polar forms will be needed frequently. Represent in polar form and graph in the complex plane as in Fig. 322 on p. 608. (Show the details of your work.)

1. $3 - 3i$

2. $2i, -2i$

3. -5

4. $\frac{1}{2} + \frac{1}{4}\pi i$

5. $\dfrac{1 + i}{1 - i}$

6. $\dfrac{3\sqrt{2} + 2i}{-\sqrt{2} - (2/3)i}$

7. $\dfrac{-6 + 5i}{3i}$

8. $\dfrac{2 + 3i}{5 + 4i}$

| 9-15 | **PRINCIPAL ARGUMENT**

Determine the principal value of the argument.

9. $-1 - i$

10. $-20 + i, -20 - i$

11. $4 \pm 3i$

12. $-\pi^2$

13. $7 \pm 7i$

14. $(1 + i)^{12}$

15. $(9 + 9i)^3$

| 16-20 | **CONVERSION TO $x + iy$**

Represent in the form $x + iy$ and graph it in the complex plane.

16. $\cos\frac{1}{2}\pi + i\sin(\pm\frac{1}{2}\pi)$

17. $3(\cos 0.2 + i\sin 0.2)$

18. $4(\cos\frac{1}{3}\pi \pm i\sin\frac{1}{3}\pi)$

19. $\cos(-1) + i\sin(-1)$

20. $12(\cos\frac{3}{2}\pi + i\sin\frac{3}{2}\pi)$

| 21-25 | **ROOTS**

Find and graph all roots in the complex plane.

21. $\sqrt{-i}$

22. $\sqrt[8]{1}$

23. $\sqrt[4]{-1}$

24. $\sqrt[3]{3 + 4i}$

25. $\sqrt[5]{-1}$

26. **TEAM PROJECT. Square Root. (a)** Show that $w = \sqrt{z}$ has the values

$$w_1 = \sqrt{r}\left[\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right],$$

$$(18) \quad w_2 = \sqrt{r}\left[\cos\left(\frac{\theta}{2} + \pi\right) + i\sin\left(\frac{\theta}{2} + \pi\right)\right]$$

$$= -w_1.$$

(b) Obtain from (18) the often more practical formula

$$(19) \quad \sqrt{z} = \pm\left[\sqrt{\tfrac{1}{2}(|z| + x)} + (\text{sign } y)i\sqrt{\tfrac{1}{2}(|z| + x)}\right]$$

where sign $y = 1$ if $y \geqq 0$, sign $y = -1$ if $y < 0$, and all square roots of positive numbers are taken with positive sign. *Hint:* Use (10) in App. A3.1 with $x = \theta/2$.

(c) Find the square roots of $4i$, $16 - 30i$, and $9 + 8\sqrt{7}i$ by both (18) and (19) and comment on the work involved.

(d) Do some further examples of your own and apply a method of checking your results.

| 27-30 | **EQUATIONS**

Solve and graph all solutions, showing the details:

27. $z^2 - (8 - 5i)z + 40 - 20i = 0$ (Use (19).)

28. $z^4 + (5 - 14i)z^2 - (24 + 10i) = 0$

29. $8z^2 - (36 - 6i)z + 42 - 11i = 0$

30. $z^4 + 16 = 0$. Then use the solutions to factor $z^4 + 16$ into quadratic factors with *real* coefficients.

31. **CAS PROJECT. Roots of Unity and Their Graphs.** Write a program for calculating these roots and for graphing them as points on the unit circle. Apply the program to $z^n = 1$ with $n = 2, 3, \cdots, 10$. Then extend the program to one for arbitrary roots, using an idea near the end of the text, and apply the program to examples of your choice.

| 32-35 | **INEQUALITIES AND AN EQUATION**

Verify or prove as indicated.

32. **(Re and Im)** Prove $|\text{Re } z| \leq |z|$, $|\text{Im } z| \leq |z|$.

33. **(Parallelogram equality)** Prove

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2).$$

Explain the name.

34. **(Triangle inequality)** Verify (6) for $z_1 = 4 + 7i$, $z_2 = 5 + 2i$.

35. **(Triangle inequality)** Prove (6).

13.3 Derivative. Analytic Function

Our study of complex functions will involve point sets in the complex plane. Most important will be the following ones.

Circles and Disks. Half-Planes

The **unit circle** $|z| = 1$ (Fig. 327) has already occurred in Sec. 13.2. Figure 328 shows a general circle of radius ρ and center a. Its equation is

$$|z - a| = \rho$$