Calculus of Multivariable Functions

5.1 FUNCTIONS OF SEVERAL VARIABLES AND PARTIAL DERIVATIVES

Study of the derivative in Chapter 4 was limited to functions of a single independent variable such as $y = f(x)$. Many economic activities, however, involve functions of more than one independent variable. $z = f(x, y)$ is defined as a *function of two independent variables* if there exists one and only one value of z in the range of f for each ordered pair of real numbers (x, y) in the domain of f. By convention, z is the *dependent variable*; x and y are the *independent variables*.

To measure the effect of a change in a single independent variable (x or y) on the dependent variable (z) in a multivariable function, the partial derivative is needed. The *partial derivative of z with respect to x* measures the instantaneous rate of change of z with respect to x while y is held constant. It is written $\partial z/\partial x$, $\partial f/\partial x$, $f_x(x, y)$, f_x, or z_x. The *partial derivative of z with respect to y* measures the rate of change of z with respect to y while x is held constant. It is written $\partial z/\partial y$, $\partial f/\partial y$, $f_y(x, y)$, f_y, or z_y. Expressed mathematically,

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \tag{5.1a}$$

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \tag{5.1b}$$

Partial differentiation with respect to one of the independent variables follows the same rules as ordinary differentiation while the other independent variables are treated as constant. See Examples 1 and 2 and Problems 5.1 and 5.23.

EXAMPLE 1. The partial derivatives of a multivariable function such as $z = 3x^2y^3$ are found as follows:

(a) When differentiating with respect to x, treat the y term as a constant by mentally bracketing it with the coefficient:

$$z = [3y^3] \cdot x^2$$

Then take the derivative of the x term, holding the y term constant,

$$\frac{\partial z}{\partial x} = z_x = [3y^3] \cdot \frac{d}{dx}(x^2)$$
$$= [3y^3] \cdot 2x$$

Recalling that a multiplicative constant remains in the process of differentiation, simply multiply and rearrange terms to obtain

$$\frac{\partial z}{\partial x} = z_x = 6xy^3$$

(b) When differentiating with respect to y, treat the x term as a constant by bracketing it with the coefficient; then take the derivative as was done above:

$$z = [3x^2] \cdot y^3$$

$$\frac{\partial z}{\partial y} = z_y = [3x^2] \cdot \frac{d}{dy}(y^3)$$
$$= [3x^2] \cdot 3y^2 = 9x^2y^2$$

EXAMPLE 2. To find the partial derivatives for $z = 5x^3 - 3x^2y^2 + 7y^5$:

(a) When differentiating with respect to x, mentally bracket all y terms to remember to treat them as constants:

$$z = 5x^3 - [3y^2]x^2 + [7y^5]$$

Then take the derivative of each term, remembering that in differentiation multiplicative constants remain but additive constants drop out, because the derivative of a constant is zero.

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{d}{dx}(5x^3) - [3y^2] \cdot \frac{d}{dx}(x^2) + \frac{d}{dx}[7y^5] \\ &= 15x^2 - [3y^2] \cdot 2x + 0 \\ &= 15x^2 - 6xy^2 \end{aligned}$$

(b) When differentiating with respect to y, block off all the x terms and differentiate as above.

$$\begin{aligned} z &= [5x^3] - [3x^2]y^2 + 7y^5 \\ \frac{\partial z}{\partial y} &= \frac{d}{dy}[5x^3] - [3x^2] \cdot \frac{d}{dy}(y^2) + \frac{d}{dy}(7y^5) \\ &= 0 - [3x^2] \cdot 2y + 35y^4 \\ &= -6x^2y + 35y^4 \end{aligned}$$

See Problem 5.1.

5.2 RULES OF PARTIAL DIFFERENTIATION

Partial derivatives follow the same basic patterns as the rules of differentiation in Section 3.7. A few key rules are given below, illustrated in Examples 3 to 5, treated in Problems 5.2 to 5.5, and verified in Problem 5.23.

5.2.1. *Product Rule*

Given $z = g(x, y) \cdot h(x, y)$,

$$\frac{\partial z}{\partial x} = g(x, y) \cdot \frac{\partial h}{\partial x} + h(x, y) \cdot \frac{\partial g}{\partial x} \qquad (5.2a)$$

$$\frac{\partial z}{\partial y} = g(x, y) \cdot \frac{\partial h}{\partial y} + h(x, y) \cdot \frac{\partial g}{\partial y} \qquad (5.2b)$$

EXAMPLE 3. Given $z = (3x + 5)(2x + 6y)$, by the product rule,

$$\frac{\partial z}{\partial x} = (3x + 5)(2) + (2x + 6y)(3) = 12x + 10 + 18y$$

$$\frac{\partial z}{\partial y} = (3x + 5)(6) + (2x + 6y)(0) = 18x + 30$$

5.2.2. *Quotient Rule*

Given $z = g(x, y)/h(x, y)$ and $h(x, y) \neq 0$,

$$\frac{\partial z}{\partial x} = \frac{h(x, y) \cdot \partial g/\partial x - g(x, y) \cdot \partial h/\partial x}{[h(x, y)]^2} \qquad (5.3a)$$

$$\frac{\partial z}{\partial y} = \frac{h(x, y) \cdot \partial g/\partial y - g(x, y) \cdot \partial h/\partial y}{[h(x, y)]^2} \qquad (5.3b)$$

EXAMPLE 4. Given $z = (6x + 7y)/(5x + 3y)$, by the quotient rule,

$$\frac{\partial z}{\partial x} = \frac{(5x + 3y)(6) - (6x + 7y)(5)}{(5x + 3y)^2}$$

$$= \frac{30x + 18y - 30x - 35y}{(5x + 3y)^2} = \frac{-17y}{(5x + 3y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{(5x + 3y)(7) - (6x + 7y)(3)}{(5x + 3y)^2}$$

$$= \frac{35x + 21y - 18x - 21y}{(5x + 3y)^2} = \frac{17x}{(5x + 3y)^2}$$

5.2.3. *Generalized Power Function Rule*

Given $z = [g(x, y)]^n$,

$$\frac{\partial z}{\partial x} = n[g(x, y)]^{n-1} \cdot \frac{\partial g}{\partial x} \tag{5.4a}$$

$$\frac{\partial z}{\partial y} = n[g(x, y)]^{n-1} \cdot \frac{\partial g}{\partial y} \tag{5.4b}$$

EXAMPLE 5. Given $z = (x^3 + 7y^2)^4$, by the generalized power function rule,

$$\frac{\partial z}{\partial x} = 4(x^3 + 7y^2)^3 \cdot (3x^2) = 12x^2(x^3 + 7y^2)^3$$

$$\frac{\partial z}{\partial y} = 4(x^3 + 7y^2)^3 \cdot (14y) = 56y(x^3 + 7y^2)^3$$

5.3 SECOND-ORDER PARTIAL DERIVATIVES

Given a function $z = f(x, y)$, *the second-order (direct) partial derivative* signifies that the function has been differentiated partially with respect to one of the independent variables twice while the other independent variable has been held constant:

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} \qquad f_{yy} = (f_y)_y = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}$$

In effect, f_{xx} measures the rate of change of the first-order partial derivative f_x with respect to x while y is held constant. And f_{yy} is exactly parallel. See Problems 5.6 and 5.8.

The *cross (or mixed) partial derivative* f_{xy} or f_{yx} indicates that first the primitive function has been partially differentiated with respect to one independent variable and then that partial derivative has in turn been partially differentiated with respect to the other independent variable:

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y\, \partial x} \qquad f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x\, \partial y}$$

In brief, a cross partial measures the rate of change of a first-order partial derivative with respect to the other independent variable. Notice how the order of independent variables changes in the different forms of notation. See Problems 5.7 and 5.9.

EXAMPLE 6. The (*a*) first, (*b*) second, and (*c*) cross partial derivatives for $z = 7x^3 + 9xy + 2y^5$ are taken as shown below.

(a) $\dfrac{\partial z}{\partial x} = z_x = 21x^2 + 9y \qquad \dfrac{\partial z}{\partial y} = z_y = 9x + 10y^4$

(b) $\dfrac{\partial^2 z}{\partial x^2} = z_{xx} = 42x \qquad \dfrac{\partial^2 z}{\partial y^2} = z_{yy} = 40y^3$

(c) $$\frac{\partial^2 z}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}(21x^2 + 9y) = z_{xy} = 9$$

$$\frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(9x + 10y^4) = z_{yx} = 9$$

EXAMPLE 7. The (*a*) first, (*b*) second, and (*c*) cross partial derivatives for $z = 3x^2y^3$ are evaluated below at $x = 4$, $y = 1$.

(a) $z_x = 6xy^3 \qquad z_y = 9x^2y^2$

$z_x(4, 1) = 6(4)(1)^3 = 24 \qquad z_y(4, 1) = 9(4)^2(1)^2 = 144$

(b) $z_{xx} = 6y^3 \qquad z_{yy} = 18x^2y$

$z_{xx}(4, 1) = 6(1)^3 = 6 \qquad z_{yy}(4, 1) = 18(4)^2(1) = 288$

(c) $z_{xy} = \dfrac{\partial}{\partial y}(6xy^3) = 18xy^2 \qquad z_{yx} = \dfrac{\partial}{\partial x}(9x^2y^2) = 18xy^2$

$z_{xy}(4, 1) = 18(4)(1)^2 = 72 \qquad z_{yx}(4, 1) = 18(4)(1)^2 = 72$

By *Young's theorem*, if both cross partial derivatives are continuous, they will be identical. See Problems 5.7 to 5.9.

5.4 OPTIMIZATION OF MULTIVARIABLE FUNCTIONS

For a multivariable function such as $z = f(x, y)$ to be at a relative minimum or maximum, three conditions must be met:

1. The first-order partial derivatives must equal zero simultaneously. This indicates that at the given point (a, b), called a *critical point*, the function is neither increasing nor decreasing with respect to the principal axes but is at a relative plateau.
2. The second-order direct partial derivatives, when evaluated at the critical point (a, b), must both be positive for a minimum and negative for a maximum. This ensures that from a relative plateau at (a, b) the function is moving upward in relation to the principal axes in the case of a minimum and downward in relation to the principal axes in the case of a maximum.
3. The product of the second-order direct partials evaluated at the critical point must exceed the product of the cross partials evaluated at the critical point.

In sum, as seen in Fig. 5-1, when evaluated at a critical point (a, b),

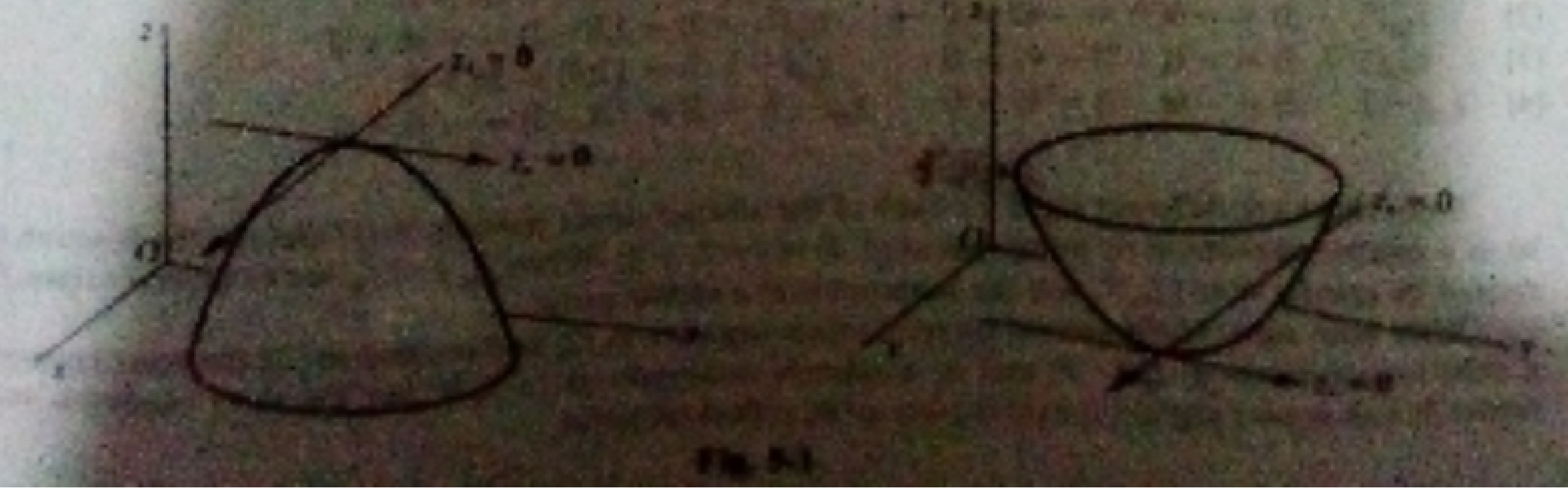

Fig. 5-1

Relative maximum	Relative minimum
1. $f_x, f_y = 0$	1. $f_x, f_y = 0$
2. $f_{xx}, f_{yy} < 0$	2. $f_{xx}, f_{yy} > 0$
3. $f_{xx} \cdot f_{yy} > (f_{xy})^2$	3. $f_{xx} \cdot f_{yy} > (f_{xy})^2$

Note the following:

(1) Since $f_{xy} = f_{yx}$ by Young's theorem, $f_{xy} \cdot f_{yx} = (f_{xy})^2$. Step 3 can also be written $f_{xx} \cdot f_{yy} - (f_{xy})^2 > 0$.

(2) If $f_{xx} \cdot f_{yy} < (f_{xy})^2$, when f_{xx} and f_{yy} have the same signs, the function is at an *inflection point*; when f_{xx} and f_{yy} have different signs, the function is at a *saddle point*, as seen in Fig. 5-2, where the function is at a maximum when viewed from one axis but at a minimum when viewed from the other axis.

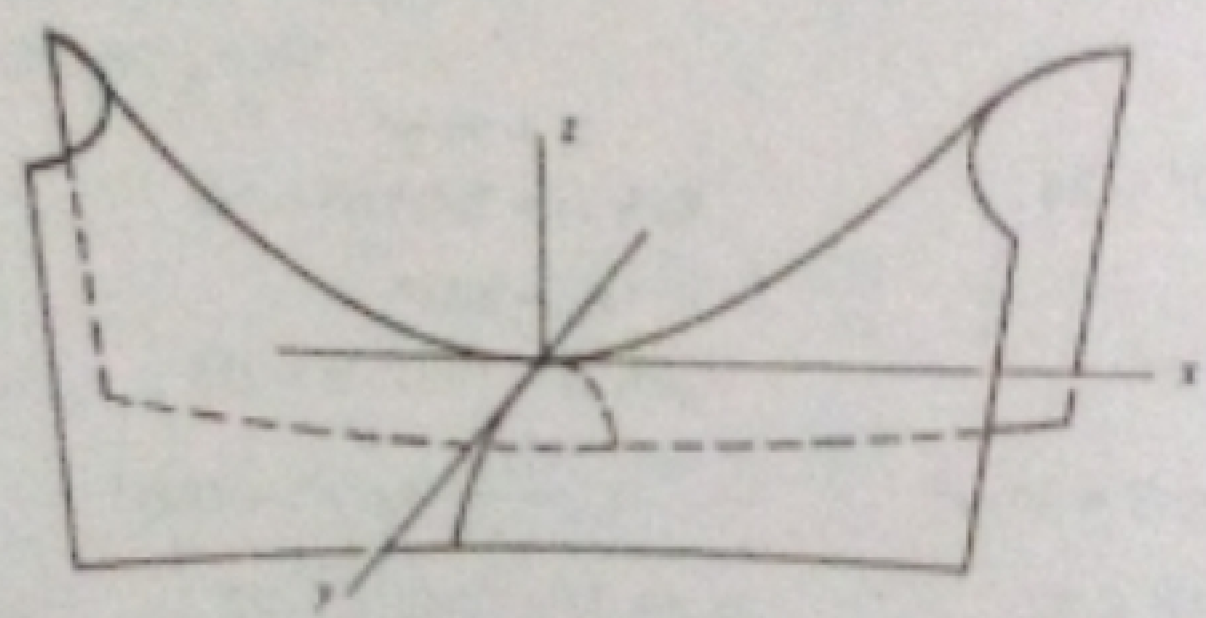

Fig. 5-2

(3) If $f_{xx} \cdot f_{yy} = (f_{xy})^2$, the test is inconclusive. See Example 8 and Problems 5.10 and 5.11; for inflection points, see Problems 5.10(*c*) and 5.11(*b*) and (*c*); for saddle points see Problems 5.10(*d*) and 5.11(*a*) and (*d*).

EXAMPLE 8. (*a*) Find the critical points. (*b*) Test whether the function is at a relative maximum or minimum, given

$$z = 2y^3 - x^3 + 147x - 54y + 12$$

(*a*) Take the first-order partial derivatives, set them equal to zero, and solve for x and y:

$$z_x = -3x^2 + 147 = 0 \qquad z_y = 6y^2 - 54 = 0 \qquad (5.5)$$
$$x^2 = 49 \qquad y^2 = 9$$
$$x = \pm 7 \qquad y = \pm 3$$

With $x = \pm 7$, $y = \pm 3$, there are four distinct sets of critical points: (7, 3), (7, −3), (−7, 3), and (−7, −3).

(*b*) Take the second-order direct partials from (5.5), evaluate them at each of the critical points, and check the signs:

$$z_{xx} = -6x \qquad z_{yy} = 12y$$

(1) $z_{xx}(7, 3) = -6(7) = -42 < 0$ $\qquad z_{yy}(7, 3) = 12(3) = 36 > 0$
(2) $z_{xx}(7, -3) = -6(7) = -42 < 0$ $\qquad z_{yy}(7, -3) = 12(-3) = -36 < 0$
(3) $z_{xx}(-7, 3) = -6(-7) = 42 > 0$ $\qquad z_{yy}(-7, 3) = 12(3) = 36 > 0$
(4) $z_{xx}(-7, -3) = -6(-7) = 42 > 0$ $\qquad z_{yy}(-7, -3) = 12(-3) = -36 < 0$

Since there are different signs for each of the second direct partials in (1) and (4), the function cannot be at a relative maximum or minimum at (7, 3) or (−7, −3). When f_{xx} and f_{yy} are of different signs, $f_{xx} \cdot f_{yy}$ cannot be greater than $(f_{xy})^2$, and the function is at a saddle point.

With both signs of the second direct partials negative in (2) and positive in (3), the function may be at a relative maximum at (7, −3) and at a relative minimum at (−7, 3), but the third condition must be tested first to ensure against the possibility of an inflection point.

(c) From (5.5), take the cross partial derivatives and check to make sure that $z_{xx}(a, b) \cdot z_{yy}(a, b) > [z_{xy}(a, b)]^2$.

$$z_{xy} = 0 \qquad z_{yx} = 0$$

$$z_{xx}(a, b) \cdot z_{yy}(a, b) \qquad > [z_{xy}(a, b)]^2$$

From (2),
$$(-42) \cdot (-36) \quad > \quad (0)^2$$
$$1512 > 0$$

From (3),
$$(42) \cdot (36) \quad > \quad (0)^2$$
$$1512 > 0$$

The function is maximized at $(7, -3)$ and minimized at $(-7, 3)$; for inflection points, see Problems 5.10(c) and 5.11(b) and (c).

5.5 CONSTRAINED OPTIMIZATION WITH LAGRANGE MULTIPLIERS

Differential calculus is also used to maximize or minimize a function subject to constraint. Given a function $f(x, y)$ subject to a constraint $g(x, y) = k$ (a constant), a new function F can be formed by (1) setting the constraint equal to zero, (2) multiplying it by λ (the *Lagrange multiplier*), and (3) adding the product to the original function:

$$F(x, y, \lambda) = f(x, y) + \lambda[k - g(x, y)] \tag{5.6}$$

Here $F(x, y, \lambda)$ is the *Lagrangian function*, $f(x, y)$ is the original or *objective function*, and $g(x, y)$ is the *constraint*. Since the constraint is always set equal to zero, the product $\lambda[k - g(x, y)]$ also equals zero, and the addition of the term does not change the value of the objective function. Critical values x_0, y_0, and λ_0, at which the function is optimized, are found by taking the partial derivatives of F with respect to *all three* independent variables, setting them equal to zero, and solving simultaneously:

$$F_x(x, y, \lambda) = 0 \qquad F_y(x, y, \lambda) = 0 \qquad F_\lambda(x, y, \lambda) = 0$$

Second-order conditions differ from those of unconstrained optimization and are treated in Section 12.5. See Example 9; Problems 5.12 to 5.14; Sections 6.6, 6.10, and 6.11; and Problems 6.28 to 6.43 and 6.45 to 6.48.

EXAMPLE 9. Optimize the function

$$z = 4x^2 + 3xy + 6y^2$$

subject to the constraint $x + y = 56$.

1. Set the constraint equal to zero,

$$56 - x - y = 0$$

Multiply it by λ and add it to the objective function to form the Lagrangian function Z.

$$Z = 4x^2 + 3xy + 6y^2 + \lambda(56 - x - y) \tag{5.7}$$

2. Take the first-order partials, set them equal to zero, and solve simultaneously.

$$Z_x = 8x + 3y - \lambda = 0 \tag{5.8}$$
$$Z_y = 3x + 12y - \lambda = 0 \tag{5.9}$$
$$Z_\lambda = 56 - x - y = 0 \tag{5.10}$$

Subtracting (5.9) from (5.8) to eliminate λ gives

$$5x - 9y = 0 \qquad x = 1.8y$$

Substitute $x = 1.8y$ in (5.10),

$$56 - 1.8y - y = 0 \qquad y_0 = 20$$

From which we find

$$x_0 = 36 \qquad \lambda_0 = 348$$

Substitute the critical values in (5.7),

$$\begin{aligned} \bar{Z} &= 4(36)^2 + 3(36)(20) + 6(20)^2 + (348)(56 - 20 - 36) \\ &= 4(1296) + 3(720) + 6(400) + 348(0) = 9744 \end{aligned}$$

In Chapter 12, Example 5, it will be shown that $\bar{Z}$ is at a minimum. Notice that at the critical values, the lagrangian function $\bar{Z}$ equals the objective function z because the constraint equals zero. See Problems 5.13 to 5.14 and Sections 6.6, 6.10, and 6.11.

Calculus of Multivariable Functions in Economics

6.1 MARGINAL PRODUCTIVITY

The *marginal physical product* of capital (MPP_K) is defined as the change in output brought about by a small change in capital when all the other factors of production are held constant. Given a production function such as

$$Q = 36KL - 2K^2 - 3L^2$$

the MPP_K is measured by taking the partial derivative $\partial Q/\partial K$. Thus,

$$MPP_K = \frac{\partial Q}{\partial K} = 36L - 4K$$

Similarly, for labor, $MPP_L = \partial Q/\partial L = 36K - 6L$. See Problems 6.1 to 6.3.

6.2 INCOME DETERMINATION MULTIPLIERS

The partial derivative can also be used to derive the various multipliers of an income determination model. Given

$$Y = C + I + G + (X - Z)$$

where

$$C = C_0 + bY \qquad G = G_0 \qquad Z = Z_0$$
$$I = I_0 + aY \qquad X = X_0$$

from Problem 2.19, it is clear that the equilibrium level of income is

$$\bar{Y} = \frac{1}{1 - b - a}(C_0 + I_0 + G_0 + X_0 - Z_0) \tag{6.1}$$

Taking the partial derivative of *(6.1)* with respect to any of the variables or parameters gives the multiplier for that variable or parameter. Thus, the government multiplier is given by

$$\frac{\partial \bar{Y}}{\partial G_0} = \frac{1}{1 - b - a}$$

The import multiplier is given by

$$\frac{\partial \bar{Y}}{\partial Z_0} = -\frac{1}{1 - b - a}$$

And the multiplier for a change in the marginal propensity to invest is given by $\partial \bar{Y}/\partial a$, where, by means of the quotient rule,

$$\frac{\partial \bar{Y}}{\partial a} = \frac{(1 - b - a)(0) - (C_0 + I_0 + G_0 + X_0 - Z_0)(-1)}{(1 - b - a)^2} = \frac{C_0 + I_0 + G_0 + X_0 - Z_0}{(1 - b - a)^2}$$

6.3 PARTIAL ELASTICITIES

Income elasticity of demand ϵ_Y measures the percentage change in the demand for a good resulting from a small percentage change in income, when all other variables are held constant. *Cross elasticity of demand* ϵ_c measures the relative responsiveness of the demand for one product to changes in the price of another, when all other variables are held constant. Given the demand function

$$Q_1 = a - bP_1 + cP_2 + mY$$

where Y = income and P_2 = the price of a substitute good, the income elasticity of demand is

$$\epsilon_Y = \frac{\partial Q_1}{Q_1} \div \frac{\partial Y}{Y} = \frac{\partial Q_1}{\partial Y}\left(\frac{Y}{Q_1}\right)$$

and the cross elasticity of demand is

$$\epsilon_c = \frac{\partial Q_1}{Q_1} \div \frac{\partial P_2}{P_2} = \frac{\partial Q_1}{\partial P_2}\left(\frac{P_2}{Q_1}\right)$$

Since a multivariate function has more than one elasticity, the various elasticities are called *partial elasticities*. See Examples 1 and 2 and Problems 6.18 to 6.21.

EXAMPLE 1. Given the demand for beef

$$Q_b = 4850 - 5P_b + 1.5P_p + 0.1Y \tag{6.2}$$

with $Y = 10\,000$, $P_b = 200$, and the price of pork $P_p = 100$. The calculations for (1) the income elasticity and (2) the cross elasticity of demand for beef are given below.

(1)
$$\epsilon_Y = \frac{\partial Q_b}{Q_b} \div \frac{\partial Y}{Y} = \frac{\partial Q_b}{\partial Y}\left(\frac{Y}{Q_b}\right) \tag{6.3}$$

From (6.2),
$$\frac{\partial Q_b}{\partial Y} = 0.1$$

and
$$Q_b = 4850 - 5(200) + 1.5(100) + 0.1(10\,000) = 5000 \tag{6.4}$$

Substituting in (6.3), $\epsilon_Y = 0.1(10\,000/5000) = 0.2$.

With $\epsilon_Y < 1$, the good is income-inelastic. For any given percentage increase in national income, demand for the good will increase less than proportionately. Hence the relative market share of the good will decline as the economy expands. Since the income elasticity of demand suggests the growth potential of a market, the growth potential in this case is limited.

(2)
$$\epsilon_c = \frac{\partial Q_b}{Q_b} \div \frac{\partial P_p}{P_p} = \frac{\partial Q_b}{\partial P_p}\left(\frac{P_p}{Q_b}\right)$$

From (6.2), $\partial Q_b/\partial P_p = 1.5$; from (6.4), $Q_b = 5000$. Thus,

$$\epsilon_c = 1.5\left(\frac{100}{5000}\right) = 0.03$$

6.8 HOMOGENEOUS PRODUCTION FUNCTIONS

A production function is said to be homogeneous if when each input factor is multiplied by a positive real constant k, the constant can be completely factored out. If the exponent of the factor is 1, the function is homogeneous of degree 1; if the exponent of the factor is greater than 1, the function is homogeneous of degree greater than 1; and if the exponent of the factor is less than 1, the function is homogeneous of degree less than 1. Mathematically, a function $z = f(x, y)$ is homogeneous of degree n if for all positive real values of k, $f(kx, ky) = k^n f(x, y)$. See Example 10 and Problem 6.44.

EXAMPLE 10. The degree of homogeneity of a function is illustrated below.

1. $z = 8x + 9y$ is homogeneous of degree 1 because

$$f(kx, ky) = 8kx + 9ky = k(8x + 9y)$$

2. $z = x^2 + xy + y^2$ is homogeneous of degree 2 because

$$f(kx, ky) = (kx)^2 + (kx)(ky) + (ky)^2 = k^2(x^2 + xy + y^2)$$

3. $z = x^{0.3}y^{0.4}$ is homogeneous of degree less than 1 because

$$f(kx, ky) = (kx)^{0.3}(ky)^{0.4} = k^{0.3+0.4}(x^{0.3}y^{0.4}) = k^{0.7}(x^{0.3}y^{0.4})$$

4. $z = 2x/y$ is homogeneous of degree 0 because

$$f(kx, ky) = \frac{2kx}{ky} = 1\left(\frac{2x}{y}\right) \qquad \text{since } \frac{k}{k} = k^0 = 1$$

5. $z = x^3 + 2xy + y^3$ is not homogeneous because k cannot be completely factored out:

$$\begin{aligned} f(kx, ky) &= (kx)^3 + 2(kx)(ky) + (ky)^3 \\ &= k^3x^3 + 2k^2xy + k^3y^3 = k^2(kx^3 + 2xy + ky^3) \end{aligned}$$

6.9 RETURNS TO SCALE

A production function exhibits *constant returns to scale* if when all inputs are increased by a given proportion k, output increases by the same proportion. If output increases by a proportion greater than k, there are *increasing returns to scale*; and if output increases by a proportion smaller than k, there are *diminishing returns to scale*. In other words, if the production function is homogeneous of degree greater than, equal to, or less than 1, returns to scale are increasing, constant, or diminishing. See Problems 6.44 and 6.57.

6.10 OPTIMIZATION OF COBB-DOUGLAS PRODUCTION FUNCTIONS

Economic analysis is frequently couched in terms of the *Cobb-Douglas production function* $q = AK^{\alpha}L^{\beta}$ ($A > 0$; $0 < \alpha, \beta < 1$), where q is the quantity of output in physical units, K the quantity of capital, and L the quantity of labor. Here α (the *output elasticity of capital*) measures the percentage change in q for a 1 percent change in K while L is held constant; β (the *output elasticity of labor*) is exactly parallel; and A is an *efficiency parameter* reflecting the level of technology.

A *strict* Cobb-Douglas function, in which $\alpha + \beta = 1$, exhibits *constant returns to scale*. A *generalized* Cobb-Douglas function, in which $\alpha + \beta \neq 1$, exhibits *increasing returns to scale* if $\alpha + \beta > 1$ and *decreasing returns to scale* if $\alpha + \beta < 1$. A Cobb-Douglas function is optimized subject to a budget constraint in Example 12 and Problems 6.45 and 6.46; second-order conditions are explained in Section 12.5. Selected properties of Cobb-Douglas functions are demonstrated and proved in Problems 6.57 to 6.62.

EXAMPLE 11. The first and second partial derivatives for (*a*) $q = AK^{\alpha}L^{\beta}$ and (*b*) $q = 5K^{0.4}L^{0.6}$ are illustrated below.

(*a*)
$$q_K = \alpha AK^{\alpha-1}L^{\beta} \qquad q_L = \beta AK^{\alpha}L^{\beta-1}$$
$$q_{KK} = \alpha(\alpha-1)AK^{\alpha-2}L^{\beta} \qquad q_{LL} = \beta(\beta-1)AK^{\alpha}L^{\beta-2}$$
$$q_{KL} = \alpha\beta AK^{\alpha-1}L^{\beta-1} \qquad q_{LK} = \alpha\beta AK^{\alpha-1}L^{\beta-1}$$

(*b*)
$$q_K = 2K^{-0.6}L^{0.6} \qquad q_L = 3K^{0.4}L^{-0.4}$$
$$q_{KK} = -1.2K^{-1.6}L^{0.6} \qquad q_{LL} = -1.2K^{0.4}L^{-1.4}$$
$$q_{KL} = 1.2K^{-0.6}L^{-0.4} \qquad q_{LK} = 1.2K^{-0.6}L^{-0.4}$$

EXAMPLE 12. Given a budget constraint of \$108 when $P_K = 3$ and $P_L = 4$, the generalized Cobb-Douglas production function $q = K^{0.4}L^{0.5}$ is optimized as follows:

1. Set up the Lagrangian function.

$$Q = K^{0.4}L^{0.5} + \lambda(108 - 3K - 4L)$$

2. Using the simple power function rule, take the first-order partial derivatives, set them equal to zero, and solve simultaneously for K_0 and L_0 (and λ_0, if desired).

$$\frac{\partial Q}{\partial K} = Q_K = 0.4K^{-0.6}L^{0.5} - 3\lambda = 0 \qquad (6.15)$$

$$\frac{\partial Q}{\partial L} = Q_L = 0.5K^{0.4}L^{-0.5} - 4\lambda = 0 \qquad (6.16)$$

$$\frac{\partial Q}{\partial \lambda} = Q_{\lambda} = 108 - 3K - 4L = 0 \qquad (6.17)$$

Rearrange, then divide (6.15) by (6.16) to eliminate λ.

$$\frac{0.4K^{-0.6}L^{0.5}}{0.5K^{0.4}L^{-0.5}} = \frac{3\lambda}{4\lambda}$$

Remembering to subtract exponents in division,

$$0.8K^{-1}L^{1} = 0.75$$

$$\frac{L}{K} = \frac{0.75}{0.8} \qquad L = 0.9375K$$

Substitute $L = 0.9375K$ in (6.17).

$$108 - 3K - 4(0.9375K) = 0 \qquad K_0 = 16$$

Then by substituting $K_0 = 16$ in (6.17),

$$L_0 = 15$$