Electrodynamics-II

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5.4.2 Magnetostatics Boundary Conditions

We shall see boundary conditions for two components of Magnetic fields

- For Perpendicular components of magnetic field
- For Tangential/parallel components of magnetic field

For perpendicular components of Magnetic Field

$$\oint \vec{B}.d\vec{a} = \int \vec{\nabla}.\vec{B}\, d\tau'$$

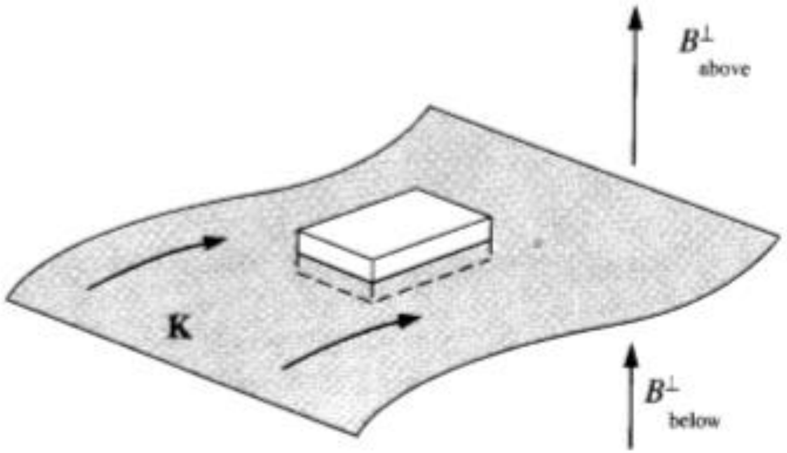

Figure 5.49

We have proved that $\vec{\nabla}.\vec{B} = 0$

So

$$\oint \vec{B}.d\vec{a} = 0$$

$$\oint \vec{B}.d\vec{a} = \int_{\text{Top}} \vec{B}.d\vec{a} + \int_{\text{bottom}} \vec{B}.d\vec{a} + \int_{\text{Side walls}} \vec{B}.d\vec{a}$$

Perpendicular Field Boundary Conditions

$$\oint \vec{B}.d\vec{a} = \int_{\text{Top}} B^{\perp}_{above} da - \int_{\text{bottom}} B^{\perp}_{below} da + \int_{\text{Side walls}} \vec{B}.d\vec{a}$$

Integral of magnetic field due to side walls is zero because $\int da \rightarrow 0$

So last term vanishes and we get

$$\oint \vec{B}.d\vec{a} = \int_{\text{Top}} B^{\perp}_{above} da - \int_{\text{bottom}} B^{\perp}_{below} da$$

$$\oint \vec{B}.d\vec{a} = B^{\perp}_{above} \int da - B^{\perp}_{below} \int da = A(B^{\perp}_{above} - B^{\perp}_{below}) = 0$$

For perpendicular components of Magnetic Field

$$\oint \vec{B}.d\vec{a} = B^{\perp}_{above} \int da - B^{\perp}_{below} \int da = A(B^{\perp}_{above} - B^{\perp}_{below}) = 0$$

Area cannot be equal to zero so

$$B^{\perp}_{above} - B^{\perp}_{below} = 0$$

$$B^{\perp}_{above} = B^{\perp}_{below}$$

It is proved that perpendicular components of magnetic fields are continuous.

Boundary conditions for parallel components components of Magnetic Field

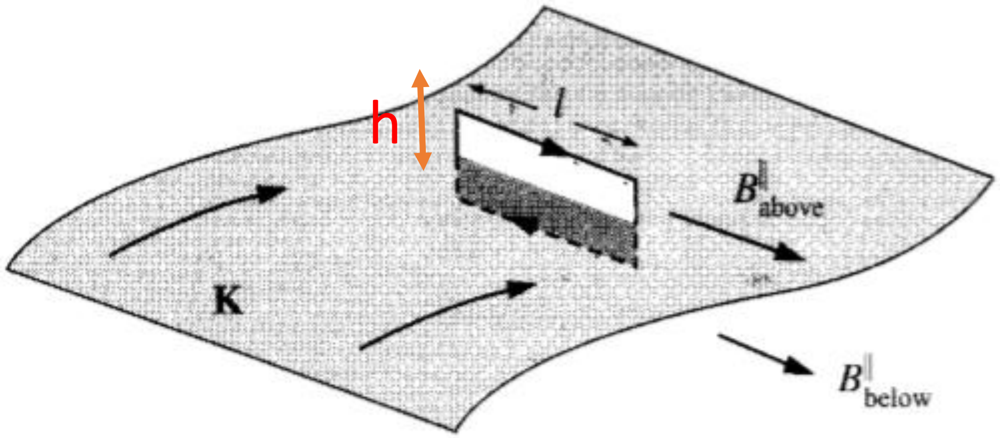

Figure 5.50

Boundary conditions for parallel components components of Magnetic Field

Using the Ampere's Law

$$\oint \vec{B}.d\vec{l} = \int_{\text{Below}} \vec{B}.d\vec{l} + \int_{\text{Top}} \vec{B}.d\vec{l} + \int_{\text{Height}} \vec{B}.d\vec{l} = \mu_0 I_{enc}$$

$$\oint \vec{B}.d\vec{a} = \int B^{\parallel}_{above} dl - \int B^{\parallel}_{below} dl + \int_{\text{Height}} \vec{B}.d\vec{l} = \mu_0 I_{enc}$$

Last term $\int \vec{B}.d\vec{l}$ *goes to zero because height of amperian loop approaches to zero*

$$B^{\parallel}_{above} \int dl - B^{\parallel}_{below} \int dl = l(B^{\parallel}_{above} - B^{\parallel}_{below}) = \mu_0 I_{enc}$$

Where $I_{en} = KL$ and L is length of Amperian loop

Boundary conditions for parallel components components of Magnetic Field

$$l(B_{above}^{\parallel} - B_{below}^{\parallel}) = \mu_0 K l$$

$$B_{above}^{\parallel} - B_{below}^{\parallel} = \mu_0 K$$

$$(B_{above}^{\parallel} - B_{below}^{\parallel}) = \mu_0 K$$

$$B_{above} - B_{below} = \mu_0(\vec{K} \times \hat{n})$$

Where $\hat{n}$ is the unit vector perpendicular to the surface pointing upward.

Multipole Expansion of the vector potential

- We want to find out the approximate formula for magnetic vector potential at very large distances $\vec{r}$ due to this current distribution.(fig.5.51)
- Inverse of Separation vector can be written in terms of Legendre Polynomials. (see section 3.4.1)

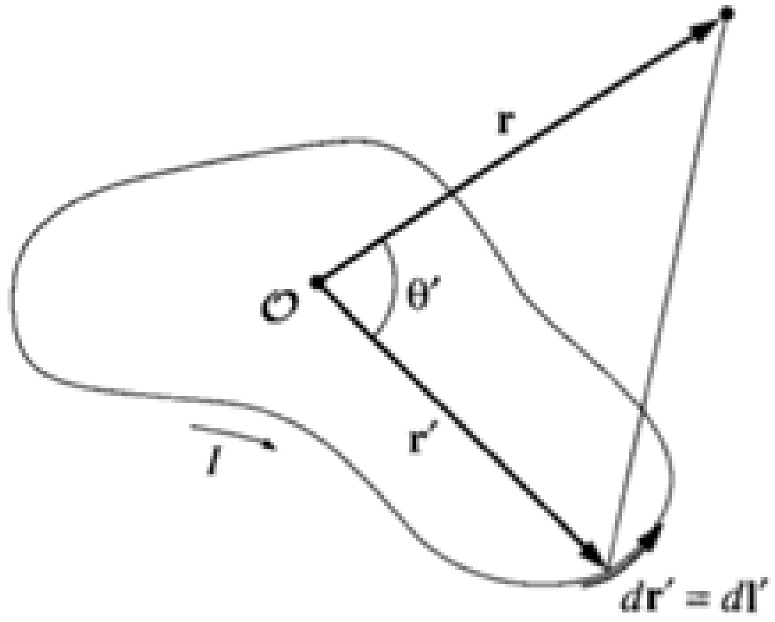

Figure 5.51

$$\frac{1}{\mathscr{r}} = \frac{1}{\sqrt{r^2 + (r')^2 - 2rr'cos\theta'}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n cos\theta'$$

Legendre Polynomial terms

$$P_0(\cos\theta) = 1$$
$$P_1(\cos\theta) = \cos\theta$$
$$P_2(\cos\theta) = \tfrac{1}{4}(1 + 3\cos 2\theta)$$
$$P_3(\cos\theta) = \tfrac{1}{8}(3\cos\theta + 5\cos 3\theta)$$
$$P_4(\cos\theta) = \tfrac{1}{64}(9 + 20\cos 2\theta + 35\cos 4\theta)$$
$$P_5(\cos\theta) = \tfrac{1}{128}(30\cos\theta + 35\cos 3\theta + 63\cos 5\theta)$$
$$P_6(\cos\theta) = \tfrac{1}{512}(50 + 105\cos 2\theta + 126\cos 4\theta + 231\cos 6\theta)$$
$$P_7(\cos\theta) = \tfrac{1}{1024}(175\cos\theta + 189\cos 3\theta + 231\cos 5\theta + 429\cos 7\theta)$$

Multipole Expansion of the vector potential

Accordingly, the vector potential of a current loop can be written

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \oint \frac{1}{\imath}\, d\mathbf{l}' = \frac{\mu_0 I}{4\pi} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \oint (r')^n P_n(\cos\theta')\, d\mathbf{l}', \tag{5.78}$$

or, more explicitly:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \left[\frac{1}{r} \oint d\mathbf{l}' + \frac{1}{r^2} \oint r' \cos\theta'\, d\mathbf{l}' + \frac{1}{r^3} \oint (r')^2 \left(\frac{3}{2}\cos^2\theta' - \frac{1}{2} \right) d\mathbf{l}' + \cdots \right]. \tag{5.79}$$

Multipole Expansion of the vector potential

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \frac{1}{r} \oint d\mathbf{l}'$$ → Magnetic Mono Pole Term

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \frac{1}{r^2} \oint r' \cos\theta' \, d\mathbf{l}'$$ → Magnetic Dipole Pole Term

As in the multipole expansion of V, we call the first term (which goes like $1/r$) the **monopole** term, the second (which goes like $1/r^2$) **dipole**, the third **quadrupole**, and so on.

Mono Pole Term

Now, it happens that the *magnetic monopole term is always zero*, for the integral is just the total vector displacement around a closed loop:

$$\oint d\mathbf{l}' = 0. \tag{5.80}$$

This reflects the fact that there are (apparently) no magnetic monopoles in nature (an assumption contained in Maxwell's equation $\nabla \cdot \mathbf{B} = 0$, on which the entire theory of vector potential is predicated).

Magnetic Moment and Magnetic Vector Potential

In the absence of any monopole contribution, the dominant term is the dipole (except in the rare case where it, too, vanishes):

$\hat{r}.\vec{r'} = r'cos\theta'$

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0 I}{4\pi r^2}\oint r' \cos\theta' \, d\mathbf{l}' = \frac{\mu_0 I}{4\pi r^2}\oint (\hat{\mathbf{r}} \cdot \mathbf{r}') \, d\mathbf{l}'. \tag{5.81}$$

This integral can be rewritten in a more illuminating way if we invoke Eq. 1.108, with $\mathbf{c} = \hat{\mathbf{r}}$:

$$\oint (\mathbf{c} \cdot \mathbf{r}) \, d\mathbf{l} = \mathbf{a} \times \mathbf{c},$$

$$\oint (\hat{\mathbf{r}} \cdot \mathbf{r}') \, d\mathbf{l}' = -\hat{\mathbf{r}} \times \int d\mathbf{a}'. \tag{5.82}$$

Where

$$\mathbf{a} \equiv \int_S d\mathbf{a}$$

Then

$$\boxed{\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2},} \tag{5.83}$$

where **m** is the **magnetic dipole moment**:

$$\boxed{\mathbf{m} \equiv I \int d\mathbf{a} = I\mathbf{a}.} \tag{5.84}$$

Where **a is vector area of loop.**