Electrodynamics II

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Comparison between Electrostatics and Magnetostatics
Magnetic Vector Potential

Electrostatics	Magnetostatics
In Electrostatics, we have studied $\vec{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{\mathscr{r}^2}$	In Magnetostatics $\vec{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\boldsymbol{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2} d\tau'$
$\vec{\nabla} \times \vec{E} = 0$ (For Electrostatic Field)	$\vec{\nabla}.\vec{B}(\boldsymbol{r}) = 0$ (For Static Magnetic Field)

Electrostatics	Magnetostatics
Any vector whose curl is equal to zero can be written as the gradient of another another scaler. i.e. $\vec{E} = -\vec{\nabla} V$ **We had defined** $V = \text{Electric Potential}$ Using the Coulomb' law $\vec{\nabla}.\vec{E} = \frac{\rho}{\epsilon_0}$ **Gauss's Law**	**Any vector whose divergence is equal to zero can be written as the curl of another vector** $\vec{B} = \vec{\nabla} \times \vec{A}$ *Here we define* $\vec{A} = Magnetic\ vector\ potential$ Using the Biot-Savart law $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ **Ampere's Law**
$\vec{\nabla}.(-\vec{\nabla} V) = \frac{\rho}{\epsilon_0}$	$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla}.\vec{A}) - \vec{\nabla}^2 \vec{A} = \mu_0 \vec{J}$ **We choose vector** $\vec{\mathrm{A}}$ **such that its** $\vec{\nabla}.\vec{\mathrm{A}} = 0$**, but curl is not zero.**

Electrostatics	Magnetostatics

$$\nabla^2 V = \frac{-\rho}{\epsilon_0}$$

Passion's Equation

$$-\nabla^2 \vec{A} = \mu_0 \vec{J}$$

$$\nabla^2 \vec{A} = -\mu_0 \vec{J}$$

In components form

$$\nabla^2 A_x \hat{x} + \nabla^2 A_y \hat{y} + \nabla^2 A_z \hat{z} = -\mu_0 (J_x \hat{x} + J_y \hat{y} + J_z \hat{z}$$

$$\left.\begin{aligned} \nabla^2 A_x &= -\mu_0 J_x \\ \nabla^2 A_y &= -\mu_0 J_y \\ \nabla^2 A_z &= -\mu_0 J_z \end{aligned}\right\}$$

$\rho \longleftrightarrow \vec{J}$

Electrostatics	Magnetostatics
Electric Potential	**Magnetic Vector Potential**
for charge density	By using the same token
$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}')d\tau'}{\mathscr{r}}$	$A_x = \frac{\mu_0}{4\pi}\int \frac{J_x d\tau'}{\mathscr{r}}$
	$A_y = \frac{\mu_0}{4\pi}\int \frac{J_y d\tau'}{\mathscr{r}}$
	$A_z = \frac{\mu_0}{4\pi}\int \frac{J_z d\tau'}{\mathscr{r}}$
	And combining all these equations We get equation of current density
	$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')d\tau'}{\mathscr{r}}$

Method-II

Derivation of magnetic vector potential

Magnetic Vector Potential

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\vec{r}\,')da'}{\mathcal{r}}$$ For Surface Charge Density	$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{K(\vec{r}\,')da'}{\mathcal{r}}$$ For Surface Current Density
$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\vec{r}\,')dl'}{\mathcal{r}}$$ For Line Charge Density	$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{I}dl'}{\mathcal{r}}$$ For Line Current Density

Content

Example 5.11

Assignment

Problem 5.22, 5.23

Example 5.11

A spherical shell, of radius R, carrying a uniform surface charge σ, is set spinning at angular velocity $\boldsymbol{\omega}$. Find the vector potential it produces at point $\mathbf{r}$ (Fig. 5.45).

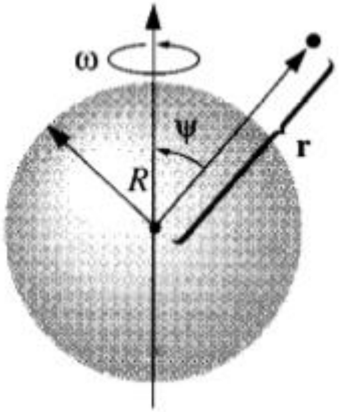

Figure 5.45

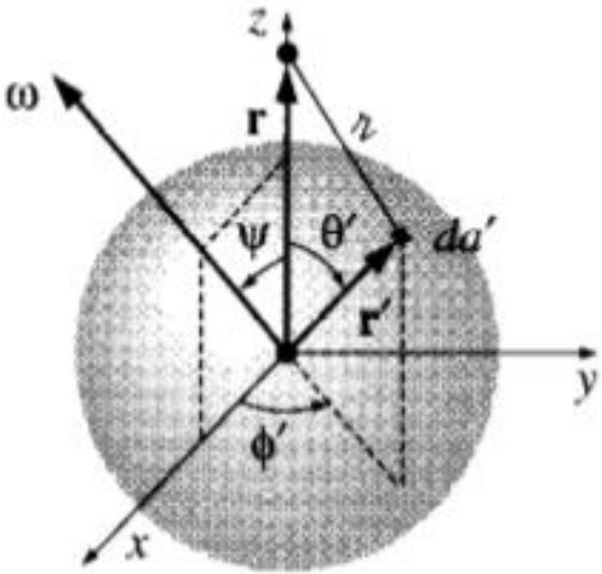

Figure 5.46

Example 5.11

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{K}(\vec{r}')da'}{\mathscr{r}}$$

$$\mathscr{r} = \sqrt{R^2 + r^2 - 2Rr\cos\theta'}$$

$$d\vec{a} = R^2 \sin\theta' d\theta' d\phi'$$

$\vec{K} = \sigma\vec{v}$ and $\vec{v} = \vec{\omega} \times \vec{r'}$

$$\vec{K} = \sigma(\vec{\omega} \times \vec{r'}) \longrightarrow \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\sigma(\vec{\omega} \times \vec{r})\, da'}{\mathscr{r}}$$

Example 5.11

$r' = R$ because r' represents the points lying on the surface of sphere

$$da' = R^2 sin\theta' d\theta' d\emptyset'$$

$$\vec{v} = \vec{\omega} \times \vec{r'} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ w sin\psi & 0 & w cos\psi \\ R sin\theta' cos\emptyset' & R sin\theta' sin\emptyset' & R cos\emptyset' \end{vmatrix}$$

$$= Rw\,[-(w cos\psi sin\theta' cos\emptyset')\hat{x} + (cos\psi sin\theta' cos\emptyset' - sin\psi cos\theta')\hat{y} + sin\psi sin\theta' sin\emptyset')\hat{z}]$$

Example 5.11

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{K}(\vec{r}')da'}{\mathscr{r}}$$

$$= \frac{\mu_0}{4\pi}\int \frac{\vec{K}(\vec{r}')R^2 sin\theta' d\theta' d\emptyset'}{\sqrt{R^2 + r^2 - 2Rrcos\theta'}}$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\sigma\vec{v}\ da'}{\mathscr{r}} = \frac{\mu_0}{4\pi}\int \frac{\sigma(\vec{\omega} \times \vec{r})R^2 sin\theta' d\theta' d\emptyset'}{\sqrt{R^2 + r^2 - 2Rrcos\theta'}}$$

Example 5.11

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\sigma(\vec{r}\times\vec{\omega})R^2 sin\theta' d\theta' d\emptyset'}{\sqrt{R^2+r^2-2Rrcos\theta'}}$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\sigma(Rw\,[-(wcos\psi sin\theta' cos\emptyset')\hat{x} + (cos\psi sin\theta' cos\emptyset' - sin\psi cos\theta')\hat{y} + sin\psi sin\theta' sin\emptyset')\hat{z}]\,)R^2 sin\theta' d\theta' d\emptyset'}{\sqrt{R^2+r^2-2Rrcos\theta'}}$$

$$\int_0^{2\pi} sin\emptyset'\, d\emptyset' = \int_0^{2\pi} cos\emptyset'\, d\emptyset' = 0$$

$$\vec{A} = -\frac{\mu_0 R^3 \sigma w sin\psi}{2}\int_0^{\pi} \frac{cos\theta' sin\theta' d\theta'}{\sqrt{R^2+r^2-2Rrcos\theta'}}\hat{y}$$

Example 5.11

$$\vec{A} = -\frac{\mu_0 R^3 \sigma w \sin\psi}{2} \int_0^{\pi} \frac{\cos\theta' \sin\theta' d\theta'}{\sqrt{R^2 + r^2 - 2Rr\cos\theta'}} \hat{y}$$

Let $u = \cos\theta'$ ⟹ $du = -\sin\theta' d\theta'$

For Limits

When $\theta' = 0$ ⟹ $u = \cos 0 = 1$

When $\theta' = \pi$ ⟹ $u = \cos\pi = -1$

$$\vec{A} = \frac{\mu_0 R^3 \sigma w \sin\psi}{2} \int_0^{\pi} \frac{\cos\theta' \left(-\sin\theta' d\theta'\right)}{\sqrt{R^2 + r^2 - 2Rr\cos\theta'}} \hat{y}$$

$$\vec{A} = \frac{\mu_0 R^3 \sigma w \sin\psi}{2} \int_1^{-1} \frac{u\,du}{\sqrt{R^2 + r^2 - 2Rru}} \hat{y}$$

Example 5.11

$$\vec{A} = -\frac{\mu_0 R^3 \sigma w sin\psi}{2} \int_{-1}^{1} \frac{udu}{\sqrt{R^2 + r^2 - 2Rru}} \hat{y}$$

$$Now \int_{-1}^{1} \frac{udu}{\sqrt{R^2 + r^2 - 2Rru}} = -\frac{R^2 + r^2 + Rru}{3R^2r^2} \sqrt{R^2 + r^2 - 2Rru}\Bigg|_{-1}^{1}$$

$$= -\frac{R^2 + r^2 + Rr(1)}{3R^2r^2} \sqrt{R^2 + r^2 - 2Rr(1)} + \frac{R^2 + r^2 + Rr(-1)}{3R^2r^2} \sqrt{R^2 + r^2 - 2Rr(-1)}$$

Example 5.11

$$= -\frac{R^2 + r^2 + Rr(1)}{3R^2r^2}\sqrt{R^2 + r^2 - 2Rr(1)} + \frac{R^2 + r^2 + Rr(-1)}{3R^2r^2}\sqrt{R^2 + r^2 - 2Rr(-1)}$$

$$= -\frac{1}{3R^2r^2}\left[(R^2+r^2 + Rr)|R - r| - (R^2+r^2 - Rr)(R + r)\right]$$

Inside the Sphere For R > r

$$= -\frac{1}{3R^2r^2}\left[(R^2+r^2 + Rr)(R - r) - (R^2+r^2 - Rr)(R + r)\right]$$

$$= -\frac{1}{3R^2r^2}\left[(R^3-rR^2 + Rr^2 - r^3 + rR^2 - Rr^2) - (R^3+rR^2 + Rr^2 + r^3 - rR^2 - Rr^2)\right]$$

Example 5.11

$$= -\frac{1}{3R^2r^2}[(R^3 - rR^2 + Rr^2 - r^3 + rR^2 - Rr^2) - (R^3 + rR^2 + Rr^2 + r^3 - rR^2 - Rr^2)]$$

$$= -\frac{1}{3R^2r^2}[(R^3 - r^3) - (R^3 + r^3)]$$

$$\boxed{\int_{-1}^{1} \frac{u\,du}{\sqrt{R^2 + r^2 - 2Rru}} = \frac{2r^3}{3R^2r^2} = \frac{2r}{3R^2}}$$

Inside the Sphere For R > r

Example 5.11

outside the sphere For r > R

$$= -\frac{1}{3R^2r^2}[(R^2+r^2+Rr)(r-R)\ -(R^2+r^2-Rr)(R+r)\ ]$$

$$= -\frac{1}{3R^2r^2}[(rR^2-R^3+r^3-Rr^2+Rr^2-rR^2)\ -(R^3+rR^2+Rr^2+r^3-rR^2-Rr^2)]$$

$$= -\frac{1}{3R^2r^2}[(-R^3+r^3)\ -(R^3+r^3)]$$

$$\int_{-1}^{1}\frac{udu}{\sqrt{R^2+r^2-2Rru}}=\frac{2R^3}{3R^2r^2}=\frac{2R}{3r^2}$$ ***outside the sphere For r > R***

Example 5.11

Outside the Sphere For $R > r$

$$\vec{A} = -\frac{\mu_0 R^3 \sigma w sin\psi}{2}\int_{-1}^{1}\frac{udu}{\sqrt{R^2 + r^2 - 2Rru}}\hat{y} = -\frac{\mu_0 R^3 \sigma w sin\psi}{2}\left(\frac{2R}{3r^2}\right)\hat{y}$$

$$\vec{A} = \frac{\mu_0 R^3 \sigma(-wrsin\psi\hat{y})}{2}\left(\frac{2R}{3r^3}\right)$$

and

$$\vec{\omega} \times \vec{r} = -\omega r sin\psi\hat{y}$$

$$\boxed{\vec{A} = \frac{\mu_0 R^4 \sigma}{3r^3}\vec{\omega} \times \vec{r}}$$

Outside the Sphere For $R > r$

Example 5.11

Inside the Sphere For $R > r$

$$\vec{A} = -\frac{\mu_0 R^3 \sigma w sin\psi}{2}\int_{-1}^{1}\frac{udu}{\sqrt{R^2+r^2-2Rru}}\hat{y} = -\frac{\mu_0 R^3 \sigma w sin\psi}{2}\left(\frac{2r}{3R^2}\right)\hat{y}$$

$$\vec{A} = \frac{\mu_0 R\sigma(-wrsin\psi\hat{y})}{3}$$

$$\boxed{\vec{A} = \frac{\mu_0 R\sigma}{3}\vec{\omega}\times\vec{r}}$$

Inside the Sphere For $r < R$

Example 5.11

In natural spherical coordinates(fig.5.45) , r has angle θ making with z axis.

let ω is along z axis, angle between ω and r is θ. Then

$$\vec{\omega} \times \vec{r} = -\omega r sin\theta \hat{\emptyset}$$

$$\vec{A} = \frac{\mu_0 R^4 \sigma}{3r^3} \vec{\omega} \times \vec{r} = \frac{\mu_0 R^4 \sigma}{3r^3} (-\omega r sin\theta \hat{\emptyset}) = -\frac{\mu_0 \omega R^4 \sigma}{3r^2} sin\theta \hat{\emptyset}$$

$$\boxed{\vec{A} = -\frac{\mu_0 \omega R^4 \sigma}{3r^2} sin\theta \hat{\emptyset}}$$

$Outside\ the\ Sphere\ For\ R > r$

Example 5.11

$$\vec{\omega} \times \vec{r} = -\omega r sin\theta \widehat{\emptyset}$$

$$\vec{A} = \frac{\mu_0 R\sigma}{3} \vec{\omega} \times \vec{r} = \frac{\mu_0 R\sigma}{3} \left(-\omega r sin\theta \widehat{\emptyset}\right) = -\frac{\mu_0 \omega R\sigma}{3} r sin\theta \widehat{\emptyset}$$

$$\boxed{\vec{A}(r,\theta,\emptyset) = -\frac{\mu_0 \omega R\sigma}{3} r sin\theta \widehat{\emptyset}}$$

$Inside\ the\ Sphere\ For\ r < R$