Electrodynamics II

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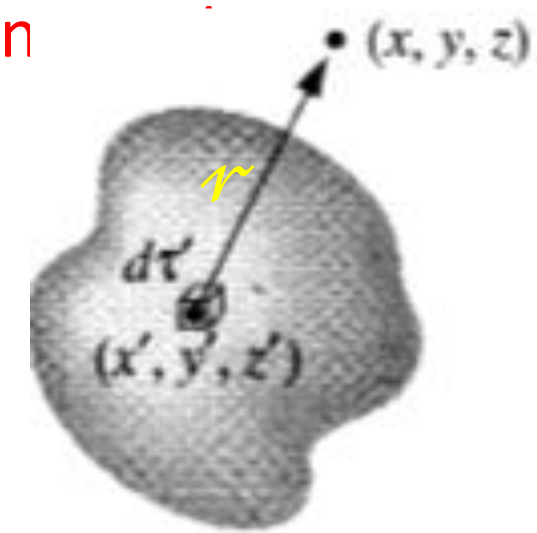
Divergence and Curl of B

The Biot-Savart law for the general case of a volume curren



$$\vec{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\boldsymbol{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2} d\tau'$$

This formula gives the magnetic field at a point $\boldsymbol{r} = (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ in terms of an integral over the current distribution $\vec{J}(\boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}')$ (Fig)

Here

$\vec{B}$ is a function of (x, y, z)

$\vec{J}$ is the function of (x', y', z')

$d\tau' = dx'dy'dz'$

$\vec{\mathscr{r}} = (x - x')\hat{x} + (y - y')\hat{y} + (z - z)\hat{z}$

Divergence and Curl of B

The integration is over the primed coordinates; the divergence and the curl are to be taken with respect to the unprimed coordinates.

Applying the divergence to above equation we obtain

$$\vec{\nabla}.\vec{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int \vec{\nabla}.\left(\frac{\vec{J}(\boldsymbol{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2}\right) d\tau'$$

Using the Product rule $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}),$

$$\vec{\nabla}.\left(\frac{\vec{J}(\boldsymbol{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}}.\,(\vec{\nabla} \times \vec{J}) - \vec{J}.\,(\vec{\nabla} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2})$$

But $\vec{\nabla} \times \vec{J} = 0$

Because $\vec{J}$ does not depend on the unprimed variables (x,y,z)

Divergence and Curl of B

Whereas

$$\vec{\nabla}.\left(\frac{\vec{J}(\boldsymbol{r'}) \times \hat{\mathscr{r}}}{\mathscr{r}^2}\right) = -\vec{J}.(\vec{\nabla} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2})$$

$$\vec{\nabla} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = 0$$

Problem:1.62

So

$$\vec{\nabla}.\vec{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int \vec{\nabla}.\left(\frac{\vec{J}(\boldsymbol{r'}) \times \hat{\mathscr{r}}}{\mathscr{r}^2}\right) d\tau' = 0$$

$$\boxed{\vec{\nabla}.\vec{B}(\boldsymbol{r}) = 0}$$

Divergence and Curl of B

Applying curl to equation 5.45

$$\vec{\nabla} \times \vec{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int \vec{\nabla} \times \left(\frac{\vec{J}(\boldsymbol{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2} \right) d\tau'$$

Using the product rule $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}).$

$$\vec{\nabla} \times \left(\frac{\vec{J}(\boldsymbol{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2} \right) = \left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} . \vec{\nabla} \right) \vec{J} - (\vec{J} . \vec{\nabla}) \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} + \vec{J} \left(\vec{\nabla} . \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) - \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} (\vec{\nabla} . \vec{J})$$

$\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} . \vec{\nabla} \right) \vec{J} = 0$ and $\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} (\vec{\nabla} . \vec{J}) = 0$

Divergence and Curl of B

$$\vec{\nabla} \times \left(\frac{\vec{J}(\mathbf{r'}) \times \hat{\mathscr{r}}}{\mathscr{r}^2} \right) = \vec{J} \left(\vec{\nabla} . \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) - (\vec{J} . \vec{\nabla}) \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

The first term involves the divergence

$$\vec{\nabla} . \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = 4\pi\delta^3(\vec{\mathscr{r}})$$

And

$$\vec{\mathscr{r}} = \vec{r} - \overrightarrow{r'}$$

$$\vec{\nabla} . \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = 4\pi\delta^3(\vec{r} - \overrightarrow{r'})$$

The second term integrates to zero, as we'll see in the next slides.

$$(\vec{J} . \vec{\nabla}) \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = 0$$

Divergence and Curl of B

$$\vec{\nabla} \times \vec{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int \vec{\nabla} \times \left(\frac{\vec{J}(\boldsymbol{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) d\tau' = \frac{\mu_0}{4\pi} \int \vec{J} \left(\vec{\nabla} . \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) d\tau'$$

$$= \frac{\mu_0}{4\pi} \int \vec{J}(\vec{r'}) 4\pi \delta^3(\vec{r} - \vec{r'}) d\tau'$$

$$\vec{\nabla} \times \vec{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int \vec{J}(\vec{r'}) 4\pi \delta^3(\vec{r} - \vec{r'}) d\tau'$$

$$\vec{\nabla} \times \vec{B}(\boldsymbol{r}) = \mu_0 \vec{J}(\vec{r})$$

Which confirms that above equation is not restricted to straight line current but hold quite generally in magnetostatics.

Divergence and Curl of B

Now we prove the 2nd term of equation 5.50 is equal to zero

$$-(\vec{J}.\vec{\nabla})\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = (\vec{J}.\vec{\nabla}')\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = (\vec{J}.\vec{\nabla}')\frac{\vec{\mathscr{r}}}{\mathscr{r}^3}$$

Using this $\hat{\mathscr{r}} = \frac{\vec{\mathscr{r}}}{\mathscr{r}}$

$$\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f),$$

$$(\vec{J}.\vec{\nabla}')\left(\frac{x-x'}{\mathscr{r}^3}\right) = \vec{\nabla}'.\left[\frac{(x-x')}{\mathscr{r}^3}\vec{J}\right] - \left(\frac{(x-x')}{\mathscr{r}^3}\right)(\vec{\nabla}'.\vec{J})$$

$$\left[-(\vec{J}.\vec{\nabla})\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right]_x = \vec{\nabla}'.\left[\frac{(x-x')}{\mathscr{r}^3}\vec{J}\right]$$

$$\int (\vec{J}.\vec{\nabla}')\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau' = \int \vec{\nabla}'.\left[\frac{(x-x')}{\mathscr{r}^3}\vec{J}\right]d\tau'$$

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$$\int (\vec{J}.\vec{\nabla}')\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau' = \int \vec{\nabla}'.\left[\frac{(x-x')}{\mathscr{r}^3}\vec{J}\right]d\tau' = \oint \frac{(x-x')}{\mathscr{r}^3}\vec{J}.\overrightarrow{da'}$$

Here $\vec{J}$ is the volume current density and its surface integral is equal to zero.

$$\int (\vec{J}.\vec{\nabla}')\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau' = 0$$